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# Spaces of urelements

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# Spaces of Urelements.

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Dedicated to Professor Prachar on his 60-th birthday.

## 1. Introduction.

We will prove a topological characterization of a class of spaces which can be constructed from a space of urelemente. Space of this kindoccur, when independence results on the axiom of choice $AC$ are derived by applying standard topological procedures to sets whose existence contradicts the $AC$. We will consider the problem of their characterization in the ordered Mostowski model only. There the space $U$ of urelemente in its order topology is a source of many independence theorems. Our main result asserts:

In the Mostowski model a Hausdorff space $X$ is a continuous one-to-one image of a Dedekind-finite subset of $U^\omega$, if and only if every infinite set $Y \subseteq X$ has an infinite compact subset.

Our notation will follow [6] and [7]. When viewed from outside the model, the set $U$ of urelemente is $\mathbf{Q}$. But in the model most subsets of $\mathbf{Q}$ are deleted so that $U$ becomes a connected, locally compact dense and Dedekind-complete linearly ordered space. As is easily seen, every infinite subset of $U$ contains a closed, nontrivial interval which is compact. So the above topological condition is satisfied. It was first introduced by Bankston [1] under the name antianticompact. It is a hereditary property. We observe that in the presence of $AC$ there are no antianticompact $T_2$ spaces.

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1.1. LEMMA. If $P(\omega)$ is well orderable, then every antianticompact $T_2$ space is Dedekind-finite.

PROOF. Let $X$ be antianticompact, $T_2$ and countable. Then the topology $\mathbf{X}$ is well orderable, too. Therefore we may perform the usual argument of constructing an infinite discrete subset of $X$, thereby obtaining a contradiction to antianticompactness. Q.E.D.

A similar proof shows, that there are no antidiscrete $T_2$ spaces, either. A related result is due to J. Tong [9]: $AC^{\omega}$ implies, that there are no antianticompact $R_0$ spaces with an ascending chain of open sets.

On the other hand, in the Mostowski model (where $P(\omega)$ is well orderable) there are many antianticompact $T_2$ spaces (Dedekind-finite, of course).

1.2 PROPOSITION. If a $T_2$ space $X$ is a continuous one to one image of a Dedekind-finite subset $D$ of $U^{\omega}$ then $Y$ is antianticompact.

PROOF. Since it is easily verified that a continuous one to one image of an antianticompact space is antianticompact, it suffices to show that $D$ is antianticompact. Be $T \subseteq D$ infinite and in $\Delta(e)$ for some finite $e \subseteq U$ ($\Delta(e)$ is the class of all sets which are supported by $e$). It was observed in [3], that there is a one to one mapping $f: I \to T$ in $\Delta(e)$, where $I \subseteq U$ is an open interval between points in $\Delta(e)$. A permutation argument shows, that $f$ is of the form $f(u) = (f_i(u))_{i\in\omega}$, where $f_i$ is the identity map or $f_i$ is a constant $a \in e$. Hence $f$ induces on $I$ the order topology which is antianticompact. So $T$ contains an infinite antianticompact subset. Q.E.D.

## 2. Main result.

It was observed in [3] that the coarsest $T_2$ topology on $U$ which is supported by $\varphi$ is the order topology $\mathbf{U}_0$. We extend this result to sets of the form $X = \text{orb}_e\, x = \{px : p \in \text{fix}\, e\}$. If supp $(x)$ denotes the least support of $X$ and $\text{supp}\, x \setminus e = \{a_i : i \in n\}$, $a_0 < a_1 < \ldots < a_{n-1}$, then there is a mapping $f: \text{orb}_e\, x \to U^n$ which is defined through $f = \{p(x, \mathbf{a}) : p \in \text{fix}\, e\}$ $(\mathbf{a} = (a_i)_{i\in n})$; $f \in \Delta(e)$ (i.e.: $e$ supports $f$). It is one to one. This canonical mapping induces a natural topology $\mathbf{X}_0$ on $X$ which is generated by the product topology $\mathbf{U}_0^n$ on $U^n$.

2.1 LEMMA. Be $X$, $\mathbf{X}_0$ and $e$ as above. If $\mathbf{X} \in \Delta(e)$ is a $T_2$ topology on $X$ then $\mathbf{X}_0 \subseteq \mathbf{X}$.

PROOF. By the foregoing remarks we may assume that $X = \mathrm{orb}_e\, \boldsymbol{a}$, where $\boldsymbol{a}: n \to U \setminus e$ is increasing (i.e.: $\boldsymbol{a}(i) < \boldsymbol{a}(i+1)$). Hence $X$ is the set of all increasing functions $x \in \prod_{i \in n} I_i$, where $I_i$ is an interval between two consecutive elements of $e$. $\mathbf{X}_0$ is the subspace topology which is inherited from $U_0^n$. It is generated by the subbase sets $O(i, a) = \{x \in X: x(i-1) < a < x(i)\}$, where $a \in U$ and $0 \leqslant i \leqslant n$ ($x(-1)$ and $x(n)$ define void clauses). If $x \in O(i, a)$, then $O(i, a) = \mathrm{orb}_{e \cup \{a\}}(x)$. From this it follows with a permutation argument, that if $O(i, a) \notin \mathbf{X}$ for some $i$ and some $a \in I_i$, then $O(i, b)^0 = \emptyset$ for all $b \in \mathrm{orb}_e\, a = I_i$ (for the other values of $b$ it follows from the definition, that $O(i, b) = \emptyset$ or $O(i, b) = X$). In order to obtain a contradiction, we assume the latter and observe that $0^- \cap X^\varrho \neq \emptyset$ whenever $0 \in \mathbf{X}$ is nonempty and $^-$ and $\varrho$ (boundary operator) are formed with respect to $(oU)^n$ ($oU$ is the order compactification of $(U, \mathbf{U}_0)$). For if $0$ is in $\Delta(f)$, then there is a $x \in 0 \setminus \bigcup \{O(i, a): a \in f \cap I_i\}$ ($(O(i, a))^0 = \emptyset$) and $\mathrm{orb}_f\, x$ (which is an intersection of at most $n$ sets $O(j, a)$, $a \in f \cap I_j$ and $j \neq i$) has boundary points in $X^\varrho$. It follows from compactness that

$$C = \cap \{O^- \cap X^\varrho: x \in O \in \mathbf{X}\}$$

is nonempty and closed. Since subsets of $(oU)^n$ are definable from a finite subset of $U$ and the ordering relation on $U$, every nonempty closed subset of $(oU)^n$ has a maximal element in the lexicographic order. Applied to $C$ this yields a mapping $f: X \to X^\varrho$ in $\Delta(e)$ such that $f(x) \in O^-$ if $x \in O \in \mathbf{X}$. Since $|\mathrm{supp}\,(fx) \setminus e| < n = |\mathrm{supp}\,(x) \setminus e|$, a standard permutation argument assures that there is a $y \in X^\varrho$ such that the set $f^{-1}(y)$ is infinite. We choose $3^n + 1$ elements $x_i$ of this set and get by $T_2$ pairwise disjoint sets $O_i$, $x_i \in O_i \in \mathbf{X}$. Then $y \in \bigcap_i O_i^-$. This gives a contradiction (hence all sets $O(i, a)$ are in $\mathbf{X}$). For if $y \in A^-$, $A \subseteq X^- \subseteq (oU)^n$, then for some $R_i \in \{<, =, >\}$ and some $a_i$, $y(i) R_i a_i$, $A^-$ contains the set $\{x \in X^-: \forall i \in n: y(i) R_i X(i) R_i a_i\}$, whence at most $3^n$ pairwise disjoint subsets of $X^-$ can have a common element $y$ in their closures (a similar estimate holds for $(oU)^n$). Q.E.D.

We next improve this lemma in the case of an antianticompact topology on $X$.

2.2. LEMMA. Let $X$ and $e$ be as above and assume that $\mathbf{X} \in \Delta(e)$ is an antianticompact $T_2$ topology on $X$. Then $\mathbf{X} = \mathbf{X}_0$.

PROOF. According to 1.2, $\boldsymbol{X}_0$ is antianticompact. In view of lemma 2.1 we prove that $\boldsymbol{X} \subseteq \boldsymbol{X}_0$. Be $x \in 0 \in \boldsymbol{X}$ let $f \supseteq e$ be a support of $x$ and $0$ and fix $c_i$, $d_i$, $i \in n$, such that $c_i < d_i < c_{i+1}$, $x \in P = \prod_{i \in n} ]c_i, d_i[ \subseteq X$ and $]c_i, d_i[ \cap g = \{x(i)\}$, where $g = f \cup g_0$, $g_0 = \{c_i, d_i \colon i \in n\}$. This is possible, since $X$ is open in $(U^n, \boldsymbol{U}_0^n)$. We will prove that $Q = O \cap P = P \in \boldsymbol{X}_0$. We set for $E \subseteq n$ and $y \in P$, $L(E, y) = \{z \in P \colon z|n \setminus E = y|n \setminus E\}$ and prove by induction on $|E|$ that $L(E, x) \subseteq Q$. $|E| = 0$ says $x \in Q$ and $|E| = n$ gives $L(n, x) = P \subseteq Q$. Assume that $L(i, x) = L(\{0, \dots, i-1\}, x) \subseteq Q$. We show that for each $y \in L(i, x) L(\{i\}, y) \subseteq Q$, whence $L(i+1, x) = L(i \cup \{i\}, x) \subseteq Q$. To this end we observe, that $\boldsymbol{X}/L(\{i\}, y)$ is a $T_2$ topology on $]c_i, d_i[$ in $\Delta(e \cup g_0 \cup y'n \setminus \{i\})$ and since $(e \cup g_0 \cup y'n \setminus \{i\}) \cap ]c_i d_i[ = \emptyset$, we may conclude from [3] that $\boldsymbol{X}|L$ is one of the following topologies: discrete, half open interval (these 3 topologies are anticompact by [8]) or the order topology which is the only antianticompact one (and therefore it is $X|L$). We next consider the interval $]a_i, b_i[$ around $y(i) = x(i)$ which corresponds to the connectedness component of $L(\{i\}, y) \cap Q$ around $y$: $a_i < x(i) < b_i$ and $a_i, b_i$ are in $[c_i, d_i] \cap \Delta(g \cup y'n)$. Since $]c_i, d_i[ \cap \Delta(g \cup y'n) = \{x(i)\}$, $a_i = c_i$, $b_i = d_i$ and $L(\{i\}, y) \cap Q = L(\{i\}, y)$. Q.E.D.

Combining these results we may conclude:

2.3 THEOREM. In the Mostowski model a Hausdorff space is antianticompact, if and only if it is a continuous one-to-one image of a Dedekind-finite subset of $U^\omega$, $U$ with the order topology.

PROOF. We consider an antianticompact $T_2$ space $(X, \boldsymbol{X})$ in $\Delta(e)$. By 2.2 to each orbit $o = \mathrm{orb}_e\, x$ there corresponds naturally an embedding (topologically) $f_0 \colon o \to U^{n(0)}$ where $f_0' o$ is homeomorphic to some orbit $\mathrm{orb}_e\, \boldsymbol{a}$, $\boldsymbol{a} \in U^{n(0)}$. Since the set of all orbits $\mathrm{orb}_e\, \boldsymbol{a}$, $\boldsymbol{a} \in U^n$, $n \in \omega$ is countable, also the set $O$ of all $e$-orbits of $X$ is countable, for otherwise there are uncountably many orbits $o(\alpha)$, $\alpha \in \omega_1$, with the same image $f_{o(\alpha)}' o(\alpha) = \mathrm{orb}_e\, \boldsymbol{a}$, whence $\{f_{o(\alpha)}^{-1}(\boldsymbol{a}) \colon \alpha \in \omega_1\}$ would be an uncountable subset of $X$, contradicting 1.1. Consequently the topological sum $D$ of $O$ can be embedded in $U^\omega$ and the functions $f_0^{-1}$ induce a continuous bijective mapping $f \colon D \to X$. Since $X$ is Dedek indfinite, so is $D$. This proves « only if ». The converse implication is 1.2. Q.E.D.

It follows, than in the Mostowski model finite products of antianticompact $T_2$ spaces are antianticompact.

## 3. Additional remarks.

Using lemma 2.1, we can answer a question from [4] concerning the following properties of a topological space $(X, \mathbf{X})$. $X$ is $A1$, if for every open covering $\mathbf{0}$ there is a neighborhood choice function $f: X \to \mathbf{0}$ such that $x \in f(x)$. $X$ is $A2$, if there is a $f: X \to \mathbf{X}$ such that $x \in f(x)$ and $f'X$ refines $\mathbf{0}$. $AC$ implies that every space is $A1$, and conversely, the assertion « every $T_2$ space is $A1$ » implies $AC$ and « every $T_2$ space is $A2$ » implies $MC$ (every set is a union of a well orderable family of finite sets). In $ZF^0$ $AC \Rightarrow MC \Rightarrow PW$, where $ZF^0$ is set theory minus foundation and $PW$ asserts that the power set of an ordinal is well orderable, in $ZF$ ($ZF^0$ + foundation) $PW \Rightarrow AC$, but in $ZF^0$ $PW \not\Rightarrow MC$, $MC \not\Rightarrow AC$. In [4] it was shown that the assertion « every hereditarily $A2$ $T_2$-space is a union of a well orderable family of discrete sets (property $D2$) » is in strength between $MC$ and $PW$. The problem was left open, if it implies $MC$ (in $ZF^0$, of course). The following partial answer was provided: In the ordered Mostowski model every hereditarily $A1 + T_2$ space is well orderable.

3.1 THEOREM. In the ordered Mostowski model every hereditarily $A2$ $T_2$-space is $D2$. Hence this assertion does not imply $MC$ in $ZF_0$.

PROOF. Be $(X, \mathbf{X}) \in \Delta(e)$. Since the family of all orbits $\mathrm{orb}_e(x)$, $x \in X$, is well orderable, it suffices to show that $\mathrm{orb}_e x$ is discrete. As was observed in 2.2, $\mathrm{orb}_e x$ is covered by a family of open sets $P = \prod_{i \in n} ]c_i, d_i[$, where $P \subseteq \mathrm{orb}_e x$. We show that $P$ is discrete. Since by lemma 2.1 $\mathbf{X}_0|P \subseteq \mathbf{X}|P$, $\mathbf{0} = \{0 \in \mathbf{X}|P\colon \forall i \in n\colon \sup O|i < d_i\}$ is an open cover of $P(O|i = \{x(i)\colon x \in 0\} \subseteq [c_i, d_i])$. Let $f$ be an $A2$ mapping for $\mathbf{0}$ in $\Delta(h)$ and consider $f_i(y) = \sup f(y)|i \in (h \cup y'n) \cap ]c_i, d_i[$. For some $y$ and all $i$ $h \cap ]c_i, d_i[ < y(i) < d_i$. Hence $f_i(y) = y(i)$ for all $i$ and therefore $V(y) = \{z \in P\colon \forall i\colon z(i) \leqslant y(i)\}$ is a neighborhood of these points $y$. Since $P = \mathrm{orb.}\, y$ and $\mathbf{X}|P \in \Delta(g)$, where $g = e \cup \{c_i, d_i\colon i \in n\}$, $V(y)$ is a neighborhood of $y$ for every point $y \in P$. Similarly $W(y) = \{z \in P\colon z(i) \geqslant y(i)$ for all $i\}$ is a neighborhood of $y$, whence $\{y\} = V(y) \cap W(y)$ is isolated. Q.E.D.

As was observed in [4], there are compact (hence $A1$) $T_2$ spaces in the Mostowski model which are not $D2$.

While antianticompact $T_2$ spaces do not exist in the presence of $AC$, the large class of anticompact spaces does not conflict with $AC$. A space is anticompact, if compact subsets are finite (example: discrete spaces or $D$-finite subsets of $\mathbf{R}$). We next investigate, if nondiscrete first countable anticompact $T_2$ spaces can exist. We shall relate this question to the countable multiple choice axiom $MC^\omega$ (if $(E_n)_{n\in\omega}$ is a countable sequence of nonempty sets, there is a sequence $(F_n)_{n\in\omega}$ of finite sets such that $\emptyset \neq F_n \subseteq E_n$. In $ZF^0$, $MC^\omega \not\Rightarrow AC^\omega$ (unknown for $ZF$) and $AC^\omega \Rightarrow MC^\omega$ ($AC^\omega$: countable $AC$).

3.2 LEMMA. (1) In $ZF_0 + MC^\omega$ a $T_2$ space with a countable local base is a Kelley $k$-space ($A$ is closed, if and only if $A \cap K$ is closed, $K$ all compact sets).

(2) In $ZF^0$ anticompact $T_2 + k$-spaces are discrete.

PROOF. For (2) see [1]. (1) is a modification of standard arguments. Be $p \in A^- \setminus A$ and consider a neighborhood base $(U_n)_{n\in\omega}$ at $p$, $U_n \supseteq U_{n+1}$. By $MC^\omega$ there is a sequence $(F_n)_{n\in\omega}$ of finite sets such that $\emptyset \neq F_n \subseteq U_n \cap A$. $K = \{p\} \cup \bigcup_{n\in\omega} F_n$ is compact, because the open sets containing $p$ are cofinite in $K$, and $p \in (K \cap A)^-$. So $K \cap A$ is not closed. Q.E.D.

3.3 THEOREM. $MC^\omega$ is equivalent to the proposition that anticompact metrizable topological groups are discrete.

PROOF. If $MC^\omega$ holds, we get « discrete » by an application of the previous lemma. For the proof of the converse, we will start with a counterexample $(E_n)_{n\in\omega}$ of $PMC^\omega$, $E_n \cap E_m = \emptyset$ for $n \neq m$, and construct an anticompact metric group with no isolated points. $PMC^\omega$ is the axiom that there is an infinite set $A \subseteq \omega$ and a sequence $(F_n)_{n\in A}$ of finite sets such that for $n \in A$, $\emptyset \neq F_n \subseteq E_n$ (« $P$ » stands for « partial »). As was shown in [5], $MC^\omega \Leftrightarrow PMC^\omega$. We set $E = \bigcup_{n\in\omega} E_n$, $E(n) = \bigcup_{m\in n} E_m$ and $X = [E]^{<\omega}$, the system of all finite subsets of $E$, $X_n = [E(n)]^{<\omega}$. On $X$ we consider the Baire-metric: $d(x,x) = 0$ and $d(x,y) = 1/(n+1)$, if $x \cap E(n) = y \cap E(n)$ and $x \cap E_n \neq y \cap E_n$. The group-multiplication is the symmetric difference $(A\setminus B) \cup (B\setminus A)$. As is easily verified, $X$ is a metric topological group without isolated points. We show that $X$ is anticompact. Let $K$ be compact. First we observe, that $X_n$ is closed and discrete, since $d(x,y) > 1/(m+1)$, whenever $x \in X_n$, $y \in X_m$, $n \leqslant m$, and because $X = \bigcup_{n\in\omega} X_n$. Hence

$K \cap X_n$ is finite. This implies that $A = \{n \in \omega\colon K \cap (X_{n+1} \setminus X_n) \neq \emptyset\}$ is finite, whence $K = \bigcup_{n \in A} (K \cap X_{n+1})$ is finite, too. For if $n \in A$, then $F_n = E_n \cap (\cup K)$ is nonempty and as $F_n \subseteq \bigcup (K \cap X_{n+1})$, $F_n$ is finite. So $(F_n)_{n \in A}$ would define a $PMC$-function of $(E_n)_{n \in \omega}$, a contradiction.
Q.E.D.

In [2] the same construction with finite sets $E_n$ was used to obtain a $\sigma$-compact group which is not Lindelöf. 3.3 shows, that the finiteness of the sets $E_n$ was essential there.

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