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An Existence Theorem for Bounded Solutions of Differential Equations in Banach Spaces.

BOGDAN RZEPECKI (*)

SUMMARY - In this note we shall give sufficient conditions for the existence of bounded solutions of the differential equation $y' = f(t, y)$, $y(0) = x_0$ on the half-line $t \geq 0$. Here f is a function with values in a Banach space satisfying some regularity Ambrosetti type condition expressed in terms of the «measure of noncompactness α ».

Let $J = [0, \infty)$, and let $(E, \|\cdot\|)$ be a Banach space. Assume that $f: J \times E \rightarrow E$ is a function which satisfies the following conditions: (1) for each fixed $x \in E$ the mapping $t \mapsto f(t, x)$ is measurable; (2) for each fixed $t \in J$ the mapping $x \mapsto f(t, x)$ is continuous; and (3) $\|f(t, x)\| \leq G(t, \|x\|)$ for $(t, x) \in J \times E$, where the function G is monotonically nondecreasing in the second variable such that $t \mapsto G(t, u)$ is locally bounded for any fixed $u \in J$ and $t \mapsto G(t, y(t))$ is measurable for each continuous bounded function y of J into itself.

Let $x_0 \in E$. By (PC) we shall denote the problem of finding a solution of the differential equation

$$y' = f(t, y)$$

satisfying the initial condition $y(0) = x_0$.

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We deal with the problem (PC) using a method developed by Ambrosetti [1]. This method is based on the properties of the measure of noncompactness α . The proof of our theorem is suggested by a paper of Stokes [7] concerning finite-dimensional vector differential equations.

The measure of noncompactness $\alpha(X)$ of a nonempty bounded subset X of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$. For properties of α the reader is referred to [2], [3], [5].

Denote by $C(J)$ the set of all continuous functions from J to E . The set $C(J)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J .

Let us put

$$X(t) = \{x(t) : x \in X\}, \quad X_t = \bigcup \{X(s) : 0 \leq s \leq t\},$$

and

$$\int_0^t f(s, X(s)) ds = \left\{ \int_0^t f(s, x(s)) ds : x \in X \right\}$$

for $t \in J$ and $X \subset C(J)$. Moreover, we use the standard notations. The closure of a subset A of a topological vector space, its convex hull and its closed convex hull be denoted, respectively, by $\bar{A}$, $\text{conv } A$ and $\overline{\text{conv}} A$. For a mapping F defined on A we denote by $F[A]$ the image of A under F .

The Ascoli theorem we state as follows: $X \subset C(J)$ is conditionally compact if and only if X is almost equicontinuous and $\overline{X(t)}$ is compact for every $t \in J$. We shall use also the following result due to Ambrosetti [1]: If I is a compact subset of J and Y is a bounded equicontinuous subset of the usual Banach space of continuous E -valued functions on I , then

$$\alpha\left(\bigcup \{Y(t) : t \in I\}\right) = \sup \{\alpha(Y(t)) : t \in I\}.$$

Our result be proved by the following fixed point theorem of Furi and Vignoli type (see e.g. [6], Theorem 2):

Let $\mathfrak{X}$ be a nonempty closed convex subset of $C(J)$. Let $\Phi: 2^{\mathfrak{X}} \rightarrow [0, \infty)$ be a function such that $\Phi(X \cup \{x\}) = \Phi(X)$, $\Phi(\overline{\text{conv}} X) = \Phi(X)$, $\Phi(X_1) \leq \Phi(X_2)$ whenever $X_1 \subset X_2$, and if $\Phi(X) = 0$ then $\bar{X}$ is compact, for every $x \in \mathfrak{X}$ and every subsets X, X_1, X_2 of $\mathfrak{X}$. Suppose

that T is a continuous mapping of $\mathfrak{X}$ into itself and $\Phi(T[X]) < \Phi(X)$ for arbitrary subset X of $\mathfrak{X}$ with $\Phi(X) > 0$. Under the hypotheses, T has a fixed point in $\mathfrak{X}$.

THEOREM. Let h, L be functions of J into itself such that h is nondecreasing with $h(0) = 0$ and $h(t) < t$ for $t > 0$, and L is measurable and integrable on compact subsets of J with $\sup \left\{ \int_0^t L(s) ds : t \in J \right\} \leq 1$. Suppose that the scalar inequality

$$g(t) \geq \|x_0\| + \int_0^t G(t, g(s)) ds$$

has a bounded solution g existing on J ; denote by Z_0 the set of all $x \in E$ with $\|x\| \leq r_0 = \sup \{g(t) : t \in J\}$. Assume, moreover, that for any $t > 0$, $\varepsilon > 0$ and $X \subset Z_0$ there exists a closed subset Q of $[0, t]$ such that $\text{mes}([0, t] \setminus Q) < \varepsilon$ and

$$\alpha(f[I \times X]) \leq \sup \{L(s) : s \in I\} \cdot h(\alpha(X))$$

for each closed subset I of Q .

Then (PC) has at least one solution y defined on J and $\|y(t)\| \leq g(t)$ for $t \in J$.

PROOF. Denote by $\mathfrak{X}$ the set of all $x \in C(J)$ such that $\|x(t)\| \leq g(t)$ on J and

$$\|x(\sigma) - x(\tau)\| \leq \left| \int_{\sigma}^{\tau} G(s, r_0) ds \right|$$

for σ, τ in J . The set $\mathfrak{X}$ is a closed convex bounded and almost equicontinuous subset of $C(J)$.

Let us put

$\Phi(X) = \sup \{\alpha(X(t)) : t \in J\}$ for a subset X of $\mathfrak{X}$. Obviously $\Phi(X) < \infty$, $\Phi(X_1) \leq \Phi(X_2)$ for $X_1 \subset X_2$, and $\Phi(X \cup \{x\}) = \Phi(X)$ for $x \in \mathfrak{X}$. Since

$$(\overline{\text{conv } X})(t) = (\overline{\text{conv } X})(t) \subset (\overline{\text{conv } X})(t) \subset \overline{\text{conv } (X(t))},$$

so

$$\alpha((\overline{\text{conv } X})(t)) \leq \alpha(\overline{\text{conv } (X(t))}) = \alpha(X(t))$$

for $t \in J$. Hence

$$\sup \left\{ \alpha \left((\overline{\text{conv}} X)(t) \right) : t \in J \right\} \leq \sup \left\{ \alpha(X(t)) : t \in J \right\} ,$$

and consequently, $\Phi(\overline{\text{conv}} X) = \Phi(X)$. If $\Phi(X) = 0$ then $\overline{X(t)}$ is compact for every $t \in J$; therefore Ascoli's theorem proves that $\overline{X}$ is compact in $C(J)$.

To apply our fixed point theorem we define the continuous mapping T as follows: for $y \in C(J)$,

$$(Ty)(t) = x_0 + \int_0^t f(s, y(s)) ds .$$

Modifying the reasoning from [7] we infer that $T[\mathfrak{X}] \subset \mathfrak{X}$.

Let X be a subset of $\mathfrak{X}$ such that $\Phi(X) > 0$. To prove the theorem it remains to be show that $\Phi(T[X]) \leq h(\Phi(X))$.

To this end, fix t in J . Let $\varepsilon > 0$, and let $\delta = \delta(\varepsilon) > 0$ be a number such that

$$\int_A G(s, r_0) ds < \varepsilon/2$$

for each measurable $A \subset [0, t]$ with $\text{mes}(A) < \delta$. By the Luzin theorem there exists a closed subset B_1 of $[0, t]$ with $\text{mes}([0, t] \setminus B_1) < \delta/2$ and the function L is continuous on B_1 . Furthermore, by our comparison condition, there exists a closed subset B_2 of $[0, t]$ such that $\text{mes}([0, t] \setminus B_2) < \delta/2$ and

$$\alpha(f[I \times X_t]) \leq \sup \{L(s) : s \in I\} \cdot h(\alpha(X_t))$$

for each closed subset I of B_2 .

Define:

$$A = A_1 \cup A_2, \quad B = [0, t] \setminus A ,$$

where $A_i = [0, t] \setminus B_i$ for $i = 1, 2$. Since L is uniformly continuous on B , for any given $\varepsilon' > 0$ there exists $\eta > 0$ such that $t', t'' \in B$ and $|t' - t''| < \eta$ implies $|\alpha(X_{t'})[L(t') - L(t'')] < \varepsilon'$. For a positive integer $m > t/\eta$, let $t_0 = 0 < t_1 < \dots < t_m = t$ be the partition of the interval $[0, t]$ with $t_j = m^{-1}t + t_{j-1}$ ($j = 1, 2, \dots, m$). Moreover, let $I_j = [t_{j-1}, t_j] \setminus A$ and let s_j be a point in I_j such that $L(s_j) = \sup \{L(s) : s \in I_j\}$.

By the integral mean-value theorem, for $x \in X$ we have

$$\begin{aligned} \int_B f(s, x(s)) \, ds &= \sum_{j=1}^m \int_{I_j} f(s, x(s)) \, ds \in \sum_{j=1}^m \text{mes}(I_j) \overline{\text{conv}} \left(\{f(s, x(s)) : s \in I_j\} \right) \subset \\ &\subset \sum_{j=1}^m \text{mes}(I_j) \overline{\text{conv}} (f[I_j \times X_t]) . \end{aligned}$$

Thus

$$\begin{aligned} \alpha(T[X](t)) &\leq \alpha \left(\int_A f(s, X(s)) \, ds \right) + \alpha \left(\sum_{j=1}^m \text{mes}(I_j) \overline{\text{conv}} (f[I_j \times X_t]) \right) \leq \\ &\leq 2 \cdot \sup \left\{ \left\| \int_A f(s, x(s)) \, ds \right\| : x \in X \right\} + \sum_{j=1}^m \text{mes}(I_j) \alpha(f[I_j \times X_t]) \leq \\ &\leq 2 \cdot \int_A G(s, r_0) \, ds + \sum_{j=1}^m \text{mes}(I_j) L(s_j) h(\alpha(X_t)) < \varepsilon + h(\alpha(X_t)) \int_B L(s) \, ds + \\ &\quad + \sum_{j=1}^m \int_{I_j} h(\alpha(X_t)) |L(s_j) - L(s)| \, ds < \varepsilon + h(\alpha(X_t)) + \varepsilon' t , \end{aligned}$$

and therefore $\alpha(T[X](t)) \leq \varepsilon + h(\alpha(X_t))$. Since

$$\alpha(X_t) = \sup \{ \alpha(X(s)) : 0 \leq s \leq t \} \leq \Phi(X) ,$$

we obtain $\alpha(T[X](t)) \leq \varepsilon + h(\Phi(X))$; as ε is arbitrary, this implies $\alpha(T[X](t)) \leq h(\Phi(X))$. Hence $\Phi(T[X]) \leq h(\Phi(X))$, and consequently T has a fixed point in $\mathfrak{X}$. The proof is complete.

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