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On the Squares of Sets of Linear Groups.

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The covering problem ($CC \supseteq G$, C-conjugacy class) for the group $PSL(2, q)$ was described in the paper [1]. In this paper we investigate squares of sets $K_w = \{g \in G: |g| = w\}$ of elements of the same order of linear groups. We will say that the group G has the property W if for each w, the set $K_w K_w$ is a subgroup of G (see [2]). We have proved that the set $K_2 K_2$ is not a subgroup in any groups $GL(n, K)$, $SL(n, K)$, $PSL(n, K)$ for $n \geqslant 3$. We have proved, too, that $K_2 K_2 \leqslant GL(2, K)$, $K_2 K_2 \leqslant SL(2, K)$ for char $K = 2$ and $K_2 K_2 \nleqslant GL(2, K)$ for char $K \neq 2$. The set $K_2 K_2$ of the group $PSL(2, q)$ is described in Theorem 5.

The following Lemma will be useful in the sequel.

Lemma 1. Let G be a group. An element $g \in K_2^m$ $(m \geqslant 2)$ if and only if there is an element $x \in K_2^{m-1}$, $x \neq g^{-1}$ such that $(gx)^2 = 1$.

Proof. If $g = s_1 s_2 \dots s_m$, $s_i \in K_2$ then $g^{-1} = xgx$ where $x = s_m s_{m-1} \dots s_2$. Conversely, if if $(gx)^2 = 1$ and $x \in K_2^{m-1}$ then $g = x^{-1}(g^{-1}x^{-1})$ and $(g^{-1}x^{-1})^2 = g^{-1}x^{-1}g^{-1}x^{-1} = g^{-1}g = 1$. Since $g^{-1}x^{-1} \neq 1$ by assumption, hence $g \in K_2^m$.

Theorem 1. The groups $GL(n, K)$, $SL(n, K)$ and $PSL(n, K)$ with $n \geqslant 3$ over any field K have not property W.

Proof. Let us consider all possible cases: (i) $|K| > 3$, (ii) $|K| = 2$, (iii) $|K| = 3$.

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Ad (i). In this case there is an element u such that $u^2 \neq 1$.

If $A = \operatorname{diag}[u^{-1}, u, 1]$, $B = \operatorname{diag}[1, u, u^{-1}]$ with $u^2 \neq 1$ then A, $B \in K_2K_2$, $AB \notin K_2K_2$ by Lemma 1 ($m = 2$). Using this fact we can see that quasidiagonal matrices $[A, 1, \ldots, 1]$, $[B, 1, \ldots, 1]$ belong to the set K_2K_2 but their product do not belong to the set K_2K_2. In the remain cases we act similarly, namely in thefase (ii) one can take

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and in the case (iii)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

Since $A_8 \simeq PSL(4, 2)$, $G_{168} \simeq PSL(3, 2)$ (see [4]) then by Theorem 1 the simple group A_8 and G_{168} have not property W.

REMARK 1. If $X \in K_2 \subset GL(2, K)$ and char $K = 2$ then $\det X = 1$.

THEOREM 2. If char $K = 2$ and $|K| > 2$ then $A \in K_2K_2 \subseteq GL(2, K)$ if and only if $\det A = 1$ i.e. $K_2K_2 = SL(2, K)$.

PROOF. Taking into consideration Corollary 4.7 p. 360 of [3] it is sufficient enough to show our proposition for matrices $A = N_1 = [0, 1; -a_2, -a_1]$, $A = N_2 = [k_1, 0; 0, k_2]$.

By Lemma 1

$$(1) \quad N_i \in K_2K_2 \iff \underset{\substack{X \in K_2 \\ X \neq N_i}}{\exists} (XN_i = N_1^{-1}X, \ \det X \neq 0), \qquad i = 1, 2.$$

If $N_i \in K_2K_2$ then $\det N_i = 1$ by Remark 1. If $\det N_1 = 1$ then the equation $XN_1 = N_1^{-1}X$ is satisfied by $X = [0, 1;\ 1, 0]$ for $a_1 \neq 0$ and $X = [u, u+1;\ u+1, u]$ ($u \neq 0$, $u \neq 1$) for $a_1 = 0$. Therefore $N_1 \in K_2K_2$. The equation $XN_2 = N_2^{-1}X$ has a solution $X \in GL(2, K)$ if and only if $k_1k_2 = 1$ i.e. $\det N_2 = 1$. We can take $X = [0, 1;\ 1, 0]$ which satisfies other condition of (1).

REMARK 2. If $|K| = 2$ then $K_2K_2 \leqslant GL(2, K) = SL(2, K) = PSL(2, K)$ what one can verify by calculations.

COROLLARY 2.1. If char $K = 2$ and $|K| > 2$ then $K_2K_2 = PSL(2, K)$.
The proof results from Theorem 2 and from the equality $SL(2, K) = PSL(2, K)$.

THEOREM 3. If char $K \neq 2$ then $K_2K_2 \not\leqslant GL(2, K)$.
Indeed, by Lemma 1 and by easy calculations we have

$$[-1, 0; 0, 1],\ [0, 1; -1, -a_1] \in K_2K_2 (a_1 \neq 0)$$

and

$$[-1, 0; 0, 1]\cdot[0, 1; -1, -a_1] = [0, -1; -1, -a_1] \notin K_2K_2 .$$

Theorem 4. The set K_2K_2 is a subgroup of the group $SL(2, K)$.

PROOF. If char $K = 2$ then Theorem 4 results from Remark 2 and from Theorem 2. If char $K \neq 2$ then only the matrix $[-1, 0; 0, -1]$ has the order 2 what one can verify by calculations.

THEOREM 5. (*a*) If in the field K (char $K \neq 2$) the element -1 is a square then $K_2K_2 = PSL(2, K)$.

(*b*) If in the field K $\left(\text{char } K \neq 2,\ |K| \neq 3\right)$ the element -1 is not a square then $K_2K_2 \not\leqslant PSL(2, K)$.

PROOF. (*a*) One can verify that $K_2 = \{[a, b; c, -a],\ a^2 + bc = -1\}$ in the group $PSL(2, K)$. By Lemma 1 we have

$$A \in K_2K_2 \iff \underset{\substack{T \in K_2 \\ T \neq A}}{\exists} A^{-1} = T^{-1}AT . \tag{2}$$

Let $T = [x, y; z, -x]$, $x^2 + yz = -1$, $A = [a_{11}, a_{12}; a_{21}, a_{22}]$. The condition (2) is equivalent to the solvability of the system equations

$$\begin{cases} x^2 + yz = -1 \\ a_{21}\, y + a_{12}\, z = (a_{22} - a_{11})\, x \end{cases} \tag{3}$$

or to the solvability of the equation

$$[(a_{11} + a_{22})^2 - 4]x^2 + 4a_{12}a_{21} = u^2 \text{ if } a_{12} \neq 0 \text{ or } a_{21} \neq 0 . \tag{4}$$

One can verify that in (2) we can take $T = [0, y; y^{-1}, 0]$ if $a_{12} = a_{21} = 0$; $T = [x, (a_{22} - a_{11})a_{21}^{-1}x; 0, -x]$, $x^2 = -1$ if $a_{21} \neq 0$ and $T = [x, 0; (a_{22} - a_{11})a_{12}^{-1}x, -x]$, $x^2 = -1$ if $a_{12} \neq 0$. Therefore $K_2K_2 = PSL(2, K)$.

(*b*) The matrices $A = [-1, 1; 1, -2]$, $T = [0, 1; -1, 0]$ fulfil the condition (2). Therefore $A \in K_2K_2$ and $A \neq I$.

If $A = [0, 1; -1, \pm 2]$ then the equation (4) has no solution and $A \notin K_2K_2$. Since K_2K_2 is a normal subset and in this case $PSL(2, K)$ is a simple group then $K_2K_2 \nleqslant PSL(2, K)$.

REMARK 3. If $|K| = 3$ then K_2K_2 is a subgroup of $PSL(2, K)$ what can be verified by calculations.

COROLLARY 5.1. $K_2K_2 = A_6$ in the alternating group A_6.

The proof results from the fact $A_6 \simeq PSL(2, K)$ (see [4]) and from (*a*) of Theorem 5.

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