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Propagation, Reflection and Refraction of Singularities for a Hyperbolic Trasmission Problem in Two Adjacent Angular Regions.

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0. Introduction.

The main purpose of this paper is to study propagation and reflection of singularities of the solution to a transmission problem for the wave equation with different sound speeds in two adjacent angular regions.

The reflection of singularities for transmission problems with smooth separation surface has been studied by M.E. Taylor [10] [11].

For manifolds with conical singularities M. Kalka-A. Menikoff [7], J. Cheeger-M. E. Taylor [2] have propagation of singularities results for the wave equation by constructing the Kernel of the fundamental solution for such operator.

On the same type of manifolds M. Rouleux [8], studies the analytic regularity of the Kernel of the fundamental solution of the wave equation, calculated by the methods of functional analysis of Cheeger and Taylor.

J. P. Varenne [12], obtains propagation, reflection and diffraction of singularities results for a mixed Cauchy problem with zero Dirichlet data relative to the wave equation in $\mathbb{R}_+ \times \Omega$, where Ω is a corner of $\mathbb{R}^2$ or a wedge of $\mathbb{R}^3$.

Varenne gives an explicit representation of the solution and determinies its wave front set by the use of the well known Hormander Kernel theorem (see [5]).

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In this paper we use the classic method of images (see I. Stakgold [9]) and the theory of pseudodifferential operators to find an explicit potential representation of solution of the following transmissione problem:

$$
(\mathrm{I}) \quad \begin{cases}
\Box v_1 = 0 & \text{in } \mathbb{R}_+ \times \Omega_1 \\
\Box_c^\alpha v_2 = 0 & \text{in } \mathbb{R}_+ \times \Omega_2 \\
v_1(0, x_1, x_2) = 0 & \dfrac{\partial v_1}{\partial t}(0, x_1, x_2) = \varphi \quad \text{in } \Omega_1 \\
v_2(0, x_1, x_2) = 0 & \dfrac{\partial v_2}{\partial t}(0, x_1, x_2) = \psi \quad \text{in } \Omega_2 \\
v_1|_{\mathbb{R}_+ \times \Gamma_1} = f_1 & v_2|_{\mathbb{R}_+ \times \Gamma_2} = f_2 \\
(v_1 - v_2)|_{\mathbb{R}_+ \times \Gamma} = g_1 & \left(\dfrac{dv_1}{d\lambda} - \dfrac{dv_2}{dN}\right)\Big|_{\mathbb{R}_+ \times \Gamma} = g_2
\end{cases}
$$

where

$$\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 > 0, x_2 > 0\}, \qquad \Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 < 0, x_2 > 0\},$$

$$\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 > 0, x_2 = 0\}, \qquad \Gamma_2 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 < 0, x_2 = 0\},$$

Γ is the common boundary of Ω_1 and Ω_2 and the data φ, ψ, f_1, f_2, g_1, g_2 are opportune distribution with compact support.

Moreover $\Box$ is the wave operator $\partial^2/\partial t^2 - \Delta$, $\Box_c^\alpha$ is the «modified» wave operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x_1^2} + \cos^2\alpha\,\frac{\partial^2}{\partial x_2^2}\right),$$

N denotes the normal unit vector to Γ interior to Ω_1 and λ is the unit vector of components $(\cos\alpha, \sin\alpha)$. Then we show that the singularities of such solution travel:

1°) along bicharacteristic lines coming out from points of the wave front set of f_1, f_2, g_1 and g_2, if possible after either a reflection on the other face or a refraction on the separating surface $\mathbb{R}_+ \times \Gamma$ according to the geometric optics;

2^o) along bicharacteristic lines which have the initial point in the wave front set of the data φ and ψ, if possible after either at the most two reflections on the faces or a refraction on the common face $\mathbb{R}_+ \times \Gamma$ otherwise a reflection on the edge.

In our case diffraction phenomena don't occur because Ω_1 and Ω_2 have amplitude $\pi/2$, analogously to what happens in [12].

Using the method shown in this work our results can be extended to Ω_1 and Ω_2 right angle wedges of $\mathbb{R}^3$ by only technical difficulties. The results of this paper are explained as follows: in section 1 we precise the notations and the problems studied; in section 2 we study the singularities of two mixed problems for the wave equation in Ω_1 and Ω_2 respectively recalling some results of J. P. Varenne [12]; in section 3 we give a potential representation of the solution of (I) when the initial data φ and ψ are zero using the theory of pseudodifferential operators; in section 4 we give a complete description of wave front set of the solution of the problem (I) by the study of wave front set of the solution obtained in section 3 and the results of the section 1.
We wish to thank Proff. J. Lewis and C. Parenti for useful conversations about this work.

1. Preliminaries.

First we precise some notations.

We define $\Omega_1, \Omega_2, \Gamma_1, \Gamma_2$ as in the introduction. Let

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\} .$$

Moreover we denote the space of extendible distributions by $\mathcal{D}'(\overline{\mathbb{R}_+ \times \Omega_i})$, $i = 1, 2$, the space of distributions with compact support in Ω_i by $\mathcal{E}'(\Omega_i)$, $i = 1, 2$, and the space of distributions on $\mathbb{R}_+$ with values in $H^s(\Gamma)$ by $\mathcal{D}'(\mathbb{R}_+; H^s(\Gamma))$ (see [6]). In the problem (I) we assume that

$$\varphi \in \mathcal{E}'(\Omega_1) , \qquad \psi \in \mathcal{E}'(\Omega_2)$$

$$f_1 \in \mathcal{D}'(\mathbb{R}_+; H^{\frac{3}{2}}(\Gamma_1)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma_1) , \quad f_2 \in \mathcal{D}'(\mathbb{R}_+; H^{\frac{3}{2}}(\Gamma_2)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma_2)$$

$$g_1 \in \mathcal{D}'(\mathbb{R}_+; H^{\frac{3}{2}}(\Gamma)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma) , \quad g_2 \in \mathcal{D}'(\mathbb{R}_+; H^{\frac{1}{2}}(\Gamma)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma) .$$

Finally we consider the operators $\square$ and $\square_c^\alpha$ as in the introduction where $c \in]0, 1[$ and $\alpha \in]-\pi/2, \pi/2[$. We seek a solution $\nu = (\nu_1, \nu_2) \in \mathfrak{D}'(\overline{\mathbb{R}_+ \times \Omega_1}) \times \mathfrak{D}'(\overline{\mathbb{R}_+ \times \Omega_2})$ of the problem (I) in the following way

$$\nu = u + w \tag{1.1}$$

where $u = (u_1, u_2)$ and u_1, u_2 are solutions of the following independent problems

$$\begin{cases} \square u_1 = 0 & & \text{in } \mathbb{R}_+ \times \Omega_1 \\ u_1(0, x_1, x_2) = 0\,, & \dfrac{\partial u_1}{\partial t}(0, x_1, x_2) = \varphi & \text{in } \Omega_1 \\ u_1|_{\mathbf{R}_+ \times \Gamma_1} \equiv 0\,, & u_1|_{\mathbf{R}_+ \times \Gamma} \equiv 0 & \end{cases} \tag{1.2}$$

$$\begin{cases} \square_c^\alpha u_2 = 0 & & \text{in } \mathbb{R}_+ \times \Omega_2 \\ u_2(0, x_1, x_2) = 0\,, & \dfrac{\partial u_2}{\partial t}(0, x_1, x_2) = \psi & \text{in } \Omega_2 \\ u_2|_{\mathbf{R}_+ \times \Gamma_2} \equiv 0\,, & u_2|_{\mathbf{R}_+ \times \Gamma} \equiv 0\,. & \end{cases} \tag{1.3}$$

Instead $w = (w_1, w_2)$ is solution of the following transmission problem:

$$\begin{cases} \square w_1 = 0 & & \text{in } \mathbb{R}_+ \times \Omega_1 \\ \square_c^\alpha w_2 = 0 & & \text{in } \mathbb{R}_+ \times \Omega_2 \\ w_1(0, x_1, x_2) = 0\,, & \dfrac{\partial w_1}{\partial t}(0, x_1, x_2) = 0 & \text{in } \Omega_1 \\ w_2(0, x_1, x_2) = 0\,, & \dfrac{\partial w_2}{\partial t}(0, x_1, x_2) = 0 & \text{in } \Omega_2 \\ w_1|_{\mathbf{R}_+ \times \Gamma_1} = f_1 & w_2|_{\mathbf{R}_+ \times \Gamma_2} = f_2 & \\ (w_1 - w_2)_{\mathbf{R}_+ \times \Gamma} = g_1\,, & \left.\left(\dfrac{dw_1}{d\lambda} - \dfrac{dw_2}{dN}\right)\right|_{\mathbf{R}_+ \times \Gamma} = g_2 - \left.\left(\dfrac{du_1}{d\lambda} - \dfrac{du_2}{dN}\right)\right|_{\mathbf{R}_+ \times \Gamma}. & \end{cases} \tag{1.4}$$

We will describe the wave front set of the solution ν, i.e. $WF(\nu)$ by $WF(u)$ and $WF(w)$, where for a vector valued distributed $\nu = (\nu_1, \nu_2)$ we put

$$WF(\nu) = WF(\nu_1) \cup WF(\nu_2)\,. \tag{1.5}$$

In the section 2 we give a complete description of $WF(u_i)$, $i = 1, 2$, recalling the results of J. P. Varenne [12].

Moreover we determine the wave front set of $(du_1/d\lambda)_{\mathbb{R}_+\times\Gamma}$ and $(du_2/dN)_{\mathbb{R}_+\times\Gamma}$.

Indeed in the section 3 we give a potential representation of the solution w of the problem (1.4), which we use to obtain the wave front set of w knowing the wave front sets of the boundary data.

Later on we denote the cotangent bundle of $\mathbb{R}_+\times\Omega_i$ by $T^*(\mathbb{R}_+\times\Omega_i)$, $i = 1, 2$, and the covariant variables of t, $s \in \mathbb{R}_+$ $x = (x_1, x_2)$, $y = (y_1, y_2) \in \Omega_i$ by $\tau, \sigma \in \mathbb{R}$, $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ respectively.

2. – By the classical method of images the solutions u_1 and u_2 of the problems (1.2) and (1.3) can be written as follows:

$$u_1(t, x_1, x_2) = \int_0^{+\infty}\int_0^{+\infty} G_1(t, x_1, x_2, y_1, y_2)\varphi(y_1, y_2)\, dy_1\, dy_2 \tag{2.1}$$

$$u_2(t, x_1, x_2) = \frac{1}{c\cos a}\int_0^{+\infty}\int_0^{+\infty} G_1\left(ct, x_1, \frac{x_2}{\cos\alpha}, y_1, \frac{y_2}{\cos\alpha}\right)\psi(y_1, y_2)\, dy_1\, dy_2 \tag{2.2}$$

where $G_1(t, x_1, x_2, y_1, y_2)$ is the Green function of the problem (1.2). Precisely

$$\begin{aligned} G_1(t, x_1, x_2, y_1, y_2) &= \frac{1}{2\pi}\frac{H\left(t-\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\right)}{\sqrt{t^2-((x_1-y_1)^2+(x_2-y_2)^2)}} - \\ &- \frac{H\left(t-\sqrt{(x_1-y_1)^2+(x_2+y_2)^2}\right)}{\sqrt{t^2-((x_1-y_1)^2+(x_2+y_2)^2)}} + \frac{H\left(t-\sqrt{(x_1+y_1)^2+(x_2+y_2)^2}\right)}{\sqrt{t^2-((x_1+y_1)^2+(x_2+y_2)^2)}} - \\ &- \frac{H\left(t-\sqrt{(x_1+y_1)^2+(x_2-y_2)^2}\right)}{\sqrt{t_2-((x_1+y_1)^2+(x_2-y_2)^2)}} \end{aligned} \tag{2.3}$$

where $H(s)$ is the Heaviside function.

In order to describe propagation and reflection of singularities of $u = (u_1, u_2)$ we give the following two theorems without proof because we refer to J. P. Varenne [12] for details.

THEOREM 2.1. If $(t, x; \tau, \xi) \in WF(u_1)$ then there exists $(y; \eta) \in \in WF(\varphi)$ such that $(t, x, y; \tau, \xi, -\eta)$ belongs to normal bundle of one of the following surfaces:

$$\Phi^{\pm}(t, x_1, x_2, y_1, y_2) = t - \sqrt{(x_1 \pm y_1)^2 + (x_2 \pm y_2)^2} = 0\ , \tag{2.4}$$

$$\Phi_{\pm}(t, x_1, x_2, y_1, y_2) = t - \sqrt{(x_1 \pm y_1)^2 + (x_2 \mp y_2)^2} = 0\ . \tag{2.5}$$

If $(t, x; \tau, \xi) \in WF(u_2)$ then there exists $(y; \eta) \in WF(\psi)$ such that $(t, x; y; \tau, \xi, -\eta)$ belongs to normal bundle of one the following surfaces:

$$\Phi_c^{\pm}(t, x_1, x_2, y_1, y_2) = t - \frac{1}{c}\sqrt{(x_1 \pm y_1)^2 + \left(\frac{x_2 \pm y_2}{\cos\alpha}\right)^2} = 0\ , \tag{2.6}$$

$$\Phi_{\mp}^{c}(t, x_1, x_2, y_1, y_2) = t - \frac{1}{c}\sqrt{(x_1 \pm y_1)^2 + \left(\frac{x_2 \mp y_2}{\cos\alpha}\right)^2} = 0\ . \tag{2.7}$$

Before enunciating the second theorem we will give some notations.

If $(0, y_1, y_2; 0\, \eta_1, \eta_2) \in T^*(\mathbb{R}_+ \times \Omega_1)$, $\eta_1^2 + \eta_2^2 > 0$, we denote the null bicharacteristic issuing from $(0, y_1, y_2; 0, \eta_1, \eta_2)$, associated to $\tau = \pm\sqrt{\eta_1^2 + \eta_2^2}$ by $\beta_0^{\pm}$:

Moreover we denote the null bicharacteristic issuing from $(0, -y_1, -y_2; 0, -\eta_1, -\eta_2)$ associated to $\tau = \pm\sqrt{\eta_1^2 + \eta_2^2}$ by $\tilde{\beta}_0^{\pm}$.

If $(0, y_1, y_2; 0, \eta_1, \eta_2) \in T^*(\mathbb{R}_+ \times \Omega_2)$, $\eta_1^2 + \cos\alpha^2\, \eta_2^2 > 0$, $\beta_{0c}^{\pm}$ and $\tilde{\beta}_{0c}^{\pm}$ are the null bicharacteristic issuing from $(0, y_1, y_2;\ 0, \eta_1, \eta_2)$ and $(0, -y_1, -y_2; 0, -\eta_1, -\eta_2)$, respectively, associated to

$$\tau = \pm c\sqrt{\eta_1^2 + \cos\alpha^2\, \eta_2^2}\ .$$

THEOREM 2.2. If $(t, x; \tau, \xi)$ $WF(u_2)$ then there is $(y, \eta) \in WF(\psi)$ such that $\tau^2 = c^2(\eta_1^2 + \cos{}^2\alpha\, \eta_2^2)$ and one of two following cases is true:

i) $\beta_{0c}^{\pm}$ hits the corner in $t = ((\cos^2\alpha)/c)(y_1\tau/\eta_1) = y_2\tau/c\eta_2$ when $\tilde{\beta}_{0c}^{\pm}$ hits also the corner and come into $\mathbb{R}_+ \times \Omega_2$ so $(t, x; \tau, \xi)$ belongs to $\beta_{0c}^{\pm}$ or $\tilde{\beta}_{0c}^{\pm}$:

ii) $\beta_{0c}^{\pm}$ does not hit the corner and $(t, x; \tau, \xi)$ joins with $(y; \eta)$ by $\beta_{0c}^{\pm}$ after at most two transversal reflections on the faces $x_1 = 0$ or $x_2 = 0$.

We choice the sign $+$ or $-$ according to τ is positive or negative.

The same results is true for $WF(u_1)$ with $\Omega_2, \psi, \beta^{\pm}_{0c}, \tilde{\beta}^{\pm}_{0c}$ replaced by $\Omega_1, \beta^{\pm}_{0c}, \tilde{\beta}^{\pm}_{0c}$ respectively and $c = \cos\alpha = 1$.

Now we will study $(du_1/d\lambda - du_2/dN)|_{\mathbf{R}_+\times\Gamma}$ to determinate its wave front set.

After differentiating u_1 and u_2 we use the integration by parts to obtain the following expression for the traces of $du_1/d\lambda$ and du_2/dN on $\mathbf{R}_+\times\Gamma$:

$$\text{(2.8)} \qquad \left.\frac{du_1}{d\lambda}\right|_{\mathbf{R}_+\times\Gamma}(t, x_2) = \cos\alpha \left.\frac{\partial u_2}{\partial x_1}\right|_{\mathbf{R}_+\times\Gamma}(t, x_2) =$$

$$= \frac{\cos\alpha}{\pi}\int_0^{+\infty}\int_0^{+\infty}\left[\frac{H(t-\sqrt{y_1^2+(x_2-y_2)^2})}{\sqrt{t^2-(y_1^2+(x_2-y_2)^2)}} - \frac{H(t-\sqrt{y_1^2+(x_2+y_2)^2})}{\sqrt{t^2-(y_1^2+(x_2+y_2)^2)}}\right]\cdot \frac{\partial\varphi}{\partial y_1}(y_1, y_2)\,dy_1\,dy_2$$

$$\text{(2.9)} \qquad \left.\frac{du_2}{dN}\right|_{\mathbf{R}_+\times\Gamma}(t, x_2) = \left.\frac{\partial u_2}{\partial x_1}\right|_{\mathbf{R}_+\times\Gamma}(t, x_2) =$$

$$= \frac{1}{\pi c^2\cos\alpha}\int_0^{+\infty}\int_0^{+\infty}\left[\frac{H\left(t-(1/c)\sqrt{y_1^2+((x_2-y_2)/\cos\alpha)^2}\right)}{\sqrt{t^2-(1/c^2)\left(y_1^2+((x_2-y_2)/\cos\alpha)^2\right)}} - \frac{H\left(t-(1/c)\sqrt{y_1^2+((x_2+y_2)/\cos\alpha)^2}\right)}{\sqrt{t^2-(1/c^2)\left(y_1^2+((x_2+y_2)/\cos\alpha)^2\right)}}\right]\frac{\partial\psi}{\partial y_1}(y_1, y_2)\,dy_1\,dy_2\,.$$

Remark that $(du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}$ and $(du_2/dN)|_{\mathbf{R}_+\times\Gamma}$ have null traces on $x_2 = 0$ and are odd extendible distributions on $\overline{\mathbf{R}\times\Gamma}$.

So we can define $(du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}$ and $(du_2/dN)|_{\mathbf{R}_+\times\Gamma}$ when $x_2 < 0$:

$$\left.\frac{du_1}{d\lambda}\right|_{\mathbf{R}_+\times\Gamma}(t, x_2) = -\left.\frac{du_1}{d\lambda}\right|_{\mathbf{R}_+\times\Gamma}(t, -x_2)$$

$$\forall t > 0,\ \forall x_2 < 0\,.$$

$$\frac{du_2}{dN}_{\mathbf{R}_+\times\Gamma}(t, x_2) = -\left.\frac{du_2}{dN}\right|_{\mathbf{R}_+\times\Gamma}(t, -x_2)$$

By (2.8) and (2.9) we have that $(du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}$ and $(du_2/dN)|_{\mathbf{R}_+\times\Gamma}$ are linear continuous transformation from $C_0^\infty(\Omega_i)$ to $\mathfrak{D}'(\mathbf{R}_+\times\mathbf{R})$, $i = 1, 2$, with densities $\partial\varphi/\partial y_1$ and $\partial\psi/\partial y_1$ respectively.

So we can write

$$\left.\frac{du_1}{d\lambda}\right|_{\mathbf{R}_+\times\Gamma} = \left\langle H_1 - H_2, \frac{\partial\varphi}{\partial y_1}\right\rangle \qquad \left.\frac{du_2}{dN}\right|_{\mathbf{R}_+\times\Gamma} = \left\langle H_1^c - H_2^c, \frac{\partial\psi}{\partial y_1}\right\rangle$$

where $H_1 - H_2$ and $H_2^c - H_1^c$ are the Kernels which are in (2.8) and (2.9) respectively.

Moreover these transformations are extendible to $\mathcal{E}'(\Omega_i)$ $i = 1, 2$, respectively.

These extensions are unique and continuous by applying a known theorem of L. Hormander [5]. Hormander's theorem also says that

$$WF\left(\left.\frac{du_1}{d\lambda}\right|_{\mathbf{R}_+\times\Gamma}\right) \subseteq WF'(H_1 - H_2)\circ WF\left(\frac{\partial\varphi}{\partial y_1}\right),$$

$$WF\left(\left.\frac{du_2}{dN}\right|_{\mathbf{R}_+\times\Gamma}\right) \subseteq WF'(H_1^c - H_2^c)\circ WF\left(\frac{\partial\psi}{\partial y_1}\right).$$

From the last remark we obtain the following

THEOREM 2.3. Let $(t, x_2; \tau, \xi_2) \in T^*(\mathbf{R}_+ \times \mathbf{R})$.

If $(t, x_2; \tau, \xi_2) \in WF\big((du_2/dN)|_{\mathbf{R}_+\times\Gamma}\big)$ then there exists $(y, \eta) \in WF(\psi)$ such that $\tau^2 = c^2(n_1^2 + \cos^2\alpha\eta_2^2)$, $\tau \neq 0$ and

i) $t = \frac{1}{c}\sqrt{y_1^2 + \left(\frac{x_2 - y_2}{\cos\alpha}\right)^2}, \qquad \xi_2 = \eta_2;$

$$\eta_2 = -\frac{(x_2 - y_2)\tau}{c^2\cos^2\alpha t}, \qquad \eta_1 = \frac{y_1\tau}{c^2 t}$$

or

ii) $t = \frac{1}{c}\sqrt{y_1^2 + \left(\frac{x_2 + y_2}{\cos\alpha}\right)^2}, \qquad \xi_2 = -\eta_2,$

$$\eta_2 = -\frac{(x_2 + y_2)\tau}{c^2\cos^2\alpha t}, \qquad \eta_1 = \frac{y_1\tau}{c^2 t}.$$

The same result is true for $WF\big((du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}\big)$ with $c = \cos\alpha = 1$ and ψ replaced by φ.

The details of the proof are omitted because they are similar to the methods used by Varenne in [12].

REMARK 2.1. By theorem 2.3 we have that if $(t, x_2; \tau, \xi_2)$ belongs to $WF\big((du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}\big)$ then its mirror image $(t, -x_2; \tau, -\xi_2)$ is also in $WF((du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma})$. Moreover $WF((du_1/d\lambda)|_{\mathbf{R}_+\times\Gamma}) \cap T^*(\mathbf{R}_+\times\Gamma)$ consists of intersection points of bicharacteristics $\beta_0^{\pm}$ or their reflected bicharacteristic lines, with the face $x_1 = 0$. Analogous remarks are true for the wave front set of $(du_2/dN)_{\mathbf{R}_+\times\Gamma}$.

REMARK 2.2. Later on we denote the Laplace transform of a distribution $g \in \mathcal{D}'(\mathbf{R}_+\times\Gamma)$ with respect to $t \in \mathbf{R}_+$, by $\bar{g}(k, x_2)$.

Here we show that $(\overline{du_1}/d\lambda)_{\mathbf{R}_+\times\Gamma}$ and $(\overline{du_2}/dN)_{\mathbf{R}_+\times\Gamma}$ belong to $H^{\frac{1}{2}}(\mathbf{R})$. Consider the odd extension of $(\partial\varphi/\partial y_1)(y_1, y_2)$ to $\mathbf{R}_+\times\mathbf{R}$, with respect to y_2. So we can write

$$\frac{du_1}{d\lambda}\bigg|_{\mathbf{R}^+\times\Gamma}(t, x_2) = \frac{\cos\alpha}{\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{H\left(t-\sqrt{y_1^2+(x_2-y_2)^2}\right)}{\sqrt{t^2-\left(y_1^2+(x_2-y_2)^2\right)}}\frac{\partial\varphi}{\partial y_1}(y_1, y_2)\,dy_1\,dy_2\,.$$

Now we calculate the Fourier transform of $(\overline{du_1}/d\lambda)|_{\mathbf{R}_+\times\Gamma}$ and we obtain

$$\left(\overline{\frac{du_1}{d\lambda}}\bigg|_{\mathbf{R}^+\times\Gamma}\right)^{\wedge}(k;\xi_2) = \cos\alpha\int_0^{+\infty}\frac{1}{\sqrt{\xi_2^2+k^2}}\cdot$$
$$\cdot\exp\left[-y_1\sqrt{k^2+\xi_2^2}\right]\left(\frac{\partial\varphi}{\partial y_1}(y_1,\cdot)\right)^{\wedge}(\xi_2)\,dy_1\,.$$

We will show

$$\left(\overline{\frac{du_1}{d\lambda}}\bigg|_{\mathbf{R}^+\times\Gamma}\right)^{\wedge}(1+|\xi_2|)^{\frac{1}{2}} \in L^2(\mathbf{R})\,.$$

Consider

$$\cos^2\alpha\int_{-\infty}^{+\infty}(1+|\xi_2|^2)^{\frac{1}{2}}\left(\int_0^{+\infty}\frac{1}{\sqrt{\xi^2+k^2}}\cdot\right.$$
$$\left.\cdot\exp\left[-y_1\sqrt{k^2+\xi_2^2}\right]\left(\frac{\partial\varphi}{\partial y_1}(y_1,\cdot)\right)^{\wedge}(\xi_2)\,dy_1\right)^2 d\xi_2 =$$
$$= \cos^2\alpha\int_{-\infty}^{+\infty}(1+|\xi_2|^2)^{\frac{1}{2}}\frac{1}{\xi^2+k^2}\cdot$$
$$\cdot\left(\int_0^{+\infty}\exp\left[-y_1\sqrt{k^2+\xi_2^2}\right]\left(\frac{\partial\varphi}{\partial y_1}(y_1,\cdot)\right)^{\wedge}(\xi_2)\,dy_1\right)^2 d\xi_2\,.$$

It is finite because $\exp[-y_1\sqrt{k^2+\xi_2^2}]((\partial\varphi/\partial y_1)(y_1,\cdot))^\wedge(\xi_2)$ is rapidly decreasing to 0, as $|\xi_2|\to+\infty$.

Analogously we can shown $(du_2/dN)|_{\mathbb{R}_+\times\Gamma}\in H^{\frac{1}{2}}(\mathbb{R})$.

3. – In this section we propose to study the singularities of (w_1, w_2) solution in $\mathcal{D}'(\overline{\mathbb{R}_+\times\Omega_1})\times\mathcal{D}'(\overline{\mathbb{R}_+\times\Omega_2})$ to the problem (1.4). This can be converted into a boundary value problem by reflecting Ω_2 on Ω_1 and setting

$$\tilde{w}_2(t, x_1, x_2) = w_2(t, -x_1, x_2) \qquad x_1\geqslant 0\,, \qquad x_2\geqslant 0$$

and

$$\tilde{f}_2(t, x_1) = f_2(t, -x_1)\,, \qquad x_1\geqslant 0\,.$$

We are led to a second order system with zero initial data of the form

$$(3.1)\qquad \begin{cases} \begin{pmatrix}\Box & 0\\ 0 & \Box_c^\alpha\end{pmatrix}\begin{pmatrix}w_1\\ \tilde{w}_2\end{pmatrix}=0 & \text{in } R_+\times\Omega_1\\[2mm] \begin{pmatrix}1 & -1\\ \dfrac{d}{d\lambda} & \dfrac{d}{dN}\end{pmatrix}\begin{pmatrix}w_1\\ \tilde{w}_2\end{pmatrix}=\begin{pmatrix}g_1\\ l\end{pmatrix} & \text{on } \mathbb{R}_+\times\Gamma\\[2mm] \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}w_1\\ \tilde{w}_2\end{pmatrix}=\begin{pmatrix}f_1\\ \tilde{f}_2\end{pmatrix} & \text{on } \mathbb{R}_+\times\Gamma_1\end{cases}$$

where $l(t, x_2)$ denote the distribution

$$g_2(t, x_2)-\left(\frac{du_1}{d\lambda}-\frac{du_2}{dN}\right)\bigg|_{\mathbb{R}_+\times\Gamma}(t, x_2)\,.$$

From now on, to make the notation simpler, we will use w_2 and f_2 instead of $\tilde{w}_2$ and $\tilde{f}_2$, respectively.

We will write explicitly a solution of (3.1) in integral form with Kernel K. From the knowledge of $WF(K)$ we will deduce information about the wave front set of (w_1, w_2).

As first stage we will find an explicit representation for $\overline{w}_1$ and $\overline{w}_2$, Laplace transform of w_1 and w_2 respectively. To do this we first split (3.1) into two distinct problems introducing two auxiliary distributions h_1 and h_2 and we find w_1 and w_2 solutions of the two mixed problems with zero initial data

$$(3.2)\qquad \begin{cases} \Box w_1 = 0 \quad \text{in } \mathbb{R}_+ \times \Omega_1 \\ w_{1/\mathbb{R}_+\times\Gamma} = h_1 \\ w_{1/\mathbb{R}_+\times\Gamma_1} = f_1 \end{cases} \qquad \begin{cases} \Box_c^\alpha w_2 = 0 \quad \text{in } \mathbb{R}_+ \times \Omega_1 \\ \dfrac{d}{dx_2} w_{2/\mathbb{R}_+\times\Gamma} = h_2 \\ w_{2/\mathbb{R}_+\times\Gamma_1} = f_2 \,. \end{cases}$$

Then we determine h_1 and h_2 in such a way that (w_1, w_2) results solution of (3.1), too.

Moreover, for the present, we suppose that $\cos\alpha = 1$.

Applying the Laplace transform to (3.2) we obtain the two auxiliary problems:

$$(3.3)\qquad \begin{cases} (k^2 - \Delta)\bar{w}_1 = 0 \quad \text{in } \Omega_1 \\ \bar{w}_{1/\Gamma} = \bar{h}_1 \\ \bar{w}_{1/\Gamma_1} = \bar{f}_1 \end{cases} \qquad \begin{matrix} (k^2_{/c^2} - \Delta)\bar{w}_2 = 0 \quad \text{in } \Omega_1 \\ \dfrac{d}{dx_1} \bar{w}_{2/\Gamma} = \bar{h}_2 \\ \bar{w}_{2/\Gamma_1} = \bar{f}_2 \end{matrix}$$

where $\bar{f}_1, \bar{f}_2, \bar{h}_1, \bar{h}_2$ are the Laplace transform of f_1, f_2, h_1, h_2 respectively.

By the method of images write the Green functions of (3.3) in the form

$$\begin{aligned} \bar{G}_1(k, x_1, x_2, y_1, y_2) &= E(k, x_1, x_2, y_1, y_2) - E(k, x_1, x_2, y_1, -y_2) + \\ &\quad + E(k, x_1, x_2, -y_1, -y_2) - E(k, x_1, x_2, -y_1, y_2) \end{aligned}$$

$$\begin{aligned} \bar{G}_2(k/c, x_1, x_2, y_1, y_2) &= E(k/c, x_1, x_2, y_1, y_2) - E(k/c, x_1, x_2, y_1, -y_2) - \\ &\quad - E(k/c, x_1, x_2, -y_1, -y_2) + E(k/c, x_1, x_2, -y_1, y_2) \end{aligned}$$

where

$$E(k, x_1, x_2, y_1, y_2) = \frac{1}{2\pi} K_0(k|(x_1, x_2) - (y_1, y_2)|)$$

and K_0 is the Mac Donald function (see [13]).

Using this Green function we can find $\bar{w}_1$ and $\bar{w}_2$ solution of (3.3) in the form

$$(3.4)\qquad \begin{aligned} \bar{w}_1(k, x_1, x_2) &= \int_0^{+\infty} -\frac{\partial}{\partial y_1} G_1(k, x_1, x_2, 0, y_2)\bar{h}_1(k, y_2)\, dy_2 + \\ &\quad + \int_0^{+\infty} -\frac{\partial}{\partial y_2} G_1(k, x_1, x_2, y_1, 0)\bar{f}_1(k, y_1)\, dy_1 \,, \end{aligned}$$

$$(3.5)\qquad \overline{w}_2(k/c, x_1, x_2) = \int_0^{+\infty} G_2(k/c, x_1, x_2, 0, y_2)\,\overline{h}_2(k, y_2)\,dy_2 + \\ + \int_0^{+\infty} -\frac{\partial}{\partial y_2} G_2(k/c, x_1, x_2, y_1, 0)\,\overline{f}_2(k, y_1)\,dy_1 .$$

Now we have to determine $(\overline{h}_1, \overline{h}_2)$ in such a way that $(\overline{w}_1, \overline{w}_2)$ verify the boundary condition initially assigned on Γ, that is

$$(3.6)\qquad (\overline{w}_1 - \overline{w}_2)_{/\Gamma} = \overline{g}_1 , \qquad \left(\frac{d}{d\lambda}\overline{w}_1 + \frac{d}{dx_2}\overline{w}_2\right)_{/\Gamma} = \overline{l} .$$

Note that we are supposing $\cos\alpha = 1$ so $d/d\lambda = d/dx_1$.
The trace of $\overline{w}_1$ on Γ is given by

$$\frac{1}{\pi}\int_0^{+\infty}[K_0(k/c|x_2 - y_2|) - K_0(k/c|x_2 + y_2|)]\,\overline{h}_2(k, y_2)\,dy_2 + \\ + \frac{1}{\pi}\int_0^{+\infty}\frac{2x_2}{\sqrt{y_1^2 + x_2^2}}\frac{\partial}{\partial r} K_0(k/c\sqrt{y_1^2 + x_2^2})\,\overline{f}_2(k, y_1)\,dy_1$$

where r denotes $\sqrt{y_1^2 + x_2^2}$.
Calculating the trace of $(d/dx_1)\overline{w}_1$ on Γ we have

$$(3.7)\qquad -\frac{1}{\pi}\int_0^{+\infty}\left[\frac{\partial}{\partial r}K_0(k|x_2 - y_2|) + \frac{\partial}{\partial r}K_0(k|x_2 + y_2|)\right]\overline{h}'_1(k, y_2)\,dy_2 + \\ + \frac{k^2}{\pi}\int_0^{+\infty}[K_0(k|x_2 - y_2|) - K_0(k|x_2 + y_2|)]\,\overline{h}_1(k, y_2)\,dy_2 + \\ + \frac{1}{\pi}\int_0^{+\infty}\frac{2x_2}{\sqrt{y_1^2 + x_2^2}}\frac{\partial}{\partial r}K_0(k\sqrt{y_1^2 + x_2^2})\,\overline{f}'_1(k, y_1)\,dy_1 .$$

To obtain (3.7) we have kept into account that the Mac Donald function K_0 verifies the Helmotz equation $(k^2 - \Delta)u = 0$ and have supposed that $\bar{h}_1(k, 0) = 0$. Extending $\bar{h}_1, \bar{h}_2, \bar{g}_1, \bar{l}$ as odd functions, so that their derivatives result even functions, we can write

$$\bar{w}_{2/\Gamma} = R_2\bar{h}_2 + T_2\bar{f}_2$$

where

$$R_2\bar{h}_2(k/c, x_2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} K_0(k/c|x_2 - y_2|)\bar{h}_2(k, y_2)\,dy_2$$

$$T_2\bar{f}_2(k/c, x_2) = \frac{1}{\pi}\int_0^{+\infty} \frac{2x_2}{\sqrt{y_1^2 + x_2^2}}\frac{\partial}{\partial r}K_0(k/c\sqrt{y_1^2 + x_2^2})\bar{f}_2(k, y_1)\,dy_1$$

$$\frac{d}{dx_1}\bar{w}_{1/\Gamma} = (-\Lambda R_1\Lambda + k^2R_1)\bar{h}_1 + T_1\bar{f}_1'$$

where Λ is the pseudodifferential operator in $OPS^1_{10}(\mathbb{R})$ with symbol $-i|\omega|$

$$R_1\bar{h}_1(k, x_2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} K_0(k|x_2 - y_2|)\bar{h}_1(k, y_2)\,dy_2\,,$$

$$T_1\bar{f}_1'(k, x_2) = \frac{1}{\pi}\int_0^{+\infty} \frac{2x_2}{\sqrt{y_1^2 + x_2^2}}\frac{\partial}{\partial r}K_0(k\sqrt{y_1^2 + x_2^2})\bar{f}_1'(k, y_1)\,dy_1\,.$$

Since $\bar{g}_1, \bar{l}, \bar{f}_1, \bar{f}_2$ are known, the boundary conditions (3.6) turn into the following integral system:

$$(3.8)\qquad \mathcal{A}\begin{pmatrix}\bar{h}_1\\ \bar{h}_2\end{pmatrix} = \begin{pmatrix} I & -R_2\\ -\Lambda R_1\Lambda + k^2R_1 & I\end{pmatrix}\begin{pmatrix}\bar{h}_1\\ \bar{h}_2\end{pmatrix} = \begin{pmatrix}\bar{g}_1 + T_2\bar{f}\\ \bar{l} - T_1\bar{f}_1'\end{pmatrix}.$$

So far we have supposed $\cos\alpha = 1$ and then $\sin\alpha = 0$.

When $\cos\alpha$ is any real number between 0 and 1 it can easily seen that (3.8) becomes

$$(3.9)\qquad \mathcal{A}_\alpha\begin{pmatrix}\bar{h}_1\\ \bar{h}_2\end{pmatrix} = \begin{pmatrix} I & -R_2^\alpha\\ \cos\alpha(-\Lambda R_1\Lambda + k^2R_1) + \sin\alpha\Lambda & I\end{pmatrix}\begin{pmatrix}\bar{h}_1\\ \bar{h}_2\end{pmatrix} = \begin{pmatrix}\bar{g}_1 + T_2^\alpha\bar{f}_2\\ \bar{l} - \cos\alpha T_1\bar{f}_1'\end{pmatrix},$$

where

$$R_2^\alpha \bar{h}_2(k/c, x_2) = \frac{1}{\pi \cos\alpha} \int_{-\infty}^{+\infty} K_0(k/c \cos\alpha |x_2 - y_2|) \bar{h}_2(k, y_2)\, dy_2$$

and

$$T_2^\alpha \bar{f}_2(k/c, x_2) = T_2 \bar{f}_2(k/c, x_2/\cos\alpha)$$

the operators R_1 and R_2^α are in the class of pseudodifferential operators $OPS_{1,0}^{-1}$ then they take $H^s(\mathbb{R})$ to $H^{s+1}(\mathbb{R})$, Λ take $H^s(\mathbb{R})$ to $H^{s-1}(\mathbb{R})$ therefore

$$A_\alpha \colon H^{\frac{3}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}) \to H^{\frac{3}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}) .$$

Since the Fourier transform of $K_0(k|x|)$ is given by

$$\left(\frac{1}{\pi} K_0(k|x|)^\wedge (\omega)\right) = \frac{1}{\sqrt{\omega^2 + k^2}}$$

the integral operator A_α has for symbol

$$\text{(3.10)} \qquad \begin{pmatrix} \dfrac{1}{\cos\alpha \sqrt{\omega^2 + k^2} - i \sin\alpha |\omega|} & -(\cos^2\alpha\omega^2 + k^2/c^2)^{-\frac{1}{2}} \\ & 1 \end{pmatrix}$$

so A_α is elliptic.

Before going on, we need to show that

$$\text{(3.11)} \qquad T_1 \bar{f}_1' \in H^{\frac{1}{2}}(\mathbb{R}) , \qquad T_2^\alpha \bar{f}_2 \in H^{\frac{3}{2}}(\mathbb{R}) .$$

To this aim we remark that $T_1 \bar{f}_1$ and $T_2^\alpha \bar{f}_2$ are odd distributions and therefore it is enough to verify that the operator T defined by

$$Tf(x) = \frac{1}{\pi} \int_0^{+\infty} \frac{x}{\sqrt{y^2 + x^2}} \frac{\partial}{\partial r} K_0(k\sqrt{y^2 + x^2}) f(y)\, dy$$

maps $H^{\frac{1}{2}}(\mathbb{R}_+)$ in $H^{\frac{1}{2}}(\mathbb{R}_+)$ and $H^{\frac{3}{2}}(\mathbb{R}_+)$ in $H^{\frac{3}{2}}(\mathbb{R}_+)$.

First observe that the operator

$$Sf(x) = \frac{1}{\pi}\int_0^{+\infty} \frac{x}{y^2+x^2} f(y)\,dy$$

with Hardy Kernel $x/(y^2+x^2)$ takes $L^2(\mathbb{R})_+$ to $L^2(\mathbb{R}_+)$ (see [3]).
Moreover, if f belongs to $C_0^\infty(\mathbb{R}_+)$

$$\frac{d}{dx} Sf(x) = \frac{1}{\pi}\int_0^{+\infty} \frac{y^2-x^2}{(y^2+x^2)^2} f(y)\,dy = \frac{1}{\pi}\int_0^{+\infty}\left(-\frac{d}{dy}\frac{y}{y^2+x^2}\right) f(y)\,dy =$$

$$= \frac{1}{\pi}\int_0^{+\infty} \frac{y}{y^2+x^2} f'(y)\,dy\,.$$

As $y/(y^2+x^2)$ is also a Hardy Kernel we obtain

(3.12) $$\left\| \frac{d}{dx} Sf(x) \right\|_{L^2} \leqslant c\|f'\|_{L^2}$$

so

$$S\colon \mathring{H}^1(\mathbb{R}_+) \to H^1(\mathbb{R}_+)\,.$$

Using the operator S we can prove

i) $T\colon H^0(\mathbb{R}_+) \to H^0(\mathbb{R}_+)$

ii) $T\colon \mathring{H}^1(\mathbb{R}_+) \to H^1(\mathbb{R}_+)$

iii) $T\colon \mathring{H}^2(\mathbb{R}_+) \to H^2(\mathbb{R}_+)$

$$\|Tf\|_{L^2}^2 < \frac{1}{\pi}\int_0^{+\infty} \mathrm{d}x \left(\int_0^{+\infty} \frac{x}{\sqrt{y^2+x^2}} \left| \frac{\partial}{\partial r} K_0(k\sqrt{y^2+x^2}) \right| |f(y)|dy \right)^2$$

Since

$$\left| \frac{\partial}{\partial r} K_0(kr) \right| = 0(r^{-1}) \qquad r \to 0$$

and

$$S\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

we obtain

$$\|Tf\|_{L^2} < c\|f\|_{L^2} .$$

To prove ii), first note that

$$Tf(x) = \frac{1}{\pi k}\int_0^{+\infty}\left(\frac{d}{dx}K_0(k\sqrt{y^2+x^2})\right)f(y)\,dy .$$

Using that $K_0(kr)$ verify the Helmotz equation and then integrating by parts we have

$$\frac{d}{dx}Tf(x) = \frac{k}{\pi}\int_0^{+\infty}K_0(k\sqrt{y^2+x^2})f(y)\,dy + \\ + \frac{1}{\pi}\int_0^{+\infty}\frac{y}{\sqrt{y^2+x^2}}\frac{\partial}{\partial r}K_0(k\sqrt{y^2+x^2})f'(y)\,dy .$$

The first addendum is a continuous operator from $L^2(\mathbb{R}_+)$ to $H^1(\mathbb{R}_+)$ and using (3.12)

$$\left\|\int_0^{+\infty}\frac{y}{\sqrt{y^2+x^2}}\frac{\partial}{\partial r}K_0(k\sqrt{y^2+x^2})f'(y)\,dy\right\|_{L^2} < c\|f'\|_{L^2}$$

iii) can be proved in a similar way.

Applying known interpolation results and i), ii), iii) we obtain (3.11).

From the ellipticity of A_α it follows that we can find one and only one

$$(\bar{h}_1, \bar{h}_2) \in H^{\frac{3}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$$

satisfying the integral system (3.9).

Moreover $\bar{h}_1$ and $\bar{h}_2$ are odd functions and $\bar{h}_1(k, 0) = 0$.

Finally we may observe that the Fourier transform of $\bar{h}_1(k, x_2)$ and $\bar{h}_2(k, x_2)$ is given by

$$\begin{pmatrix} (a(k,\omega))^{-1} & \left((\omega^2 + (k/c\cos\alpha)^2)^{\frac{1}{2}} + (\omega^2+k^2)^{\frac{1}{2}} - i\,\mathrm{tg}\,\alpha|\omega|\right)^{-1} \\ -\left(\cos\alpha(\omega^2+k^2)^{\frac{1}{2}} - i\sin\alpha|\omega|\right)(a(k,\omega))^{-1} & (a(k,\omega))^{-1} \end{pmatrix} \cdot$$

$$\cdot\begin{pmatrix} (\bar{g}_1 + T_2^\alpha \bar{f}_2)^\wedge \\ (\bar{l} - \cos\alpha T_1 \bar{f}_1')^\wedge \end{pmatrix} = \mathcal{A}(k,\omega)\begin{pmatrix} (\bar{g}_1 + T_2^\alpha \bar{f}_2)^\wedge(k,\omega) \\ (\bar{l} - \cos\alpha T_1 \bar{f}_1')^\wedge(k,\omega) \end{pmatrix}$$

where $a(k, \omega)$ is the determinant of the matrix (3.10).

Applying the Paley-Wiener-Schwartz theorem (see [5]) it follows that $\mathcal{A}(k, \omega)$ is the Laplace-Fourier transform of a distribution with compact support. Then the inverse Laplace-Fourier transform of $\bar{h}_1(k, \omega)$ and $\bar{h}_2(k, \omega)$, which we will denote by $h_1(t, x_2)$ and $h_2(t, x_2)$ is obtained by convolutions of distributions in $\mathcal{E}'(\mathbb{R}_+ \times \mathbb{R})$ with elements of $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$.

Therefore

$$h_1, h_2 \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}) .$$

To this point, we are able to calculate w_1 and w_2 as inverse Laplace transform of $\bar{w}_1$ and $\bar{w}_2$, defined by (3.4) and (3.5) respectively.

Let $G_i(t, x_1, x_2, y_1, y_2)$, $i = 1, 2$, the inverse Laplace transform of $\bar{G}_i$, $i = 1, 2$. They can be calculated explicitly by the following equality

$$\mathcal{L}^{-1}\big(K_0(k|x-y|)\, t)\big) = \begin{cases} 0 & \text{if } 0 < t < |x-y| , \\ (t^2 - (x-y)^2)^{-\frac{1}{2}} & \text{if } t > |x-y| . \end{cases} \tag{3.13}$$

The first addendum of $G_1(t, x_1, x_2, y_1, y_2)$, is, for instance,

$$\frac{H\big(t - \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\big)}{\sqrt{t^2 - \big((x_1 - y_1)^2 + (x_2 - h_2)^2\big)}}$$

and the other three are of the same type.

Therefore we have

$$w_1(t, x_1, x_2) = \frac{\cdot}{\pi} \int_0^{+\infty} ds \int_{-\infty}^{+\infty} dy_2 \frac{H(t - s - r_1)\, x_1}{((t-s)^2 - r_1^2)^{\frac{3}{2}}} \cdot \tag{3.14}$$

$$\cdot \big(h_1(s, y_2) - h_1(t - r_1, y_2)\big) +$$

$$+ \frac{\cdot}{\pi} \int_0^{+\infty} ds \int_0^{+\infty} dy_1 \left\{ \frac{H(t - s - r_2^-)\, r_2}{((t-s) - (r_2^-)^2)^{\frac{3}{2}}} \big(f_1(s, y_1) - f_1(t - r_2^-, y_1)\big) - \right.$$

$$\left. - \frac{H(t - s - r_2^+)\, x_2}{((t-s)^2 - (r_2^+)^2)^{\frac{3}{2}}} \big(f_1(s, y_1) - f_1(t - r_2^+, y_1)\big) \right\} =$$

$$= w_1^1(t, x_1, x_2) + w_1^2(t, x_1, x_2)$$

where

$$r_1 = \sqrt{x_1^2 + (x_2 - y_2)^2}\,, \qquad r_2^- = \sqrt{(x_1 - y_1)^2 + x_2^2}\,, \qquad r_2^+ = \sqrt{(x_1 + y_1)^2 + x_2^2} \tag{3.15}$$

and analogously

$$w_2(t, x_1, x_2) = \frac{1}{\pi \cos\alpha} \int_0^{+\infty} ds \int_{-\infty}^{+\infty} dy_2 \, \frac{H(t - s - r_{1,\alpha})}{((t-s)^2 - (r_{1,\alpha})^2)^{\frac{1}{2}}} h_2(s, y_2) + \tag{3.16}$$

$$+ \frac{1}{\pi c^2 \cos\alpha} \int_0^{+\infty} ds \int_0^{+\infty} dy_1 \left\{ \frac{H(t - s - r_{2,\alpha}^-)\, x_2}{((t-s)^2 - (r_{2,\alpha}^-)^2)^{\frac{3}{2}}} \big(f_2(s, y_1) - f_2(t - r_{2,\alpha}^-, y_1)\big) + \right.$$

$$\left. + \frac{H(t - s - r_{2,\alpha}^+)\, x_2}{((t-s)^2 - (r_{2,\alpha}^+)^2)^{\frac{3}{2}}} \big(f_2(s, y_1) - f_2(t - r_{2,\alpha}^+, y_1)\big) \right\} =$$

$$= w_2^1(t, x_1, x_2) + w_2^2(t, x_1, x_2)$$

where

$$r_{1,\alpha} = \frac{1}{c}\sqrt{x_1^2 + \left(\frac{x_2 - y_2}{\cos\alpha}\right)^2}\,, \qquad r_{2,\alpha}^- = \frac{1}{c}\sqrt{(x_1 - y_1)^2 + \left(\frac{x_2}{\cos\alpha}\right)^2}\,, \qquad r_{2,\alpha}^+ = \frac{1}{c}\sqrt{(x_1 + y_1)^2 + \left(\frac{x_2}{\cos\alpha}\right)^2} \tag{3.17}$$

We can summarize the results till now found in the following

Theorem 3.1. *The distribution $w = (w_1, w_2) \in (\mathcal{D}'(\overline{\mathbb{R}_+ \times \Omega_1}))^2$ defined by (3.14) and (3.16) is solution to the problem (3.1) with boundary data (f_1, f_2) on $\mathbb{R}_+ \times \Gamma_1$ in the space $(\mathcal{D}'(\mathbb{R}_+; H^{\frac{3}{2}}(\Gamma_1)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma_1))^2$, (g_1, g_2) on $\mathbb{R}_+ \times \Gamma$ in the space $(\mathcal{D}'(\mathbb{R}_+; H^{\frac{3}{2}}(\Gamma)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma)) \times (\mathcal{D}'(\mathbb{R}_+; H^{\frac{1}{2}}(\Gamma)) \cap \mathcal{E}'(\mathbb{R}_+ \times \Gamma))$ provided (h_1, h_2) is the inverse Laplace transform of $(\bar{h}_1, \bar{h}_2)$ solution in $H^{\frac{3}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ of the integral system (3.9)*

4. We are now interested in describing C^∞-singularities of the solution $w = (w_1, w_2)$ of problem (3.1).

We are going to give a complete description of $WF(w_1)$.

The results for $WF(w_2)$ shall be analogous.

w_1 is sum of the two distributions w_1^1 and w_1^2.

w_1^1 is of the type

$$w_1^1(t, x_1, x_2) = \langle K(t, x_1, x_2; s, y_2), h_1(s, y_2)\rangle \tag{4.1}$$

that it is a distribution with Kernel

$$K(t, x_1, x_2; s, y_2) = \frac{H(t-s-r_1)\,x_1}{(t-s-r_1)^{\frac{3}{2}}}$$

where r_1 is defined by (3.15).

Let

$$X = \mathbb{R}_+ \times \Omega_1\,, \qquad Y = \mathbb{R}_+ \times \mathbb{R} \tag{4.2}$$

then

$$K \in \mathcal{D}'(X \times Y)\,.$$

Our purpose is to determine $WF(w_1^1)$ from the knowledge of $WF'(K)$ and $WF(h_1)$.

THEOREM 4.1. Let $\Phi(t, x_1, x_2; s, y_2) = t - s - \sqrt{x_1^2 + (x_2 - y_2)^2}$. If the point $(t, x, s, y_2; \tau, \xi, \sigma, \eta_2) \in T^*(X \times Y)$ belongs to $WF(K)$, then $(t, x; s, y_2)$ belongs to the surface $\Phi(t, x; s, y_2) = 0$ in $X \times Y$ and $(\tau, \xi, \sigma, \eta_2) = \tau \operatorname{grad} \Phi(t, x, s, y_2)$, $\tau \neq 0$.

PROOF. Since

$$K(t, x; s, y_2) = \frac{\partial}{\partial x_1} \frac{H(t-s-r_1)}{(t-s-r_1)^{\frac{1}{2}}}$$

the wave front set of K is included in the wake front set of

$$\frac{H(t-s-r_1)}{(t-s-r_1)^{\frac{1}{2}}}$$

which results the distribution $H(t)/t^{\frac{1}{2}} = t_+^{-\frac{1}{2}}$ concentrated on surface $\Phi = 0$.

Let $\tilde{\Phi}$ be the following diffeomorphism

$$\tilde{\Phi}(t, x, s, y_2) = (t - x - r_1, x, s, y_2)$$

and $\tilde{\Phi}_*$ the pullback of $\tilde{\Phi}$. From the equality

$$\tilde{\Phi}^*(t_+^{-\frac{1}{2}}) = \frac{H(t-s-r_1)}{(t-s-r_1)^{\frac{1}{2}}}$$

follows by a known result

$$WF\left(\frac{H(t-s-r_1)}{(t-s-r_1)^{\frac{1}{2}}}\right) = \tilde{\Phi}^*(WF(t_+^{-\frac{1}{2}})) \tag{4.3}$$

where $\tilde{\Phi}^*$ is the induced diffeomorphism between bundles. Since

$$WF(t_+^{-\frac{1}{2}}) = \{(t, x, s, y; \tau, \xi, \sigma, \eta) \in T^*(X\times Y)|t = 0, \tau \neq 0,\ \xi_1 = \xi_2 = \sigma = \eta = 0\}$$

the statement of the theorem can be deduced from (4.3).

We can obtain $WF(w_1^1)$ in terms of wave front set of the boundary data g_1, l, f_1, f_2. Indeed.

$$WF'_Y(K) = \emptyset; \quad WF'_X(K) = \emptyset$$

and the projection

$$\pi_X\colon \operatorname{supp}(K) \to X$$

is proper, i.e. the inverse image of each compact set is compact.

Then by a known result (see [1])

$$WF(w_1^1) \subseteq WF'(K) \circ WF(h_1)\,. \tag{4.4}$$

Moreover from (3.9) and ellipticity of A_α it follows

$$WF\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = WF\begin{pmatrix} g_1 + \mathfrak{L}^{-1}(T_2^\alpha \bar{f}_2) \\ l - \cos\alpha\, \mathfrak{L}^{-1}(T_1 \bar{f}_1') \end{pmatrix}. \tag{4.5}$$

In the following theorem 4.2 using (4.4) and theorem 4.1, we give a description of $WF(w_1^1)$ in terms of bicharacteristic lines issuing from points of the following set

$$\Sigma = \{(t, x; \tau, \xi) \in T^*(\mathbb{R}_+ \times \mathbb{R}^2)\colon \tau \neq 0, x_1 = \xi_1 = 0, \tau^2 > \xi_2^2\}$$

Σ can be considered a subset of $T^*(\mathbb{R}^+ \times \mathbb{R})$.

Fixed $\varrho_0 = (s, y_2; \sigma, \eta_2) \in \Sigma$ let ϱ_0^* be it image point $(s, -y_2; \sigma, -\eta_2)$.

Note that if ϱ_0 belongs to $WF(h_1)$ then ϱ_0^* is also a point of the wave front set of h_1 because h_1 is odd in the space variable.

Denote by $\beta_1(\varrho_0)$ the bicharacteristic outgoing from $\varrho_0 \in \Sigma$. Its projection on $\mathbb{R}_+ \times \mathbb{R}^2$ has equation

$$(4.6) \qquad \begin{cases} t = s + s' \\ x_1 = -\dfrac{\xi_1}{\sigma} s' \\ x_2 = -\dfrac{\eta_2}{\sigma} s' + y_2 \end{cases} \qquad s' \geqslant 0$$

where $\xi_1 = \mp \sqrt{\sigma^2 - \eta_2^2}$ according to σ is positive or negative.

REMARK 4.1. Suppose $\varrho_0 = (s, y_2; \sigma, \eta_2) \in \Sigma$, $\sigma > 0$, $y_2 > 0$.
Then if $\eta_2 \leqslant 0$ the bicharacteristic $\beta_1(\varrho_0)$ lies in $T^*(\mathbb{R}_+ \times \Omega_1)$ whereas $\beta_1(\varrho_0^*)$ is always out of $T^*(\mathbb{R}_+ \times \Omega_1)$; if $\eta_2 > 0$ $\beta_1(\varrho_0)$ is in $T^*(\mathbb{R}_+ \times \Omega_1)$ until $t < s + y_2\sigma/\eta_2$ whereas $\beta_1(\varrho_0^*)$ goes into $T^*(\mathbb{R}_+ \times \Omega_1)$ when $t > s + y_2\sigma/\eta_2$. Indeed the projections of $\beta_1(\varrho_0)$ and $\beta_1(\varrho_0^*)$ intersect on the face $x_2 = 0$ when $t - s = s' = y_2\sigma/\eta_2$.

THEOREM 4.2. Let $(t, x; \tau, \xi) \in T^*(\mathbb{R}_+ \times \Omega_1)$ belonging to the wave front set of w_1^1, defined by (3.14).

Then $\tau \neq 0$, $\xi_1 \neq 0$, $\tau^2 = \xi_1^2 + \xi_2^2$ and there is $\varrho_0 = (s, y_2; \sigma, \eta_2) \in T^*(\mathbb{R}_+ \times \Gamma)$ belonging to $\Sigma \cap WF(h_1)$ such that $(t, x; \tau, \xi) \in \beta_1(\varrho_0)$ or $(t, x; \tau, \xi) \in \beta_1(\varrho_0^*)$.

PROOF. Using (4.4) we can state that if $(t, x; \tau, \xi) \in WF(w_1^1)$ then

$$\exists (s, y_2; \sigma, \eta_2) \in WF(h_1) \ni' (t, x, s, y_2; \tau, \xi, -\sigma, -\eta_2) \in WF(K) .$$

By thoerem 4.1 it implies that

$$t - s = \sqrt{x_1^2 + (x_2 - y_2)^2}, \qquad \tau \neq 0, \qquad \sigma = \tau,$$
$$\xi_2 = \eta_2 = -\frac{(x_2 - y_2)}{t - s}, \qquad \xi_1 = -\frac{x_1}{t - s}\tau .$$

Three cases may occur

i) $(s, y_2; \sigma, \eta_2) \in T^*(\mathbb{R}_+ \times \Gamma)$

ii) $(s, -y_2; \sigma, -\eta_2) \in T^*(\mathbb{R}_+ \times \Gamma)$

iii) $(s, y_2; \sigma, \eta_2) \in T^*(\mathbb{R}_+ \times \mathbb{R})$, $y_2 = 0$.

In the first case, by remark 4.1, $(t, x;\ \tau, \xi)$ belongs to the bicharacteristic β_1 outgoing from $(s, y_2;\ \sigma, \eta_2)$ and the point ϱ_0 in the statement is just $(s, y_2;\ \sigma, \eta_2)$.

If ii) happens the point ϱ_0 is $(s, -y_2;\ \sigma, -\eta_2)$ and $(t, x;\ \tau, \xi) \in \beta_1(\varrho_0^*)$. When $y_2 = 0$ then $\eta_2 \neq 0$. Moreover $(t, x;\ \tau, \xi)$ lies on the bicharacteristic β_1 outgoing from $(s, 0;\ \sigma, \eta_2)$ if $\eta_2/\sigma < 0$ whereas it belongs to the bicharacteristic outgoing from $(s, 0;\ \sigma, -\eta_2)$ if $\eta_2/\sigma > 0$.

REMARK 4.2. Theorem 4.2 gives a complete enough description of $WF(w_1^1)$ provided that we know the wave font set of the auxiliary distribution h_1.

By (4.5) and theorem 2.3 we need only information about $WF(\mathcal{L}^{-1}(T_2^\alpha \bar{f}_2))$ and $WF(\mathcal{L}^{-1}(T_1 \bar{f}_1'))$

THEOREM 4.3. Let $(t, x_2;\ \tau, \xi_2) \in T^*(\mathbb{R}_+ \times \mathbb{R})$.

If $(t, x_2;\ \tau, \xi_2) \in WF(\mathcal{L}^{-1}(T_2^\alpha \bar{f}_2))$ then $\tau \neq 0$, $\tau^2/c^2 > \cos \alpha^2 \xi_2^2$ and there exists $(s, y_1;\ \sigma, \eta_1) \in WF(f_2)$ such that

$$t - s = \frac{1}{c}\sqrt{y_1^2 + \left(\frac{x_2}{\cos \alpha}\right)^2}, \qquad \sigma = \tau, \qquad \xi_2 = -\frac{x_2}{c^2 \cos^2 \alpha}\,\frac{\tau}{t - s},$$

$$\eta_1 = \frac{y_1}{c^2}\,\frac{\tau}{t - s}.$$

For $WF(\mathcal{L}^{-1}(T_1 \bar{f}_1'))$ an analogous result is valid with $c = \cos \alpha = 1$

PROOF.

$$\mathcal{L}^{-1}(T_2^\alpha \bar{f}_2)(t, x_2) = \frac{2}{\pi c^2 \cos \alpha}\int_0^{+\infty} ds \int_0^{+\infty} dy_1\, \frac{H(t - s - r)\, x_2}{((t - s)^2 - r^2)^{\frac{3}{2}}}\,[f_2(s, y_1) - f_2(t - r, y_1)]$$

where

$$r = \frac{1}{c}\sqrt{y_1^2 + \left(\frac{x_2}{\cos \alpha}\right)^2}$$

so it is a distribution of the type

$$Af = \langle a(t, x_2, s, y_1), f(s, y_1)\rangle$$

with Kernel $a \in \mathcal{D}'((\mathbb{R} \times \mathbb{R}) \times (\mathbb{R}_+^2))$.

Since the operator

$$A: C_0^\infty(\mathbb{R}_+^2) \to \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$$

is linear and continuous it can be extended continuously as

$$A: \mathcal{E}'(\mathbb{R}_+^2) \to \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$$

Moreover

$$WF(Af) \subseteq WF'(a) \circ WF(f) \tag{4.7}$$

Is not difficult to see, using the same tecnique of theorem 4.1, that in this case

$$WF'(a) \subseteq \Big\{(t, x_2, s, y_1; \tau, \xi_2, \sigma, \eta_1) \in T^*((\mathbb{R}_+ \times \mathbb{R}) \times \mathbb{R}_+^2) | \tau \neq 0 ,$$
$$t - s = r , \quad \sigma = \tau , \quad \xi_2 = - \frac{x_2}{c^2 \cos \alpha} \frac{\tau}{t-s} , \quad \eta_1 = \frac{y_1}{c^2} \frac{\tau}{t-s} \Big\}$$

and then from (4.7) the thesis follows.

It is enough to apply again Hormander's theorem concerning the distribution with Kernel to prove the following theorem 4.4. First introduce some notations.

Let $\varrho_0 = (s, y_1; \sigma, \eta_1) \in T^*(\mathbb{R}_+ \times \Gamma_1)$ $\sigma \neq 0$, $\sigma^2 > \eta_1^2$. We will denote by $\gamma_1(\varrho_0)$ the bicharacteristic outgoing from ϱ_0 whose projection on $\mathbb{R}_+ \times \mathbb{R}^2$ has equation

$$\begin{cases} t = s + s' \\ x_1 = -\dfrac{\eta_1}{\sigma} s' + y_1 , \qquad s' \geq 0 \\ x_2 = -\dfrac{\xi_2}{\sigma} s' \end{cases}$$

with $\xi_2 = \mp \sqrt{\sigma^2 - \eta_1^2}$ according to σ is positive or negative.

THEOREM 4.4. Let $(t, x; \tau, \xi) \in T^*(\mathbb{R}_+ \times \Omega_1)$ belonging to the wave front set of w_1^2, defined by (3.14).

Then $\tau \neq 0$ $\xi_2 \neq 0$ $\tau^2 = \xi_1^2 + \xi_2^2$ and there exists $\varrho_0 = (s, y_1; \sigma, \eta_1) \in WF(f_1)$ such that $(t, x; \tau, \xi) \in \gamma_1(\varrho_0)$ or $(t, x; \tau, \xi) \in \gamma_1(\varrho_0^*)$ where $\varrho_0^* = (s, -y_1; \sigma, -\eta_1)$.

To complete the study of propagation and reflection of singularities of the solution $w = (w_1, w_2)$ to the problem (3.1) observe that the results concerning $WF(w_2)$ are of the same type of those established in the previous theorems 4.2 and 4.4 where h_1 and f_1 are replaced by h_2 and f_2, respectively and the bicharacteristics β_1 and γ_1 are replaced by $\beta_1^{c,\alpha}$ and $\gamma_1^{c,\alpha}$ whose projections on $\mathbb{R}_+ \times \mathbb{R}^2$ have equations of the type

$$\begin{cases} t = s + s' \\ x_1 = -c^2 \dfrac{\xi_1}{\sigma} s' \\ x_2 = -c^2 \cos^2\alpha \dfrac{\eta_2}{\sigma} s' + y_2 \end{cases} \qquad \begin{cases} t = s + s' \\ x_1 = -c^2 \dfrac{\eta_1}{\sigma} s' + y_1 \\ x_2 = -c^2 \cos^2\alpha \dfrac{\xi_2}{\sigma} s' \end{cases} \qquad s' \geqslant 0$$

respectively.

It easy to verify that if $(t, x_2; \tau, \xi_2) \in WF(\mathcal{L}^{-1}(T_2^\alpha \bar{f}_2)) \cap T^*(\mathbb{R}_+ \times \Gamma)$ then it is the intersection point of the bicharacteristic $\gamma_1^{c,\alpha}$, outgoing from some $\varrho_0 \in WF(f_2)$ with the face $x_1 = 0$.

Analogously $WF(\mathcal{L}^{-1}(T_1 \bar{f}_1')) \cap T^*(\mathbb{R}_+ \times \Gamma)$ consists of intersection points of $\gamma_1(\varrho_0)$, $\varrho_0 \in WF(f_1)$, with the face $x_1 = 0$.

Putting together the result of theorems 2.1, 4.2, 4.4, using (1.5) after reflecting again w_2, f_2 on $\mathbb{R}_+ \times \Omega_2$, we obtain a complete description of the wave front set of the solution v to the problem (I).

If we look for a solution to the problem (I) in an opportune distribution space uniqueness results can be used (use [1] [6]). In this case the solution v, found by us explicitely, is the unique solution of the problem (I).

REMARK 4.3. Suppose that $\varrho_0 \in WF(f_1)$ and the bicharacteristic $\gamma_1(\varrho_0)$ intersects the face $x_1 = 0$ in $\bar{\varrho}_0$. Then it reflects on $x_1 = 0$ giving origin to the bicharacteristic $\gamma_1(\varrho_0^*)$. On the other hand $\bar{\varrho}_0 \in WF(\mathcal{L}^{-1}(T_1 \bar{f}_1'))$. It may happen that $\bar{\varrho}_0 \in WF(h_1)$ so from this point starts the bicharacteristic $\beta_1(\bar{\varrho}_0)$.

One could ask if the propagation of the singularity ϱ_0 of the data f_1 happens on $\gamma_1(\varrho_0^*)$ or $\beta_1(\bar{\varrho}_0)$. It easy verify that in this case, $\gamma_1(\varrho_0^*)$ and $\beta_1(\bar{\varrho}_0)$ coincide. The same happens in other cases analogous to this.

Now examin another eventuality.

Let $\varrho = (y_1, y_2, \eta_1, \eta_2) \in WF(\varphi)$, $\eta_1^2 + \eta_2^2 > 0$ and suppose the bicharacteristic β_0 issuing from ϱ hits the corner in ϱ_0 and reflects along $\tilde{\beta}_0$.

By remark 2.1 ϱ_0 belongs also to $WF((du_1/d\lambda)|_{\mathbb{R}_+ \times \Gamma})$ and then ϱ_0 may be in $WF(h_1)$. So this singularity should propagate along $\beta_1(\varrho_0)$

or $\beta_1(\varrho_0^*)$, by theorem 4.2. It can be verify that $\tilde{\beta}_0$ coincides with $\beta_1(\varrho_0^*)$ and along this line the singularity ϱ_0 propagates.

We conclude this section resuming how the singularities of the data g_1, g_2 on the face $\mathbb{R}_+ \times \Gamma$, common boundary to $\mathbb{R}_+ \times \Omega_1$, and $\mathbb{R}_+ \times \Omega_2$, propagate.

Let $\varrho_0 = (s, y_2; \sigma, \eta_2) \in WF(g_1) \cup WF(g_2)$.

It is known that over any $\varrho_0 \in T^*(\mathbb{R}_+ \times \Gamma)$ pass either 0, 2 (1 incoming, 1 outgoing) or 4 (2 incoming, 2 outgoing) bicharacteristics according to $c^2 \cos\alpha^2 \eta_2^2 > \sigma^2$, $\eta_2^2 > \sigma^2 > c^2 \cos\alpha^2 \eta_2^2$ or $\sigma^2 > \eta_2^2$.

If only two bicharacteristics pass over ϱ_0, they must be incident (incoming) and reflected (outgoing) bicharacteristics for the slow speed region that is Ω_2.

When the bicharacteristics incident in ϱ_0 do not issue from any singularity of the data φ, ψ, f_1, f_2 then the solution $\nu = (\nu_1, \nu_2)$ is smooth along these incoming rays but the singularity ϱ_0 of the data (g_1, g_2) propagate along the outgoing rays.

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