

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

ERMANN A TOMAINI

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Rendiconti del Seminario Matematico della Università di Padova,
tome 76 (1986), p. 59-74

http://www.numdam.org/item?id=RSMUP_1986__76__59_0

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Infinite Boundary Value Problems for Surfaces of Prescribed Mean Curvature.

ERMANN A TOMAINI (*)

The Dirichlet problem for surfaces of prescribed mean curvature consists in determining a function $u \in C^2(\Omega)$ satisfying the equation

$$L(u) = (1 + |Du|^2) \Delta u - \sum_{i,j=1}^n D_i u D_j u D_{ij} u - nH(x)(1 + |Du|^2)^{\frac{3}{2}} = 0$$

in a bounded domain Ω and taking a given boundary value φ on $\partial\Omega$. We consider Dirichlet problem where infinite boundary values are admitted on subsets of the boundary. Jenkins and Serrin developed an existence and uniqueness theory for this problem in the case $H \equiv 0$ and $n = 2$ [9], while Spruck extended Jenkins-Serrin's results to the case $H = \text{constant}$ and $n = 2$ [10].

In [1] Massari proved an existence and uniqueness theorem in the case H not constant, arbitrary n and the boundary value φ is finite in Γ_0 , $+\infty$ in Γ_1 , $-\infty$ in Γ_2 , where Γ_0 , Γ_1 , Γ_2 are open, disjoint subsets of $\partial\Omega$ and $\Gamma_0 \neq \emptyset$. In this work we complete Massari's results studying the case $\Gamma_0 = \emptyset$.

In the first section we recall some basic results of U. Massari [1], [2] Giusti [3], Miranda [5], Emmer [4] which we shall use. In the second section we prove an existence and uniqueness theorem for the following Dirichlet problem

$$\begin{aligned} L(u) &= 0 && \text{in } \Omega \\ \lim_{y \rightarrow x} u(y) &= +\infty && \forall x \in \Gamma_1 \\ \lim_{y \rightarrow x} u(y) &= -\infty && \forall x \in \Gamma_2 \end{aligned}$$

(*) Indirizzo dell'A.: Istituto di Matematica, Via Machiavelli, 44100 Ferrara, Italy.

where Γ_1 and Γ_2 are open, not empty and disjoint subsets of $\partial\Omega$. I wish to thank Professor U. Massari for his help and encouragement during the preparation of this paper.

1. A) Let Ω be a bounded domain (*i.e.* open and connected) in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$ and

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup N$$

where $\Gamma_0, \Gamma_1, \Gamma_2$ are open, not empty, disjoint C^1 sets and N is a closed set such that $H_{n-1}(N) = 0$.

B) Let A_0, A_1, A_2 open disjoint sets with Lipschitz continuous boundary such that

$$H_{n-1}(\partial\Omega \cap \partial A_i) = 0 \quad i = 0, 1, 2.$$

C) Let H a bounded Lipschitz continuous function in $\bar{\Omega}$ such that for every Caccioppoli set $B \subset \Omega$ $\text{meas } B \neq 0$, $\text{meas } B \neq \text{meas } \Omega$ we have

$$\left| n \int_B H(x) dx \right| < \int_{\mathbb{R}^n} |D\varphi_B|$$

where $\int_{\mathbb{R}^n} |D\varphi_B|$ is the perimeter of B ; *i.e.* the total variation of the vector valued measure $D\varphi_B$:

$$\int_{\mathbb{R}^n} |D\varphi_B| = \sup \left\{ \int \varphi_B \operatorname{div} g dx : g \in [C_0^1(\mathbb{R}^n)]^n, |g| \leq 1 \right\}$$

D) Let $\Omega_0 = \Omega \cup A_0$ be a connected set and

$$\Lambda(x) \geq \frac{n|H(x)|}{n-1} \quad \forall x \in \Gamma_0 \quad \Gamma_0 = A_0 \cap \partial\Omega$$

$$\Lambda(x) = \frac{nH(x)}{n-1} \quad \forall x \in \Gamma_1 \quad \Gamma_1 = A_1 \cap \partial\Omega$$

$$\Lambda(x) = -\frac{nH(x)}{n-1} \quad \forall x \in \Gamma_2 \quad \Gamma_2 = A_2 \cap \partial\Omega$$

where $\Lambda(x)$ is the mean curvature of $\partial\Omega$ at x .

Under these hypotheses there exist minima for the functionals

$$(1.1) \quad I(v, \varphi) = \int_{\Omega} \sqrt{1 + |\overline{Dv}|^2} + n \int_{\Omega} H(x)v(x) dx + \int_{\partial\Omega} |v - \varphi| dH_{n-1}$$

$$(1.2) \quad I'(v, \varphi) = \int_{\Omega} \sqrt{1 + |\overline{Dv}|^2} - n \int_{\Omega} H(x)v(x) dx + \int_{\partial\Omega} |v - \varphi| dH_{n-1}$$

for every $\varphi \in L^1(\partial\Omega)$. Such minima belong to $C^{2,\alpha}(\Omega) \cap BV(\Omega)$.

If φ is a bounded function then the minimum is bounded; if $\varphi \in C(\Gamma_0)$ then the minimum takes the boundary value φ in Γ_0 . (see [6], [7]).

Let $\{\Omega_h\}_{h \in \mathbb{N}}$ be a sequence of smooth sets with

$$\Omega_1 \subset \subset \Omega_2 \subset \subset \dots; \quad \bigcup_{h=1}^{\infty} \Omega_h = \Omega; \quad H_{n-1}(\partial\Omega) = \lim_{h \rightarrow \infty} H_{n-1}(\partial\Omega_h).$$

We need to recall the following results:

THEOREM 1.1 [1]. Let $u \in C^2(\Omega)$ be a solution of equation $L(u) = 0$ such that

$$\lim_{y \rightarrow x} u(y) = +\infty \quad \forall x \in \Gamma_1.$$

Then for every open set A with Lipschitz continuous boundary such that

$$H_{n-1}(\partial A \cap \partial\Omega) = 0$$

we have

$$\lim_{h \rightarrow \infty} \int_{\partial\Omega_h \cap A} Tuv dH_{n-1} = H_{n-1}(\Gamma_1) \quad \Gamma_1 = A \cap \partial\Omega$$

where $Tu = Du/\sqrt{1 + |\overline{Du}|^2}$ and ν is the exterior normal to $\partial\Omega$.

THEOREM 1.2 [1]. Let $u \in C^2(\Omega)$ be a solution of $L(u) = 0$ such that

$$\lim_{y \rightarrow x} u(y) = -\infty \quad \forall x \in \Gamma_2.$$

Then for every open set A with Lipschitz continuous boundary such that

$$H_{n-1}(\partial A \cap \partial \Omega) = 0$$

we have

$$\lim_{h \rightarrow \infty} \int_{\partial \Omega_h \cap A} T u \cdot \nu dH_{n-1} = -H_{n-1}(\Gamma_2) \quad \Gamma_2 = A \cap \partial \Omega$$

THEOREM 1.3 [1]. If $\Gamma_2 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\varphi: \Gamma_0 \rightarrow \mathbf{R}$ is a bounded continuous function in Γ_0 then the set $P = \{x \in \Omega: u(x) = +\infty\}$ minimizes the functional

$$(1.3) \quad \mathcal{F}_1(B) = \int_{\Omega} |D\varphi_B| + n \int_{\Omega} H(x) \varphi_B(x) dx + \int_{\partial \Omega} |\varphi_B - \varphi_{\Gamma_1}| dH_{n-1}.$$

REMARK 1.1. If $\Gamma_1 = \emptyset$, an analogous result is valid for the set $N = \{x \in \Omega: u(x) = -\infty\}$. In particular N minimizes the functional

$$(1.4) \quad \mathcal{F}_2(B) = \int_{\Omega} |D\varphi_B| - n \int_{\Omega} H(x) \varphi_B(x) dx + \int_{\partial \Omega} |\varphi_B - \varphi_{\Gamma_1}| dH_{n-1}.$$

THEOREM 1.4 [1]. Let φ be continuous in $x_0 \in \Gamma_0$, u minimizes the functional $I(v, \varphi)$ and $t > \varphi(x_0)$. Let $R > 0$ be such that

$$\varphi(x) < t - R \quad \forall x \in B_R(x_0) \cap \partial \Omega$$

$$B_R(x_0) \cap \Omega = \{(x', x_n) \in (B' \times \mathbf{R}) \cap B_R(x_0): x_n > \psi(x')\}$$

where $B' \subset \mathbf{R}^{n-1}$ is an open set and $\psi: B' \rightarrow \mathbf{R}$ is a Lipschitz continuous function. Then the set

$$E = \{(x, y) \in \Omega \times \mathbf{R}: y < u(x)\}$$

minimizes the functional

$$\int_A |D\varphi_F| + n \int_A \varphi_F(x, y) \hat{H}(x) dx dy$$

where $A = B_R(x_0, t) \subset \mathbf{R}^{n+1}$ and $\hat{H}$ is the following function

$$\hat{H}(x', x_n) = \begin{cases} H(x', x_n) & \text{if } x_n \geq \psi(x') \\ H(x', \psi(x')) & \text{if } x_n < \psi(x'). \end{cases}$$

THEOREM 1.5 [1]. If $I_2 = \emptyset$, $I_0 \neq \emptyset$, $\varphi: I_0 \rightarrow \mathbf{R}$ is a below bounded continuous function and $\emptyset$ is the unique minimum of $\mathcal{F}_1(B)$, then there exists $u \in C^2(\Omega)$ such that

$$\begin{aligned} L(u) &= 0 & \text{in } \Omega \\ u &= \varphi & \text{in } I_0 \\ \lim_{y \rightarrow x} u(y) &= +\infty & \forall x \in I_1 \end{aligned}$$

THEOREM 1.6 [1]. If $I_1 = \emptyset$, $I_0 \neq \emptyset$, $\varphi: I_0 \rightarrow \mathbf{R}$ is an upper bounded continuous function and if $\emptyset$ is the unique minimum of $\mathcal{F}_2(B)$, then there exists $u \in C^2(\Omega)$ such that

$$\begin{aligned} L(u) &= 0 & \text{in } \Omega \\ u &= \varphi & \text{in } I_0 \\ \lim_{y \rightarrow x} u(y) &= -\infty & \forall x \in I_2. \end{aligned}$$

THEOREM 1.7 [1]. Let $\varphi: I_0 \rightarrow \mathbf{R}$ be a continuous function, $I_0 \neq \emptyset$. Necessary and sufficient condition for the existence of a solution $u \in C^2(\Omega)$ to Dirichlet problem

$$\begin{aligned} L(u) &= 0 & \text{in } \Omega \\ u &= \varphi & \text{in } I_0 \\ \lim_{y \rightarrow x} u(y) &= +\infty & \forall x \in I_1 \\ \lim_{y \rightarrow x} u(y) &= -\infty & \forall x \in I_2 \end{aligned}$$

is that the unique minimum of the functionals $\mathcal{F}_1(B)$ and $\mathcal{F}_2(B)$ is the empty set.

THEOREM 1.8 [1]. Let $u_1, u_2 \in C^2(\Omega)$ be two solutions of the problem

$$\begin{aligned} L(u) &= 0 & \text{in } \Omega \\ u &= \varphi & \text{in } \Gamma_0 \\ \lim_{y \rightarrow x} u(y) &= +\infty & \forall x \in \Gamma_1 \\ \lim_{y \rightarrow x} u(y) &= -\infty & \forall x \in \Gamma_2. \end{aligned}$$

Then

- i) if $\Gamma_0 \neq \emptyset$ then $u_1 = u_2$ in Ω ;
- ii) if $\Gamma_0 = \emptyset$ then $u_1 = u_2 + \text{constant}$ in Ω .

THEOREM 1.9 [2]. Let E a set minimizing the functional

$$\int_K |D\varphi_E| + \int_K \varphi_E(x) A(x) dx$$

in an open set $\Omega \subset \mathbf{R}^n$ with $n \geq 2$ and $|A(x)| \leq A$. If $x \in \partial E$ and $\overline{B_\varrho(x)} \subset \Omega$, $\varrho > 0$, then we have

$$(1.5) \quad \varrho^{1-n} \int_{B_\varrho(x)} |D\varphi_E| + (n-1)A\omega_n \varrho \geq \omega_{n-1}$$

THEOREM 1.10 [4]. If the set E is a local minimum for the functiona

$$\mathfrak{L}(E) = \int \varphi_E(x) H(x) dx + \int |D\varphi_E|$$

in the open set $\Omega \subset \mathbf{R}^n$ with obstacle L and if $\partial L \cap \Omega$ is of class C^1 , then there exists an open set $\Omega_0 \subset \Omega$ with $\partial L \cap \Omega \subset \Omega_0$ such that $\partial E \cap \Omega_0$ is of class C^1 .

We recall the definition of generalized solution, introduced by M. Miranda (see [5]).

DEFINITION 1.1. A function $u: \Omega \rightarrow \overline{\mathbf{R}}$ is called a generalized solution of equation

$$\operatorname{div} Tu = nH(x)$$

if the set $E = \{(x, y) \in \Omega \times \mathbf{R}: y < u(x)\}$ minimizes the functional

$$(1.6) \quad \int |D\varphi_E| + n \int H(x) \varphi_E(x, y) dx dy$$

in $\Omega \times \mathbf{R}$.

That means that for every set $V \subset \Omega \times \mathbf{R}$, coinciding with E outside some compact set $K \subset \Omega \times \mathbf{R}$ we have

$$\int_K |D\varphi_E| + n \int_K H(x) \varphi_E(x, y) dx dy \leq \int_K |D\varphi_V| + n \int_K H(x) \varphi_V(x, y) dx dy.$$

We note that the function $u(x)$ can take the values $\pm \infty$.

It follows from [8] theorem 2.3 that every classical solution of $\operatorname{div} Tu = nH(x)$ is a generalized solution and reciprocally, every local bounded generalized solution is a classical solution of $\operatorname{div} Tu = nH(x)$. We introduced the sets:

$$P = \{x \in \Omega: u(x) = +\infty\}; \quad N = \{x \in \Omega: u(x) = -\infty\};$$

$$G = \Omega - (P \cup N) - \partial P \cap \partial N.$$

We have the following results

(1.7) the function $u(x)$ is regular in G and is a classical solution of $\operatorname{div} Tu = nH(x)$.

(1.8) Let $\{u_k\}$ be a sequence of generalized solution of $\operatorname{div} Tu = nH(x)$ in Ω and let E_k be the corresponding domains (1.6). Then a subsequence of E_k will converge in $L^1_{\text{loc}}(\Omega \times \mathbf{R})$ to a set $E = \{(x, y) \in \Omega \times \mathbf{R}: y < u(x)\}$ and $u(x)$ is a generalized solution of $\operatorname{div} Tu = nH(x)$. We say in this case that a subsequence of $\{u_k\}$ converges locally to $u(x)$.

THEOREM 1.11 [3]. Let u, v be two C^2 -functions in Ω such that $\operatorname{div} Tu \leq \operatorname{div} Tv$ in Ω . Suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1 open set in $\partial\Omega$ and that $u, v \in C(\Omega \cup \Gamma_1)$, $u(x) \geq v(x)$ in Γ_1 and

$$\lim_{t \rightarrow 0^+} \int_{\partial\Omega_t - A} (1 - Tu \cdot v) dH_{n-1} = 0$$

for every open set $A \supset \Gamma_1$. Then

- a) if $\Gamma_1 \neq \emptyset$ then $u \geq v$ in Ω ;
- b) if $\Gamma_1 = \emptyset$ then $u = v + \text{constant}$ in Ω .

2. THEOREM 2.1. We suppose that $\Gamma_0 = \emptyset$, $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$. If $\emptyset$ and Ω are the unique minima for the functionals

$$\mathcal{F}_1(B) = \int_{\Omega} |D\varphi_B| + n \int_{\Omega} H(x) \varphi_B(x) dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_1}| dH_{n-1}$$

$$\mathcal{F}_2(B) = \int_{\Omega} |D\varphi_B| - n \int_{\Omega} H(x) \varphi_B(x) dx + \int_{\partial\Omega} |\varphi_B - \varphi_{\Gamma_2}| dH_{n-1}$$

then there exists a solution to the Dirichlet problem

$$\begin{aligned} \operatorname{div} Tu &= nH(x) && \text{in } \Omega \\ \lim_{y \rightarrow x} u(y) &= +\infty && \forall x \in \Gamma_1 \\ \lim_{y \rightarrow x} u(y) &= -\infty && \forall x \in \Gamma_2. \end{aligned}$$

REMARK 2.1. It follows from the hypothesis on the functionals $\mathcal{F}_1(B)$ and $\mathcal{F}_2(B)$ that

$$(2.1) \quad n \int_{\Omega} H(x) dx + H_{n-1}(\Gamma_2) = H_{n-1}(\Gamma_1)$$

REMARK 2.2. It follows from theorem 1.8 that the solution of the problem is unique up to an additive constant.

We prove theorem 2.1 in two steps.

1st step. Let u_h be the solution of the problem

$$\begin{aligned} \operatorname{div} Tu_h &= nH(x) && \text{in } \Omega \\ u_h(x) &= h && \text{in } \Gamma_1 \\ u_h(x) &= 0 && \text{in } \Gamma_2. \end{aligned}$$

For every $h \in \mathcal{A}$ we can find a constant c_h with $0 < c_h < h$ such that

$$(2.2) \quad \begin{cases} \text{meas}(\{x \in \Omega: u_h(x) \geq c_h\}) \geq \frac{|\Omega|}{4} \\ \text{meas}(\{x \in \Omega: u_h(x) \leq c_h\}) \geq \frac{|\Omega|}{4}. \end{cases}$$

We set $v_h = u_h - c_h$ then v_h is a generalized solution of

$$\begin{aligned} \operatorname{div} T v_h &= n H(x) & \text{in } \Omega, \\ v_h &= h - c_h & \text{in } \Gamma_1, \\ v_h &= -c_h & \text{in } \Gamma_2. \end{aligned}$$

It follows from (1.8) that a subsequence of $\{v_h\}$ will converge locally to a generalized solution $v(x)$ of

$$(2.3) \quad \operatorname{div} T v = n H(x).$$

We prove that the sets P_v and N_v are empty and hence the set G is Ω . Then v is a locally bounded function in Ω and it is a classical solution of (2.3). First we prove that

$$\begin{aligned} a) \quad \lim_{h \rightarrow \infty} c_h &= +\infty \\ b) \quad \lim_{h \rightarrow \infty} (h - c_h) &= +\infty. \end{aligned}$$

a) If $\lim_{h \rightarrow \infty} c_h = c_0$ with $c_0 \in \mathbb{R}$, then passing possibly to a subsequence we can suppose that $v_h \rightarrow u$, solution of the problem

$$(2.4) \quad \begin{cases} \operatorname{div} T u = n H(x) & \text{in } \Omega \\ \lim_{v \rightarrow x} u(y) = +\infty & \forall x \in \Gamma_1 \\ u = -c_0 & \text{in } \Gamma_2. \end{cases}$$

Let $\{\Omega_h\}$ be a sequence of smooth open sets with

$$\Omega_1 \subset \subset \Omega_2 \subset \subset \dots; \quad \Omega = \bigcup_{h=1}^{\infty} \Omega_h; \quad H_{n-1}(\partial \Omega) = \lim_{h \rightarrow \infty} H_{n-1}(\partial \Omega_h).$$

If we integrate (2.4) in Ω_h we get

$$(2.5) \quad n \int_{\Omega_h} H(x) dx = \int_{\partial\Omega_h \cap A_1} Tu \cdot \nu dH_{n-1} + \int_{\partial\Omega_h \cap A_2} Tu \cdot \nu dH_{n-1}.$$

This is possible because u is solution of problem (2.4) and from theorem 1.3 the set P_u minimizes the functional $\mathcal{F}_1(B)$, hence $P_u = \emptyset$ or $P_u = \Omega$. But from (2.2)

$$\begin{cases} \text{meas}(\{x \in \Omega : v_h(x) \geq 0\}) \geq \frac{|\Omega|}{4} \\ \text{meas}(\{x \in \Omega : v_h(x) \leq 0\}) \geq \frac{|\Omega|}{4} \end{cases}$$

so we get $P_u \neq \Omega$ that is $P_u = \emptyset$.

Moreover $N_u = \emptyset$ because in problem (2.4) $\Gamma_2 = \emptyset$. We have

$$\lim_{y \rightarrow x} u(y) = +\infty \quad \forall x \in \Gamma_1$$

$$\partial\Omega \cap A_1 = \Gamma_1; \quad H_{n-1}(\partial\Omega \cap \partial A_1) = 0$$

and from theorem 1.1 we get

$$\lim_{h \rightarrow \infty} \int_{\partial\Omega_h \cap A_1} Tu \cdot \nu dH_{n-1} = H_{n-1}(\Gamma_1)$$

On the other hand

$$\minlim_{h \rightarrow \infty} \int_{\partial\Omega_h \cap A_2} Tu \cdot \nu dH_{n-1} > -H_{n-1}(\Gamma_2).$$

In fact if

$$\minlim_{h \rightarrow \infty} \int_{\partial\Omega_h \cap A_2} Tu \cdot \nu dH_{n-1} = -H_{n-1}(\Gamma_2).$$

it follows from theorem 1.11 that every solution w of equation

$$\text{div } Tw = -nH(x) \quad \text{in } \Omega$$

with

$$w \leq -u \quad \text{in } \partial\Omega \setminus \Gamma_2$$

must be

$$w \leq -u \quad \text{in } \Omega.$$

This is a contradiction because for every boundary value $\varphi \in C(\Gamma_2)$ a minimum of $I'(v, \varphi)$ takes it. Passing to the limit as $h \rightarrow \infty$ we have

$$\begin{aligned} n \int_{\Omega} H(x) dx \geq \lim_{h \rightarrow \infty} \int_{\partial\Omega_h A_1} Tu \cdot \nu dH_{n-1} + \minlim_{h \rightarrow \infty} \int_{\partial\Omega_h \cap A_2} Tu \cdot \nu dH_{n-1} > \\ > H_{n-1}(\Gamma_1) - H_{n-1}(\Gamma_2). \end{aligned}$$

This contradicts (2.1).

b) We prove that $\lim_{h \rightarrow \infty} (h - c_h) = +\infty$

If $\lim_{h \rightarrow \infty} (h - c_h) = \gamma_0$ with $\gamma_0 \in \mathbf{R}$, then passing possibly to a subsequence we can suppose that $v_h \rightarrow u$, solution of the problem

$$(2.7) \quad \begin{cases} \operatorname{div} Tu = nH(x) & \text{in } \Omega \\ u = \gamma_0 & \text{in } \Gamma_1 \\ \lim_{y \rightarrow x} u(y) = -\infty & \forall x \in \Gamma_2. \end{cases}$$

Arguing the same way of (a) we get

$$n \int_{\Omega} H(x) dx < H_{n-1}(\Gamma_1) - H_{n-1}(\Gamma_2)$$

contradicting (2.1).

It follows that a subsequence of $\{v_h\}$ will converge locally to a generalized solution v of problem

$$\begin{aligned} \operatorname{div} Tv &= nH(x) & \text{in } \Omega \\ \lim_{y \rightarrow x} v(y) &= +\infty & \forall x \in \Gamma_1 \\ \lim_{y \rightarrow x} v(y) &= -\infty & \forall x \in \Gamma_2. \end{aligned}$$

From theorem 1.3 the set $P = \{x \in \Omega: v(x) = +\infty\}$ minimizes the functional $\mathcal{F}_1(B)$ and the set $N = \{x \in \Omega: v(x) = -\infty\}$ the functional $\mathcal{F}_2(B)$. Therefore P and N are $\emptyset$ or Ω , but

$$\text{meas}(\{x \in \Omega: v_h(x) \geq 0\}) \geq \frac{|\Omega|}{4}$$

$$\text{meas}(\{x \in \Omega: v_h(x) \leq 0\}) \geq \frac{|\Omega|}{4}$$

and hence $P = N = \emptyset$.

We get $G = \Omega$ and v is a classical solution of the equation

$$\text{div } Tv = nH(x) \quad \text{in } \Omega.$$

2nd step. We prove that the function v takes on the required boundary value, more precisely we prove

$$\text{i) } \lim_{y \rightarrow x} v(y) = +\infty \quad \forall x \in \Gamma_1$$

$$\text{ii) } \lim_{y \rightarrow x} v(y) = -\infty \quad \forall x \in \Gamma_2$$

i) Let $x_0 \in \Gamma_1$ and let $\{x_h\}$ be a sequence of points in Ω such that $x_h \xrightarrow{h \rightarrow \infty} x_0$. We suppose that $v(x_h) \xrightarrow{h \rightarrow \infty} t \in \mathbb{R}$. The function $v_h = u_h - c_h$ minimizes $I(v, \varphi_h)$ where

$$\varphi_h(x) = \begin{cases} h - c_h & \text{if } x \in \Gamma_1 \\ -c_h & \text{if } x \in \Gamma_2. \end{cases}$$

Then $-v_h$ minimizes the functional $I'(v, -\varphi_h)$. Let $r > 0$ such that

$$h - c_h > t + r.$$

It follows from theorem 1.4 that the set

$$E_h = \{(x, y) \in \Omega \times \mathbb{R}: y < -v_h(x)\}$$

minimizes the functional

$$(2.8) \quad \int_A |D\varphi_F| - n \int_A \varphi_F(x, y) H(x) dx dy$$

in $A = B_r(x_0, -t)$ in the class $\{F \in \mathbf{R}^{n+1}: F \in \Omega \times \mathbf{R} \ F \Delta E_h \subset A\}$. The same minimal property is true for limit set

$$E = \{(x, y) \in \Omega \times \mathbf{R}: y < -v(x)\}.$$

For r small enough $(\partial\Omega \times \mathbf{R}) \cap A$ is of class C^1 , it follows from theorem 1.10 that E has boundary of class C^1 in a neighborhood of $(\partial\Omega \times \mathbf{R}) \cap A$ and ∂E and $(\partial\Omega \times \mathbf{R})$ have the same normal in the contact points. Making smaller the open set we can suppose

$$\begin{aligned} (\Omega \times \mathbf{R}) \cap A &= \{(x', x_n) \in (B' \times \mathbf{R}) \cap A: x_n > \psi(x')\} \\ E \cap A &= \{(x', x_n) \in (B' \times \mathbf{R}) \cap A: x_n > g(x')\} \end{aligned}$$

where $B' \subset \mathbf{R}^n$ in an open set, $x' = (x_1, \dots, x_{n-1})$, $g, \psi \in C^1(B')$. In every point of Γ_1 mean curvature is

$$(2.9) \quad \Lambda(x) = \frac{nH(x)}{n-1}.$$

The weak form of (2.9) is that for every $\chi \in C_0^1(B')$

$$(2.10) \quad \int_{B'} \frac{D\psi \cdot D\chi}{\sqrt{1 + |D\psi|^2}} dx' + n \int_{B'} H(x', \psi(x')) \chi(x') dx' = 0$$

On the other hand we have for every $\chi \in C_0^1(B')$, $\chi \geq 0$

$$(2.11) \quad \frac{d}{dt} \left[\int_{B'} \sqrt{1 + |D(g + t\chi)|^2} dx' - n \int_{B'} dx' \int_{g+t\chi} H(x', x_n) dx_n \right]_{t=0} \geq 0.$$

Hence

$$(2.12) \quad \int_{B'} \frac{Dg \cdot D\chi}{\sqrt{1 + |Dg|^2}} dx' + n \int_{B'} H(x', g(x')) \chi(x') dx' \geq 0.$$

Subtracting (2.12) and (2.10) we get

$$\begin{aligned} \int_{B'} D\chi \left(\frac{Dg}{\sqrt{1 + |Dg|^2}} - \frac{D\psi}{\sqrt{1 + |D\psi|^2}} \right) dx' + n \int_{B'} \chi(x') [H(x', g(x')) - \\ - H(x', \psi(x'))] dx' \geq 0 \end{aligned}$$

for every $\chi \in C_0^1(B')$, $\chi \geq 0$.

Therefore $g - \psi$ is a supersolution of an elliptic equation and $g - \psi \geq 0$ in B' ; $g - \psi = 0$ in the contact points.

It follows from maximum principle that $g - \psi = 0$ in B' and $\partial E = \partial \Omega \times \mathbf{R}$: a contradiction because

$$(x_h, -v(x_h)) \in \partial E \cap (\Omega \times \mathbf{R}).$$

We suppose now that $v(x_h) \xrightarrow{h \rightarrow \infty} -\infty$.

Let $r > 0$ be such that

$$v(x_h) < (k - c_k) - r$$

from theorem 1.4 the set $E_k = \{(x, y) \in \Omega \times \mathbf{R}: y < -v_k(x)\}$ minimizes the functional

$$(2.13) \quad \int_A |D\varphi_F| - n \int_A H(x) \varphi_F(x, y) dx dy$$

in $A = B_r(x_0, -v(x_h))$.

Set

$$E_{h,k} = \{(x, y) \in \Omega \times \mathbf{R}: y < -v_k(x) + v(x_h)\}$$

we get

$$\int_{B_r(x_0, -v(x_h))} |D\varphi_{E_k}| - n \int_{B_r(x_0, -v(x_h))} H(x) \varphi_{E_k}(x, y) dy dx = \int_{B_r(x_0, 0)} |D\varphi_{E_{h,k}}| - n \int_{B_r(x_0, 0)} H(x) \varphi_{E_{h,k}}(x, y) dx dy$$

and $E_{h,k}$ minimizes (2.15) in $B_r(x_0, 0)$.

Passing to the limit as $k \rightarrow +\infty$, the set $E_h = \{(x, y) \in \Omega \times \mathbf{R}: y < -v(x) + v(x_h)\}$ minimizes (2.15) in $B_r(x_0, 0)$.

Because $x_h \xrightarrow{h \rightarrow \infty} x_0$, choose $\sigma \in (0, r/2)$ there exists $h_0 > 0$ such that for every $h \geq h_0$

$$|x_h - x_0| < \sigma$$

and for these h we get

$$\text{meas}(B_\sigma(x_h, 0) \cap E_h) \leq \text{meas}(B_r(x_0, 0) \cap E_h).$$

On the other hand from the theorem 1.9 we have for every $0 < t < r$

$$(2.14) \quad t^{-n} \int_{B_t(x_h, 0)} |D\varphi_{E_h}| + nt \|H\|_{L^\infty(B_t(x_h, 0))} \geq \omega_n.$$

From the minimal property of E_h with $F = E_h - B_t(x_h, 0)$ we get

$$\int_{B_t(x_h, 0)} |D\varphi_{E_h}| - n \int_{B_t(x_h, 0)} H(x) \varphi_{E_h}(x, y) dx dy \leq \int_{\partial B_t(x_h, 0)} |D\varphi_F| - \int_{\partial B_t(x_h, 0)} |D\varphi_{E_h}| \leq \int_{\partial B_t(x_h, 0)} \varphi_{E_h} dH_n$$

from (2.14)

$$\begin{aligned} \int_{\partial B_t(x_h, 0)} \varphi_{E_h} dH_n &\geq (\omega_n - nt \|H\|_{L^\infty(B_t(x_h, 0))}) t^n - n \|H\|_{L^\infty(B_t(x_h, 0))} \text{meas}(B_t(x_h, 0)) \geq \\ &\geq (\omega_n - c(n, \|H\|_{L^\infty(B_t(x_h, 0))}) t) t^n. \end{aligned}$$

Integrating between 0 and σ we get

$$\begin{aligned} \text{meas}(B_\sigma(x_h, 0) \cap E_h) &= \int_0^\sigma dt \int_{\partial B_t(x_h, 0)} \varphi_{E_h}(x, y) dH_n \geq \\ &\geq \left(\frac{\omega_n}{n+1} - c(n, \|H\|_{L^\infty}) \sigma \right) \sigma^{n+1} > 0 \end{aligned}$$

while it is

$$\varphi_{E_h}(x, y) \rightarrow 0 \quad \text{a.e. } (x, y) \in \mathbb{R}^{n+1}.$$

Hence

$$v(x_h) \xrightarrow{h \rightarrow \infty} +\infty.$$

ii) Let $x_0 \in \Gamma_2$ and let $\{x_h\}$ be a sequence of points in Ω such that $x_h \xrightarrow{h \rightarrow \infty} x_0$. Arguing the same way of (i) we can prove that

$$\lim_{h \rightarrow \infty} v(x_h) = -\infty.$$

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Manoscritto pervenuto in redazione il 24 gennaio 1985.