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On Random Boundary Value Problems for Ordinary Differential Equations.

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SUMMARY - In this paper we prove that for a certain class of random boundary value problems the uniqueness of the solutions in this class implies the existence of the solutions.

1. Introduction.

The problem of concluding the existence from the uniqueness of the solutions of general boundary value problems for the ordinary differential equations comes from A. Lasota and S. N. Chow [1]. They have used the Brouer's open mapping theorem [2] as a main technique. Some other authors considered this problem, e.g. for functional differential equations [3].

In this paper we examine the general boundary value problem for random differential equations. The solutions of our problem is a stochastic process and we define the solution in the sense of almost all trajectories [4], [5]. According to paper [1] we make use of the Brouer's theorem. We prove the measurability of solutions using the technique of measurable selections [6], [7]. To show the measurability of the graph of multivalued function we apply Orlicz's method [8].

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2. Notations and definitions.

Let (Ω, Σ, P) be a complete probability space. We consider a function $f:[0,1]\times R^d\times\Omega \to R^d$ continuous with respect to (t, x) and such that the mapping $\Omega \ni \omega \to f(\cdot,\cdot,\omega) \in C([0,1)\times R^d; R^d) = \mathcal{C}$ is measurable. We define $f(t,x)(\omega) = f(t,x,\omega)$. We examine a system of differential equations

$$\dot{\xi} = f(t, \xi)\,. \tag{1}$$

DEFINITION 1. A stochastic process $\xi:[0,1)\times\Omega \to R^d$ is SP solution of equation (1) iff

(i) $\xi(t,\omega)$ is measurable for every $t \in [0,1]$,

(ii) $\xi(\cdot,\omega) \in C^1([0,1]; R^d) = \mathcal{C}^1$ for a.e. $\omega \in \Omega$ and

(iii) $\dot{\xi}(t,\omega) = f\big(t, \xi(t,\omega), \omega\big)$ for a.e. $t \in [0,1]$.

ASSUMPTION (C). For every $t_0 \in [0,1]$ and every random variable $\xi_0:\Omega \to R^d$ equation (1) has exactly one SP solution satisfying condition

$$\xi(t_0) = \xi_0 \qquad P\text{ - a.e.} \tag{2}$$

We denote by $\mathcal{L} = \mathcal{L}(\mathcal{C}^1, R^d)$ the space of all linear and continuous operators with norm topology. Let U be an arbitrary open set in $\mathcal{L}$. We take a measurable mapping $L:\Omega \to U$.

Let η be a d-dimensional random variable.

DEFINITION 2. A stochastic process $\xi:[0,1]\times\Omega \to R^d$ satisfies condition

$$L\xi = \eta \tag{3}$$

iff $\xi(\cdot,\omega) \in \mathcal{C}^1$ for a.e. $\omega \in \Omega$ and

$$L(\omega)\xi(\cdot,\omega) = \eta(\omega)\,.$$

We introduce for arbitrary sets X, Y the function $\Phi:X \to \mathcal{P}(Y)\setminus\{\emptyset\}$ ($\mathcal{P}(Y)$ denotes the class of all subsets of Y, $\emptyset$ is the empty set).

DEFINITION 3. By the graph of Φ we mean

$$\text{graph } \Phi = \{(x, y): x \in X,\ y \in \Phi(x)\} \subset X \times Y .$$

Let now X and Y be the measurable spaces.

DEFINITION 4. A function $\varphi: X \to Y$ is called a measurable selection of Φ if it is measurable and for all $x \in X$, $\varphi(x) \in \Phi(x)$.

The Borel σ-field of arbitrary metric space X is denoted by $\mathcal{B}(X)$.

3. The main theorem.

We shall prove the following

THEOREM. Let us assume that equation (1) satisfies assumption (C). Let for every measurable $L: \Omega \to U$ and every random variable $\eta: \Omega \to R^d$ equation (1) has at most one SP solution satisfying (3). Then, for every L and every η problem (1), (3) has SP solution.

To prove the theorem we shall need some lemmas concerning the measurability of the graph of multivalued functions.

4. The auxiliary lemmas.

Let us consider the set $\hat{C}$ of all continuous functions $\bar{f}: [0, 1] \times R^d \to R^d$ such that for every $t_0 \in [0, 1]$ and $x_0 \in R^d$ problem

$$\dot{x} = \bar{f}(t, x) , \tag{4}$$

$$x(t_0) = x_0 \tag{5}$$

has exactly one solution.

For every $l \in U$ and $\bar{x} \in R^d$ we define condition

$$l(x) = \bar{x} . \tag{6}$$

LEMMA 1. The set A of such triples $(f, l, \bar{x})$ for which problem (4) (6) have at least two different solutions is of Borel type in $C \times U \times R^d$.

PROOF of LEMMA 1. Let $A(N, q)$ denote the set of such triples for which there exist two different solutions x_1 and x_2 of problem (4), (6) and which are bounded by $N \in \mathbf{N}$ and such that for a certain $t \in [0, 1]$

$$(7) \qquad |x_1(t) - x_2(t)| \geqslant \frac{1}{q}, \qquad q \in \mathbf{N}.$$

It is obvious that $A = \bigcup_{N,q} A(N, q)$.

It is sufficient to prove that $A(N, q)$ is a Borel set. We shall prove that it is closed. To do this we take the sequences $\{\bar{f}_n\}$, $\{l_n\}$, $\{\bar{x}_n\}$ which converge to $\bar{f}$, l, $\bar{x}$ almost uniformly, in $\mathfrak{L}$ and in R^d, respectively.

Let $x_1^{(n)}$, $x_2^{(n)}$ be two different solutions of problem

$$(8) \qquad \dot{x} = \bar{f}_n(t, x),$$

$$(9) \qquad l_n x = \bar{x}_n$$

such that there exist $t_n \in [0, 1]$ and

$$(10) \qquad |x_1^{(n)}(t_n) - x_2^{(n)}(t_n)| \geqslant \frac{1}{q}.$$

After passing to subsequences, if it is necessary, we can assume that $t_n \to t_0$, $x_i^{(n)}(t_n) \to x_i^0$ (x_i are bounded by N). We have denoted by x_i, for $i = 1, 2$, the solutions of equation (4) with conditions

$$(11) \qquad x_i(t_0) = x_i^0.$$

These solutions are unique, therefore $x_i^{(n)} \to x_i$ uniformly. But from this we conclude that x_i are solutions of (4), (6) and they satisfy

$$|x_1(t_0) - x_2(t_0)| \geqslant \frac{1}{q}. \qquad \text{q.e.d.}$$

COROLLARY 1. The set of such triples $(\omega, l, \bar{x})$ for which equation

$$(12) \qquad \dot{x} = f(t, x, \omega)$$

with condition (6) has two different solutions is $\Sigma \times \mathfrak{B}(U) \times \mathfrak{B}(R^d)$ measurable.

LEMMA 2. The set of all quadruples $(x, l, \bar{x}, f)$ from $\mathcal{C}^1 \times U \times R^d \times \mathcal{C}$ such that x is the solution of (4), (6) is the Borel set.

PROOF of LEMMA 2. We consider the mapping $\Phi: \mathcal{C}^1 \times U \times R^d \times \mathcal{C} \to \to ([0,1], R^d)$ defined by $\Phi(x, l, \bar{x}, f) = (\varphi_1, z)$ where

$$\varphi_1(t) = \dot{x}(t) - f\big(t, x(t)\big), \qquad z = lx - \bar{x}.$$

The mapping Φ is continuous and the above set in our lemma which we are looking for is the inverse image of zero point. q.e.d.

5. Proof of Theorem.

We denote by Γ the set of such $\omega \in \Omega$ for which one can find such $\bar{x} \in R^d$ and $l \in U$ that problem (12), (6) has two different solutions. Let $H: \Gamma \to \mathfrak{I}(U \times R^d)$ be the mapping which for every $\omega \in \Gamma$ maps the nonempty set of such pairs $(l, \bar{x})$ for which problem (12), (6) has two different solutions. Applying Corollary 1, the graph H is the measurable set and applying Theorem 3 ([7]) it has a measurable selection.

Let $h: \Gamma \to U \times R^d$ be this selection and

$$h(\omega) = \big(l(\omega), \bar{x}(\omega)\big).$$

Problem (12), (6) does not have unique solution for $l = l(\omega)$, $\bar{x} = \bar{x}(\omega)$.

Let $\Psi: \Gamma \to \mathfrak{I}(\mathcal{C}^1)$ be a mapping which for every $\omega \in \Gamma$ maps the set of solutions of equation (12) with condition

$$l(\omega, x) = \bar{x}(\omega). \tag{13}$$

Making use of Lemma 2 the graph Ψ is the measurable set, therefore Ψ has a measurable selection ψ. But for every $\omega \in \Gamma$ we have card $\Psi(\omega) > 1$ so for $\bar{\Psi}(\omega) = \Psi(\omega) \setminus \{\psi(\omega)\}$ we can find the measurable selection $\bar{\psi}$ of this mapping.

We fix arbitrary $L_0 \in U$ and we define the mapping L by

$$L(\omega) = \begin{cases} l(\omega) & \text{for } \omega \in \Gamma, \\ L_0 & \text{for } \omega \notin \Gamma \end{cases}$$

and we define the random variable

$$\eta(\omega) = \begin{cases} \bar{x}(\omega) & \text{for } \omega \in \Gamma, \\ 0 & \text{for } \omega \, \bar{x} \, \Gamma. \end{cases}$$

Using Theorem 2.1 ([1]) for $\omega \notin \Gamma$ we conclude that equation (12) with condition

$$L_0(x) = 0 \tag{14}$$

has the solution which we define by $\xi(t, \omega)$.

By concluding analogously as in Lemma 2 one can find that the mapping $\omega \to \xi(\cdot, \omega)$ is measurable.

Defining

$$\xi_1(t, \omega) = \begin{cases} \psi(\omega) t & \text{for } \omega \in \Gamma, \\ \xi(t, \omega) & \text{for } \omega \notin \Gamma \end{cases}$$

and

$$\xi_2(t, \omega) = \begin{cases} \bar{\psi}(\omega) t & \text{for } \omega \in \Gamma, \\ \xi(t, \omega) & \text{for } \omega \notin \Gamma, \end{cases}$$

we obtain two different solutions of problem (1), (3). Because of our assumptions it is implied that $P(\Gamma) = 0$. But this ensures the existence of the unique SP solution of problem (1), (3). q.e.d.

REMARK 1. As a particular case of the general random boundary value problem one can consider a Cauchy problem. One can show that if there exists such $M > 0$ that

$$|f(t, x, \omega) - f(t, y, \omega)| \leqslant M|x - y| \tag{15}$$

for every $t \in [0, 1]$ and every $\omega \in \Omega$ then, the random Cauchy problem has exactly one solution. Also the problems « close » (in a certain sense) to the Cauchy problem have unique solutions. The condition (15) has limited applicability because M cannot depend on ω.

REMARK 2. Our problem may be more interesting if the open set U considered in this paper is replaced by the random set $U(\omega)$. It could give more interesting theorems for linear equations.

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