

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 72 (1984), p. 231-247

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Amalgams of Torsion-Free Nilpotent Groups of Class Three.

BERTHOLD J. MAIER (*)

SUMMARY - If A, B are groups with a common subgroup D then a group C is called a (strong) amalgam of A with B over D , if $A, B \leq C$ and $D \leq A \cap B$ in C ($D = A \cap B$). A group D is called a (strong) amalgamation base, if there exists a (strong) amalgam of A with B over D for all A and B containing D . In this paper we present necessary and sufficient conditions for the existence of (strong) amalgams in the class of torsion-free nilpotent groups of class three. We also characterize the (strong) amalgamation bases. Finally we determine those groups in this class that are algebraically closed in the sense of Abraham Robinson.

1. Introduction.

In the class of all groups the free product with amalgamated subgroups (cf. [5]) is a strong amalgam. So, strong amalgams exist in this class, and every group is a strong amalgamation base. For most other classes of groups, however, amalgams do not exist in general (cf. [5] A.8). In these cases we have to establish conditions as to when they do exist. A famous example of that kind is Higman's theorem on amalgamation of finite p -groups [4]. As to nilpotent groups of fixed class the class two case was dealt with repeatedly (e.g. in [12], [1], [8], [10]). The conditions that we found in [8] were rather technical, although fairly intelligible. They turned out to be quite simple

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in the case of torsion-free groups ([8], Satz 1). The same holds true in $\mathfrak{N}_3^+$, the class of torsion-free nilpotent groups of class at most three, which we are going to consider in this paper.

THEOREM 3.1. Let $A, B \in \mathfrak{N}_3^+$ and $D \leq A, B$ a common subgroup. There exists an amalgam of A with B over D in $\mathfrak{N}_3^+$, if and only if

$$(*) \quad \begin{aligned} \langle A_3, [B_2 \cap D, A] \rangle \cap D &\leq Z(B), & A_2 \cap D &\leq Z_2(B) & \text{and} \\ \langle B_3, [A_2 \cap D, B] \rangle \cap D &\leq Z(A), & B_2 \cap D &\leq Z_2(A). \end{aligned}$$

THEOREM 3.3. There exists a strong amalgam of A with B over D in $\mathfrak{N}_3^+$ if and only if $(*)$ and $(A^n \cap D) \cap (B^n \cap D) = D^n$, $n \in \mathbb{N}$.

Here $A = A_1 \geq A_2 \geq A_3 \geq 1$ and $1 \leq Z_1(A) \leq Z_2(A) \leq Z_3(A) = A$ denote the lower and the upper central series, respectively, of a group $A \in \mathfrak{N}_3^+$; $Z_1(A) = Z(A)$ is the center of A and $Z_{n+1}(A)/Z_n(A) = Z(A/Z_n(A))$; further $A_{n+1} = [A_n, A]$, where $[X, Y]$, the commutator group of the subgroups X, Y , is the subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$, $x \in X, y \in Y$. Finally A^n denotes the set of all n -th powers of elements of A .

In $(*)$ $B_2 \cap D$ may be viewed as a subgroup of $D \leq A$, hence, $[B_2 \cap D, A]$ makes sense in the group A ; then $\langle A_3, [B_2 \cap D, A] \rangle \cap D$ is considered as a subgroup of $D \leq B$, and the first condition $\langle A_3, [B_2 \cap D, A] \rangle \cap D \leq Z(B)$ makes sense in B . This condition is obviously necessary, as the left hand side is contained in C_3 of any amalgam C and so in $Z(C) \cap B \leq Z(B)$ if $C \in \mathfrak{N}_3^+$. Similar considerations can be made as to the other conditions.

Let us note that $(*)$ is equivalent to the following condition which reminds of Higman's condition in [4].

()** A and B have central series of length 3 which intersect in the same subgroups with D .

If $(*)$ holds, then

$$\begin{aligned} G_1 &= A, & G_2 &= \langle A_2, B_2 \cap D \rangle, \\ G_3 &= \langle [G_2, A], \langle B_3, [A_2 \cap D, B] \rangle \cap D \rangle \end{aligned}$$

and

$$\begin{aligned} H_1 &= B, & H_2 &= \langle B_2, A_2 \cap D \rangle, \\ H_3 &= \langle [H_2, B], \langle A_3, [B_2 \cap D, A] \rangle \cap D \rangle \end{aligned}$$

are two central series of A and B , respectively, such that

$$G_2 \cap D = \langle A_2 \cap D, B_2 \cap D \rangle = H_2 \cap D$$

and

$$G_3 \cap D = \langle \langle A_3, [B_2 \cap D, A] \rangle \cap D, \langle B_3, [A_2 \cap D, B] \rangle \cap D \rangle = H_3 \cap D.$$

Conversely, if $(**)$ holds and $A = G_1 \geq G_2 \geq G_3 \geq 1$, $B = H_1 \geq H_2 \geq H_3 \geq 1$ are central series of A and B with $G_2 \cap D = H_2 \cap D$, $G_3 \cap D = H_3 \cap D$, then $A_2 \cap D \leq G_2 \cap D = H_2 \cap D \leq Z_2(B)$ and similarly $B_2 \cap D \leq Z_2(A)$. This implies $[A_2 \cap D, B] \leq [H_2, B] \leq H_3$ and hence $\langle B_3, [A_2 \cap D, B] \rangle \cap D \leq H_3 \cap D = G_3 \cap D \leq Z(A)$ which is the third part of $(*)$. The last part is proved analogously.

An easy application of these theorems gives the characterization of the amalgamation bases in $\mathfrak{N}_3^+$.

THEOREM 4.1. A group D is an amalgamation base in $\mathfrak{N}_3^+$, if and only if $I(D_2) = Z_2(D)$ and $I(D_3) = Z(D)$.

Here $I(X) = \{x \in D \mid x^n \in X \text{ for some } n \in \mathbb{N}\}$ denotes the isolator of a subgroup X in D (cf. [3], p. 19).

THEOREM 4.2. A group D is a strong amalgamation base in $\mathfrak{N}_3^+$, if and only if $D_2 = Z_2(D)$, $D_3 = Z(D)$, and D is divisible.

Bacsich [2] showed that a structure is a strong amalgamation base in a first-order axiomatizable class, if and only if it is an amalgamation base and algebraically closed in the sense of A. Robinson (cf. § 4 for a definition). Note that this definition of algebraically closed is different from the one that was considered in [7].

As $D_2 = I(D_2)$ and $D_3 = I(D_3)$ for a divisible group $D \in \mathfrak{N}_3^+$, the following may be suspected.

THEOREM 4.3. A group D is algebraically closed in $\mathfrak{N}_3^+$ in the sense of A. Robinson, if and only if D is divisible.

That the condition is necessary follows from two theorems of Malcev's: (1) roots are unique in torsion-free nilpotent groups and (2) a torsion-free nilpotent group can be embedded into a divisible one of the same class; moreover the group and the embedding are unique up to isomorphism. (Cf. [3], Lemma 2.1 and Theorem 2.4.)

The plan of the paper is as follows. In § 2 we provide some embedding lemmas that we shall need in the construction of the amalgams. A finitely generated torsion-free nilpotent group G has a series of subgroups $1 = G_1 < G_2 < \dots < G_m = G$ such that G_{n+1} is the split extension of G_n with an infinite cyclic group $\langle c_n \rangle$. This split extension is uniquely determined by the automorphism that c_n induces on G_n and as well by the commutators of c_n with the elements of G_n . Such extensions of nilpotent groups to yield certain commutator relations have been dealt with in [6] and we will use them as our main tool. If we start with $A \geq D$, then we want to find a series of split extensions $D = A_1 < A_2 < \dots < A_n = A$. This is always possible if A and D are divisible, since then $D \cap Z_{i+1}(A)/D \cap Z_i(A)$ is divisible and isomorphic to a direct factor of $Z_{i+1}(A)/Z_i(A)$, $0 \leq i < 3$, the factors of the upper central series of A . The only difference is now that our split extensions are by groups isomorphic to the additive group $\mathbb{Q}^+$ of the rationals. Therefore, we will start with Malcev's theorem and embed A and B into their divisible hulls and also D . It may happen that an element in D has an n -th root both in A and in B . Because of the uniqueness of roots in torsion-free nilpotent groups, these roots have to be identified in the amalgam and, therefore, we cannot obtain strong amalgams in all cases where amalgams are possible. The idea for the construction of the amalgam is then to form inductively amalgams B_i of A_i with B over D . The induction step can be done by lifting the split extension $A_i < A_{i+1}$ to a split extension B_{i+1} of B_i . The group B_{i+1} can be viewed as an amalgam of A_{i+1} with $B \leq B_i$ over $D = A_{i+1} \cap B$. In § 3 we perform the construction of the amalgam and in § 4 we give the applications as to amalgamation bases and to algebraically closed groups in the sense of Abraham Robinson.

We close this section with some notation. $\mathfrak{N}_n$ is the class of all nilpotent groups of class at most n and $\mathfrak{N}_n^+$ the subclass of torsion-free groups. $\langle X \rangle$ denotes the subgroup generated by a set X and « $\leq$ » denotes inclusion of subgroups. Iterated commutators are used left-normed, e.g. $[x, y, z] = [[x, y], z]$. If G is a nilpotent group of class at most n then $G = G_1 \geq G_2 \geq \dots \geq G_{n+1} = 1$ and $G = Z_n(G) \geq \dots \geq Z_0(G) = 1$ denote the lower and upper central series of G , respectively. We also write Z_k instead of $Z_k(G)$. Recall that $G_{\mu+1} \leq Z_{n-\mu}$, $0 \leq \mu \leq n$ for $G \in \mathfrak{N}_n$. The normal subgroup that is generated by a subgroup $M \leq G$ will be denoted by M^G . We shall write $\mathfrak{C}_G(x)$ for the centralizer of an element x in the group G . By abuse of notation 1 will denote the natural number one, the identity element and the trivial subgroup of any group.

2. Embedding lemmas.

The first lemma states conditions as to which a nilpotent group $G \in \mathfrak{N}_n$ can be embedded in some $H \in \mathfrak{N}_n$ such that a subgroup in the lower half of the upper central series of G is contained in the corresponding term of the lower central series of H .

LEMMA 2.1. Let $M \leq G \in \mathfrak{N}_n$ and $\mu \geq (n-1)/2$. There exists a group $H \geq G$ in $\mathfrak{N}_n$ such that $M \leq H_{\mu+1}$, if and only if M^σ is abelian and $M \leq Z_{n-\mu}(G)$.

PROOF. If $M \leq H_{\mu+1}$ and $H \in \mathfrak{N}_n$, then

$$M \leq H_{\mu+1} \cap G \leq Z_{n-\mu}(H) \cap G \leq Z_{n-\mu}(G)$$

and

$$[M^\sigma, M^\sigma] \leq [H_{\mu+1}, H_{\mu+1}] \leq H_{2\mu+2} \leq H_{n+1} = 1.$$

Conversely, if $M \leq G$, then there exists a group $H \geq G \times \langle c \rangle$, $\langle c \rangle$ infinite cyclic, with elements h_g such that $[h_g, c, \dots, c] = g$, $g \in M$, where c is iterated μ -times in the commutator ([6], Satz 1). The two conditions on M guarantee that H can be chosen in $\mathfrak{N}_n$ ([6], Satz 3.1). $\square$

We need the following two cases for $n = 3$.

COROLLARY 2.2. Let $G \in \mathfrak{N}_3^+$.

1) If $M \leq Z_2(G)$ and M is abelian, then there exists $H \geq G$ in $\mathfrak{N}_3^+$ such that $H_2 \cap G = G_2 M$, $H_3 \cap G = G_3[M, G]$, $Z(H) \cap G = Z(G)$.

2) If $M \leq Z(G) \cap G_2$, then there exists $H \geq G$ in $\mathfrak{N}_3^+$ such that $H_2 \cap G = G_2$, $H_3 \cap G = G_3 M$, $Z(H) \cap G = Z(G)$.

PROOF. 1) As M is abelian $M^\sigma = M[M, G] \leq MZ(G)$ is abelian, too, and by 2.1 there exists $H \geq G$ in $\mathfrak{N}_3$ with $M \leq H_2$. The other assertions follow by checking the central series given in Hilfssatz 8 of [6]. 2) is proved similarly. $\square$

Note that the second embedding in 2.2 can also be performed by a direct product with amalgamated central subgroups. The next two lemmas are again consequences of Satz 3 in [6]. They have been

proved in [7] with the minor difference that linearly independent mod G_2 was stated as mod Z_2 . This, however, was only needed to assure that the factor G/Z_2 is torsion-free, which is true as well for G/G_2 , if G is divisible.

LEMMA 2.3. Let $G \in \mathfrak{N}_3^+$ be divisible; $x_0, \dots, x_n$ linearly independent mod G_2 ; $g_1, \dots, g_k \in G_2 \cap \mathfrak{C}_G(x_0)$; $z_1, \dots, z_n \in G_3$. There exists a group $H \geq G$ in $\mathfrak{N}_3^+$ with an element $h \in H_2 \setminus \mathfrak{C}_H(x_0)$ such that h is linearly independent of $g_1, \dots, g_k$ mod $Z(H)$ and $[h, x_i] = z_i$, $i = 1, \dots, n$.

LEMMA 2.4. Let $G \in \mathfrak{N}_3^+$ be divisible with $G_3 = Z(G)$; $x_1, \dots, x_n$ linearly independent mod G_2 , $g_i \in G_2$, $z_i, z_{jk} \in G_3$, $1 \leq i \leq n + m$, $1 \leq j, k \leq n$; let $X = \langle g_i, [x_j, x_k] \rangle$; $1 \leq i \leq n + m$, $1 \leq j, k \leq n$. The following are equivalent.

- 1) There exists $H \geq G$ in $\mathfrak{N}_3^+$ with an element h such that $[h, x_j] = g_j$, $[h, g_i] = z_i$, $[h, [x_j, x_k]] = z_{jk}$, $1 \leq i \leq n + m$, $1 \leq j, k \leq n$.
- 2) a) $[g_j, x_k] = [g_k, x_j]z_{jk}$, $1 \leq j, k \leq n$,
 b) $g_i \rightarrow z_i$ and $[z_j, x_k] \rightarrow z_{jk}$, $1 \leq i \leq n + m$, $1 \leq j, k \leq n$, induce a homomorphism of $X/X \cap Z$ onto $\langle z_i, z_{jk} \rangle$; $1 \leq i \leq n + m$, $1 \leq j, k \leq n$.

By Corollary 2.2 we can embed $G \in \mathfrak{N}_3^+$ into an $H \in \mathfrak{N}_3^+$ such that some elements in $Z(G)$ or $Z_2(G)$ are contained in H_3 or H_2 , respectively. Lemma 2.4 gives us one of the other possibilities: $G \in \mathfrak{N}_3^+$ can be embedded into an $H \in \mathfrak{N}_3^+$ such that some $x_1 \notin I(G_2)$ is not contained in $Z_2(H)$: For this we may assume that G is divisible and $x_1 \notin G_2$, and further $Z(G) = G_3$ by 2.2. Choose $g_1 \in G_2 \setminus Z(G)$, $z_1 = 1$, then we obtain $H \geq G$ in $\mathfrak{N}_3^+$ and an element $h \in H$ with $[h, x_1] = g_1$; as $g_1 \notin Z(G) \geq Z(H) \cap G$, we have that $x_1 \notin Z_2(H)$. The next lemma will give the other case: embed $G \in \mathfrak{N}_3^+$ into an $H \in \mathfrak{N}_3^+$ such that some $x \notin I(G_3)$ is not contained in $Z(H)$. Recall that $I(X)$ is the isolator of a subgroup X .

One of our conditions for the amalgam reads $A_3 \cap D \leq Z(B)$. In Lemma 2.4 we shall need that $Z(A) = A_3$. To provide this situation without violating the first condition, we shall use an embedding such that some $x \in Z(A) \setminus A_3$ is no longer in the center. We first state a fact that is used in the proof.

FACT 2.5. A divisible group $G \in \mathfrak{N}_2^+$ of finite rank and commutator subgroup G_2 of rank one is isomorphic to the direct product of m copies

of $\mathbf{Q}^+$ with a direct product with amalgamated centers of n copies of $U(3, \mathbf{Q})$. $m \geq 0$ and $n \geq 1$ are invariants for these groups. Here $U(3, \mathbf{Q})$ denotes the group of upper triangular 3×3 matrices with coefficients in $\mathbf{Q}$ and 1's on the main diagonal.

This follows from the classification of the central extensions of abelian groups and of the alternating bilinear forms on finite-dimensional vector spaces. (Cf. [11], Theorems 5.4 and 5.7.)

LEMMA 2.6. Let $G \in \mathfrak{N}_3^+$ be finitely generated and $g \notin I(G_3)$. There exists $H \geq G$ in $\mathfrak{N}_3^+$ such that $g \notin Z(H)$, $H_2 \cap G = G_2$, $H_3 \cap G = G_3$.

The condition $g \notin I(G_3)$ is necessary as $G_3 \leq Z(H)$ and $Z(H)$ is isolated for any $H \geq G$ in $\mathfrak{N}_3^+$ (cf. [3] Theorem 0.4).

PROOF. Taking the divisible hull of G we may assume that $g \notin G_3$. If even $g \notin G_2$ then we can use Lemma 2.4 as mentioned above. Hence, we may confine to the case $g \in G_2 \setminus G_3$ and $G \in \mathfrak{N}_3^+$ divisible of finite rank. As G_2 is divisible abelian, let N be a complement of the direct factor $\langle\langle g \rangle\rangle$ generated by g in G_2 . Because $g \notin G_3$, $N \triangleleft G$ and $G/N \in \mathfrak{N}_2^+$ with commutator subgroup of rank one. Enlarging N we can assume that the invariant m in 2.5 is zero for G/N . So G/N is isomorphic to the direct product of $n \geq 1$ copies of $U(3, \mathbf{Q})$ with centers amalgamated over $\langle\langle gN \rangle\rangle \cong \mathbf{Q}^+$. Also G/N is isomorphic to $\mathbf{Q}^{2n+1}$ with a multiplication

$$\begin{aligned} (r_1, s_1, \dots, r_n, s_n, t)(r'_1, s'_1, \dots, r'_n, s'_n, t') = \\ = (r_1 + r'_1, s_1 + s'_1, \dots, r_n + r'_n, s_n + s'_n, t + t' - \sum r'_i s_i), \end{aligned}$$

where gN is identified with $(0, \dots, 0, 1)$. This group $\mathbf{Q}^{2n+1}$ is embedded into a group on $\mathbf{Q}^{3n+2}$ with multiplication $\circ$

$$\begin{aligned} (r_1, s_1, u_1, \dots, r_n, s_n, u_n, t, v) \circ (r'_1, s'_1, u'_1, \dots, t', v') = \\ = (r_1 + r'_1, s_1 + s'_1, u_1 + u'_1, \dots, t + t' - \sum r'_i s_i, v + v' - \sum r'_i u_i). \end{aligned}$$

The additional components are denoted by $u_1, \dots, u_n, v$. $(\mathbf{Q}^{3n+2}, \circ)$ is still a group in $\mathfrak{N}_2^+$ and gN is identified with $(0, \dots, 0, 1, 0)$. Now we define a split extension of $\mathbf{Q}^{3n+2}$ such that the element corresponding to gN is no longer contained in the center. Our procedure parallels the embedding of each of the $U(3, \mathbf{Q})$ into some $U(4, \mathbf{Q}) \in \mathfrak{N}_3^+$. We define a bijective map φ on $\mathbf{Q}^{3n+2}$ by

$$(r_1, s_1, u_1, \dots, t, v)^\varphi = (r_1, s_1, u_1 + s_1, \dots, t, v + t)$$

and show that it is a homomorphism:

$$\begin{aligned}
 & [(r_1, s_1, u_1, \dots, t, v) \circ (r'_1, s'_1, u'_1, \dots, t', v')]^\varphi = \\
 & = (r_1 + r'_1, s_1 + s'_1, u_1 + u'_1, \dots, t + t' - \sum r'_i s_i, v + v' - \sum r'_i u_i)^\varphi = \\
 & = (r_1 + r'_1, s_1 + s'_1, u_1 + u'_1 + s_1 + s'_1, \dots, t + t' - \sum r'_i s_i, \\
 & \quad v + v' - \sum r'_i u_i + t + t' - \sum r'_i s_i), \\
 & (r_1, s_1, u_1, \dots, t, v)^\varphi \circ (r'_1, s'_1, u'_1, \dots, t', v')^\varphi = \\
 & = (r_1, s_1, u_1 + s_1, \dots, t, v + t) \circ (r'_1, s'_1, u'_1 + s'_1, \dots, t', v' + t') = \\
 & = (r_1 + r'_1, s_1 + s'_1, u_1 + s_1 + u'_1 + s'_1, \dots, t + t' - \sum r'_i s_i, \\
 & \quad v + t + v' + t' - \sum r'_i (u_i + s_i)).
 \end{aligned}$$

Let F denote the split extension of $(\mathbb{Q}^{3n+2}, \circ)$ by $\langle \varphi \rangle$. Then $[gN, \varphi] = [(0, \dots, 0, 1, 0), \varphi] = (0, \dots, 0, 1) \neq 1$. It follows from the definition of φ that $F \in \mathfrak{N}_3^+$. Now let $H = G \times F$ and embed G into H by $x \rightarrow (x, xN)$. Then $[(g, gN), \varphi] \neq 1$ in H , and because of the embedding of G identically onto the first factor of H it also follows that $H_2 \cap G = G_2$ and $H_3 \cap G = G_3$. $\square$

For further applications we state the following corollary.

COROLLARY 2.7. Let $G \in \mathfrak{N}_3^+$, $g \notin I(G_3)$ and $g_1 \in G_3$. There exists a group $H \geq G$ in $\mathfrak{N}_3^+$ with an element $h \in H$ such that $[g, h] = g_1$, $H_2 \cap G = G_2$, and $H_3 \cap G = G_3$.

PROOF. As $\mathfrak{N}_3^+$ is first-order axiomatizable, we may assume by the compactness theorem (cf. [9] Theorem 1.6.2) that G is finitely generated: the conditions $g \notin I(G_3)$ and $g_1 \in G_3$ also hold in all suitably chosen subgroups of G and $H_2 \cap G = G_2$ as well as $H_3 \cap G = G_3$ can be characterized by sets of first-order sentences with constants in G . We now recheck the proof of 2.6. If $g \notin I(G_2)$, then $g_1 \in G_3$ could be chosen arbitrarily. In the other case we had $[(g, gN), \varphi] = (1, f)$ with $1 \neq f = (0, \dots, 0, 1) \in F$, and $(1, f) \in Z(H)$. As $\langle (1, f) \rangle \cap G = 1$ and $g_1 \in G_3 \in Z(H)$, the subgroup $X = \langle (g_1, g_1 N)^{-1} (1, f) \rangle$ is central in H with $G \cap X = 1$. Thus, G is embedded into H/X and $[(g, gN), \varphi] = (1, f) = (g_1, g_1 N)$ in $H/X \in \mathfrak{N}_3^+$. $\square$

Finally we prove the result described above.

LEMMA 2.8. Let $G \in \mathfrak{N}_3^+$ be divisible. There exists a group $H \geq G$ in $\mathfrak{N}_3^+$ such that $Z(H) = H_3$, $H_3 \cap G = G_3$, $H_2 \cap G = G_2$.

PROOF. By the compactness theorem and the method of diagrams of model theory, it again suffices to prove the theorem for a G of finite rank. We first construct an $F \in \mathfrak{N}_3^+$ such that $Z(F) \cap G = G_3$ and $F_2 \cap G = G_2$. Like G , the center $Z(G)$ has finite rank. Let $z_1, \dots, z_n$ be representatives of the direct factors in a complement of G_3 in $Z(G)$. For each $z_i \notin G_3$ there exists $F^i \geq G$ in $\mathfrak{N}_3^+$ by 2.6 such that $z_i \notin Z(F^i)$, $F^i_2 \cap G = G_2$. If we set $F = F^1 \times \dots \times F^n$ and embed G diagonally, then F meets the two conditions above as $z_1, \dots, z_n \notin Z(F)$ and $Z(F)$ is isolated. Finally we choose as in 2.2 a group $H \geq F$ in $\mathfrak{N}_3^+$ such that $Z(H) = H_3 = Z(F)$, $H_2 \cap F = F_2$: This H satisfies all assertions in the lemma. $\square$

3. The amalgamation theorem.

THEOREM 3.1. Let D be a common subgroup of $A, B \in \mathfrak{N}_3^+$. There exists an amalgam of A with B over D in $\mathfrak{N}_3^+$, if and only if

$$\langle A_3, [B_2 \cap D, A] \rangle \cap D \leq Z(B), \quad A_2 \cap D \leq Z_2(B)$$

(*) and

$$\langle B_3, [A_2 \cap D, B] \rangle \cap D \leq Z(A), \quad B_2 \cap D \leq Z_2(A).$$

Let us note that the theorem is false, if the first condition is weakened to $A_3 \cap D \leq Z(B)$, as can be seen by examples.

PROOF. To show that the conditions are necessary, assume that $C \in \mathfrak{N}_3^+$ is an amalgam of A with B over D . Then $\langle A_3, [B_2 \cap D, A] \rangle \cap D \leq C_3 \cap D \leq Z(C) \cap D \leq Z(B)$ and $A_2 \cap D \leq C_2 \cap D \leq Z_2(C) \cap D \leq Z_2(B)$. The other two conditions are shown in the same way.

Now, let us prove that the conditions are sufficient.

STEP 1. It suffices to prove the theorem for finitely generated groups. As in the proof of the necessity of the conditions we see that (*) also holds in all subgroups of A and B . The existence of the amalgam of A with B over D will follow from the existence of the amalgam of all finitely generated subgroups by the compactness theorem

of model theory and the method of diagrams. A purely algebraic proof of this step could run as follows: We proved in [8] Lemma 5 that the restriction to finitely generated subgroups is justified at least in the case of varieties of groups, *e.g.* in the class $\mathfrak{N}_3$. Thus, the existence of amalgams in $\mathfrak{N}_3^+ \subset \mathfrak{N}_3$ for all finitely generated subgroups of A and B will yield an amalgam of A with B over D in $\mathfrak{N}_3$. The factor group of such an amalgam by its torsion subgroup is an amalgam in $\mathfrak{N}_3^+$, as the torsion subgroup meets A , B , and D trivially.

STEP 2. There exist groups $A' \geq A$, $B' \geq B$ of finite rank in $\mathfrak{N}_3^+$ such that

$$+ (A', B', D): A'_3 = Z(A'), B'_3 = Z(B'); A'_i \cap D = B'_i \cap D, i = 2, 3.$$

If we denote the condition $(*)$ in the theorem by $*(A, B, D)$, then it is easy to see that $+(A', B', D)$ implies $*(A', B', D)$. Actually the new condition is much stronger, as it says that the common subgroup D intersects the lower central series of A and B in exactly the same manner. This will allow us to join $A \setminus D$ to B by finitely many split extensions in the subsequent steps.

We first deal with the case $i = 2$ of the last two conditions. Note that $A_2 \cap D \leq Z_2(B)$ and $A_2 \cap D$ is abelian, as it is a subgroup of $A_2 \leq A \in \mathfrak{N}_3^+$ and $[A_2, A_2] \leq A_4 = 1$. By 2.2 we obtain $B^* \geq B$ in $\mathfrak{N}_3^+$ such that $B_2^* \cap B = B_2 \cdot A_2 \cap D$, $B_3^* \cap B = B_3[A_2 \cap D, B]$, $Z(B^*) \cap B = Z(B)$. Let us see that $*(A, B^*, D)$ holds:

$$A_2 \cap D \leq B_2^* \cap D \leq Z_2(B^*)$$

$$\begin{aligned} \langle A_3, [B_2^* \cap D, A] \rangle \cap D &= (A_3[(B_2 \cdot A_2 \cap D) \cap D, A]) \cap D \\ &= (A_3[B_2 \cap D \cdot A_2 \cap D, A]) \cap D \\ &= (A_3[B_2 \cap D, A]) \cap D \leq Z(B) \leq Z(B^*) \end{aligned}$$

$$B_2^* \cap D = (B_2 \cdot A_2 \cap D) \cap D = B_2 \cap D \cdot A_2 \cap D \leq Z_2(A)$$

$$\langle B_3^*[A_2 \cap D, B^*] \rangle \cap D = B_3^* \cap D = (B_3[A_2 \cap D, B]) \cap D \leq Z(A).$$

As subgroups of finitely generated nilpotent groups are again finitely generated, we may assume that B^* is finitely generated, since we only

have to write finitely many elements of $A_2 \cap D$ as commutators. Thus we arrived at $B^* \geq B$ in $\mathfrak{N}_3^+$ with $A_2 \cap D \leq B_2^*$ and $*(A, B^*, D)$. In the same way we now obtain $A^* \geq A$ in $\mathfrak{N}_3^+$ with $B_2^* \cap D \leq A_2^*$ and $*(A^*, B^*, D)$, but then $A_2^* \cap D = B_2^* \cap D$. Observe that the conditions concerning the centers now read $A_3^* \cap D \leq Z(B^*)$ and $B_3^* \cap D \leq Z(A^*)$, and these two subgroups are both contained in A_2^* and in B_2^* .

Next we deal with the case $i = 3$. As $A_3^* \cap D \leq Z(B^*)$ we obtain $B' \geq B^*$ in $\mathfrak{N}_3^+$ by 2.2 such that $B'_2 \cap B^* = B_2^*$, $B'_3 \cap B^* = B_3^* \cdot A_3^* \cap D$. Let us check $*(A^*, B', D)$:

$$A_2^* \cap D = B_2^* \cap D = B'_2 \cap D, \quad A_3^* \cap D \leq B'_3$$

and

$$B'_3 \cap D = (B_3^* \cdot A_3^* \cap D) \cap D = B_3^* \cap D \cdot A_3^* \cap D \leq Z(A^*).$$

Again B' can be assumed to be finitely generated. Finally we obtain $A' \geq A^*$ in $\mathfrak{N}_3^+$ with $*(A', B', D)$ and $A'_i \cap D = B'_i \cap D$, $i = 2, 3$.

STEP 3. If we replace A' and B' by their divisible hulls then $*(A', B', D)$ remains true. Next we also replace D by its divisible hull D' . This group is uniquely determined by D and thus A' and B' contain the same divisible hull D' of D . Note that this replacement of D by its divisible hulls in A' and B' respectively is the only point in the proof where elements of $A \setminus D$ and $B \setminus D$ may be identified so that the amalgam becomes a non-strong one. Still $*(A', B', D')$ holds as well as $A'_i \cap D' = B'_i \cap D'$, $i = 2, 3$, as the lower and upper central series of the divisible hulls are just the isolators of the respective series in the original groups. A' and B' being divisible of finite rank we may now apply Lemma 2.8 and assume that $+(A', B', D')$ holds. The algebraist may enjoy that this reference does not rely on the model theoretic part of the proof of 2.8.

STEP 4. We may now suppose that A, B, D are divisible of finite rank such that $+(A, B, D)$ holds. Then $A/A_2, A_2/A_3$, and A_3 are divisible torsion-free abelian groups of finite rank and $A_i \cap D/A_{i+1} \cap D \cong (A_i \cap D)A_{i+1}/A_{i+1}$, $i = 1, 2, 3$ are direct factors of these three groups, being itself divisible. We choose representatives of bases in the complements of these direct factors. We proceed in the sequence $i = 3, 2, 1$ and call the occurring elements in A $a_1, \dots, a_n$. If we

let D^i be the divisible hull of $\langle D, a_1, \dots, a_i \rangle$ in A then $D^0 = D$, $D^n = A$ and $[D^i, a_{i+1}] \leq D^i$, $i = 0, \dots, n-1$, by the choice of the sequence $a_1, \dots, a_n$. We now construct inductively amalgams B^i of D^i with B over D in $\mathfrak{R}_3^+$. In the induction step we form an amalgam of $\langle D^i, a_{i+1} \rangle$ with B^i over D^i . The divisible hull of such an amalgam is an amalgam of D^{i+1} with B^i over D^i , but it may as well be viewed as an amalgam of D^{i+1} with $B \leq B^i$ over $D = D^i \cap B$. Observe that the amalgams constructed in these steps are all strong ones, indeed B^{i+1} is a split extension of B^i by some $\langle a_{i+1} \rangle$ such that the subgroup $\langle D^i, a_{i+1} \rangle$ is isomorphic to the split extension of D^i by a_{i+1} in A . In fact, we are lifting split extensions here. To perform this inductively, we shall assume that $+(A, B^i, D^i)$ holds for all amalgams B^i . Thus $B^0 = B$ and $D^0 = D$ is the starting point that we have provided above. Dropping the i 's we start with $+(A, B, D)$ and $a \in A$ such that $[D, a] \leq D$ and construct an amalgam C of B with $\langle D, a \rangle$ over D , such that $+(A, C, \langle D, a \rangle)$ holds true. According to the sequence as the a 's occur we distinguish the three cases $a \in A_3$, $a \in A_2 \setminus A_3$, and $a \in A \setminus A_2$.

STEP 5. Case $a \in A_3$. As $A_3 \cap D$ is divisible and $a \notin D$, it holds that $\langle A_3 \cap D, a \rangle \cong A_3 \cap D \times \langle a \rangle$, and $\langle D, a \rangle \cong D \times \langle a \rangle$, too. Let us choose $X \in \mathfrak{R}_3^+$ with $Z(X) = X_3$ and some $1 \neq x \in X_3$. If we set $C = B \times X$, then $x \in C_3$ and $\langle D, x \rangle \cong D \times \langle x \rangle \cong D \times \langle a \rangle$, where the left hand side is read in C and the right hand side in A . Therefore, we may identify a and x . Finally we show $+(A, C, \langle D, a \rangle)$: $Z(A) = A_3$ by hypothesis. $C_3 = B_3 \times X_3 = Z(B) \times Z(X) = Z(C)$.

$$A_i \cap \langle D, a \rangle = \langle A_i \cap D, a \rangle = \langle B_i \cap D, a \rangle = C_i \cap \langle D, a \rangle, \quad i = 2, 3.$$

STEP 6. Case $a \in A_2 \setminus A_3$. Since A_2 is abelian and $A_2 \cap D$ is divisible, we have $\langle A_2 \cap D, a \rangle \cong A_2 \cap D \times \langle a \rangle$. Further, $\langle D, a \rangle$ is a split extension of D by $\langle a \rangle$, as $[D, a] \leq D$ and $D \cap \langle a \rangle = 1$. The group D being divisible of finite rank, we can choose representatives $d_1, \dots, d_k$ in D of a base of $D/A_2 \cap D \cong DA_2/A_2$. The operation of a on D is determined by its operation on the d_i , and, as well, by the commutators $[a, d_i]$, $i = 1, \dots, k$. If we find $C \geq B$ with an element $x \in C_2 \setminus Z(C)$ such that $[x, d_i] = [a, d_i] \in A_3 \cap D = B_3 \cap D$ and x linearly independent of $D \cap B_2 \bmod Z(C)$, then we have $\langle D, x \rangle \cong \langle D, a \rangle$. Such a group exists by lemma 2.3 over $B \times \langle x_0 \rangle$ with $\langle x_0 \rangle$ infinite cyclic. Applying lemma 2.8 we may also assume that $C_3 = Z(C)$; observe that x remains non-central under this enlargement. If we identify x

in C with a in A , we can check the remaining conditions of $+(A, C, \langle D, a \rangle)$:

$$A_2 \cap \langle D, a \rangle = A_2 \cap D \times \langle a \rangle = B_2 \cap D \times \langle a \rangle = C_2 \cap \langle D, a \rangle$$

$$A_3 \cap \langle D, a \rangle = A_3 \cap D = B_3 \cap D = C_3 \cap D = C_3 \cap \langle D, a \rangle.$$

STEP 7. Case $a \in A \setminus A_2$. As $[D, a] \leq D$ and $D \cap \langle a \rangle = 1$, the group $\langle D, a \rangle$ is again a split extension of D by $\langle a \rangle$ and determined by commutators $[a, d_i]$, $[a, d'_j]$, $i = 1, \dots, k$, $j = 1, \dots, k'$, where the d_i , d'_j are representatives of bases of $D/A_2 \cap D \cong DA_2/A_2$ and $A_2 \cap D$, respectively. As $A_2 \cap D = B_2 \cap D$ the d_i and d'_j are representatives of bases of $D/B_2 \cap D \cong DB_2/B_2$, and $B_2 \cap D$, too. The conditions in Lemma 2.4.2 for the existence of a group $C \geq B$ with an element x such that $[x, d_i] = [a, d_i]$ and $[x, d'_j] = [a, d'_j]$, $i = 1, \dots, k$, $j = 1, \dots, k'$, are formulated only in the subgroup $D \leq B$. They hold again by 2.4 as the element a realizes such an extension in A . Once more we use 2.8 to obtain $C_3 = Z(C)$. Identifying x and a we have $\langle D, x \rangle \cong \cong \langle D, a \rangle$ as the two elements induce the same automorphism on D . Finally also $+(A, C, \langle D, a \rangle)$ holds, as

$$A_i \cap \langle D, a \rangle = A_i \cap D = B_i \cap D = C_i \cap D = C_i \cap \langle D, a \rangle, \quad i = 2, 3. \quad \square$$

The reason, why we cannot obtain a strong amalgam, although there exists an amalgam, is the uniqueness of roots in torsion-free nilpotent groups. So, if there exist $a \in A \setminus D$ and $b \in B \setminus D$ such that $a^n = b^n \in D$, then a and b have to be identified in any amalgam. In our construction this was done implicitly in step 3, when we took the divisible hull D' of D . Elements a and b as above lie in the divisible hull of D both in A' and in B' . They are identified as the following steps performed a strong amalgam of A' with B' over D' . Therefore we have also proved the following.

PROPOSITION 3.2. Let $D \leq A$, $B \in \mathfrak{N}_3^+$. If there exists an amalgam of A with B over D in $\mathfrak{N}_3^+$, then there exists such an amalgam with $A \cap B = (A \cap D') \cap (B \cap D')$. Here D' denotes the divisible hull of D .

THEOREM 3.3. Let $D \leq A$, $B \in \mathfrak{N}_3^+$. There exists a strong amalgam of A with B over D if and only if $(*)$ in 3.1 and $(A^n \cap D) \cap (B^n \cap D) = D^n$, $n \in \mathbb{N}$.

PROOF. As to necessity we have to prove $(A^n \cap D) \cap (B^n \cap D) \leq D^n$, the converse being trivial. So, let C be a strong amalgam of A with B over D in $\mathfrak{N}_3^+$ and $a^n = b^n \in D$ for some $a \in A$, $b \in B$; then $a = b$ by the uniqueness of roots. Thus $a = b \in A \cap B = D$ in the strong amalgam C and $a^n \in D^n$. As to sufficiency, the last condition is equivalent to $(A \cap D') \cap (B \cap D') = D$ and the existence of a strong amalgam follows from 3.1 and 3.2. $\square$

COROLLARY 3.4. Let $D \leq A \in \mathfrak{N}_3^+$ and B be an isomorphic copy of A over D . Then there exists an amalgam of A with B over D in $\mathfrak{N}_3^+$. This amalgam may be chosen as a strong one, if and only if D is isolated in A .

PROOF. 3.1 and 3.3: $(*)$ holds as $A_i \cap D = B_i \cap D$, $i = 2, 3$ in this special case. $\square$

4. Amalgamation bases.

In this section all isolators are taken with respect to the group D .

THEOREM 4.1. A torsion-free group D is an amalgamation base in $\mathfrak{N}_3^+$ if and only if $I(D_i) = Z_{4-i}(D)$, $i = 2, 3$, i.e. the isolators of the lower central series of D coincide with the upper central series.

Note that $I(D_3) = Z(D)$ will imply $D \in \mathfrak{N}_3^+$.

PROOF. To prove the sufficiency of the conditions, let $D \leq A$, $B \in \mathfrak{N}_3^+$. We have to show that $(*)$ in 3.1 holds. $I(D_2) \leq I(A_2 \cap D) \leq \leq Z_2(A) \cap D \leq Z_2(D)$, as the second center $Z_2(A)$ is isolated. As $I(D_2) = Z_2(D)$, by hypothesis, we conclude $A_2 \cap D \leq I(A_2 \cap D) = = I(D_2) \leq I(B_2) \leq Z_2(B)$. Similarly $I(B_2 \cap D) = I(D_2)$ and $B_2 \cap D \leq \leq Z_2(A)$. Therefore, $[B_2 \cap D, A] \leq [I(D_2), A] \leq I(A_3)$ and

$$I(D_3) \leq I(\langle A_3[B_2 \cap D, A] \rangle \cap D) \leq I(I(A_3) \cap D) \leq Z(A) \cap D \leq Z(D).$$

By hypothesis, $I(D_3) = Z(D)$ and so $\langle A_3[B_2 \cap D, A] \rangle \cap D \leq I(D_3) \leq \leq Z(B)$. The last condition follows analogously.

We now show that both conditions are necessary. For both cases we are going to construct groups $A, B \geq D$ such that $(*)$ in 3.1 is violated and thus no amalgam does exist. First let $I(D_2) \neq Z_2(D)$ and let $g \in Z_2(D) \setminus I(D_2)$. Now let $X = U(4, \mathbb{Z})$ and $x \notin Z_2(X)$, $y \in X_2 \setminus X_3$.

Let A be an amalgam of D with X over $\langle g \rangle \cong \langle x \rangle$ and B an amalgam of D with X over $\langle g \rangle \cong \langle y \rangle$. Both amalgams exist by 3.1 as $D_2 \cap \langle g \rangle = 1$, $X_2 \cap \langle x \rangle = 1$, and $X_2 \cap \langle y \rangle = \langle y \rangle \cong \langle g \rangle \leq Z_2(D)$, $X_3 \cap \langle y \rangle = 1$. In A holds $g = x \notin Z_2(A)$, whereas $g = y \in B_2$ in B . As $g \in D$ we have $B_2 \cap D \not\leq Z_2(A)$ and $(*)$ does not hold for the groups A , $B \geq D$ in $\mathfrak{N}_3^+$. The same procedure, amalgamating an element $g \in Z(D) \setminus I(D_3)$ with an element in $X_2 \setminus Z(X)$ and with an element in X_3 , gives rise to groups $A, B \geq D$ violating $(*)$. Thus, both conditions are necessary. $\square$

THEOREM 4.2 A torsion-free group D is a strong amalgamation base in $\mathfrak{N}_3^+$ if and only if D is divisible and $D_i = Z_{4-i}(D)$, $i = 2, 3$.

PROOF. The conditions are sufficient by 4.1 and 3.2, as a divisible group is its own divisible hull. Conversely, the divisibility of D is necessary because of the uniqueness of roots, and the other conditions by 4.1, since $D_i = I(D_i)$, $i = 2, 3$ for a divisible group D in $\mathfrak{N}_3^+$. $\square$

Bacsich [2] showed that an amalgamation base in a first-order definable class is a strong amalgamation base, if and only if it is algebraically closed in the sense of Abraham Robinson ([9], p. 157). In this section we will show that the additional condition «divisible» in 4.2 is equivalent to algebraically closed in $\mathfrak{N}_3^+$. Note, however, that this notion of algebraically closed is different from the one that was dealt with in [7]. We give a definition of Robinson's notion in the present context: Let $G \in \mathfrak{N}_3^+$, $a_1, \dots, a_m \in G$ and $\theta(x_1, \dots, x_{n+m})$ an existential formula with free variables $x_1, \dots, x_{n+m}$ in the language $\{1, \cdot, -^1\}$ of group theory. θ is called algebraic over G if there exists a $k \geq 1$, such that in every $H \geq G$ in $\mathfrak{N}_3^+$ there exist at most k n -tuples $h_1, \dots, h_n$, such that $\theta(h_1, \dots, h_n, a_1, \dots, a_m)$ holds in H ; the least possible bound $k \geq 1$ is called the degree of θ . Then G is algebraically closed, if every algebraic formula over G has a solution in G .

THEOREM 4.3. A group G is algebraically closed in $\mathfrak{N}_3^+$ in the sense of Abraham Robinson if and only if G is divisible.

PROOF. If G is not divisible and $g \in G$ has no p -th root for some prime p , then $x^p = g$ is an algebraic formula of degree one, for g has a p -th root in the divisible hull, and at most one in any extension of G by the uniqueness of roots. Thus the condition is necessary. Conversely, assume $G \in \mathfrak{N}_3^+$ is divisible and $\theta(\bar{x}, \bar{a})$ is an algebraic formula of degree k over G that has no solution in G itself. Then there exists $H \geq G$ in $\mathfrak{N}_3^+$

with an n -tuple $\bar{h}$ such that $\theta(\bar{h}, \bar{a})$ holds in H . G is isolated in H . If $H_{0i} \geq G_{0i}$, $i = 0 \dots 2^k - 1$, are isomorphic copies of $H \geq G$ then there exist strong amalgams H_{1i} of $H_{0,2i}$ with $H_{0,2i+1}$ over $G_{0,2i} \cong G_{0,2i+1}$, $i = 0 \dots 2^{k-1} - 1$. In the same way we may form strong amalgams H_{2i} of $H_{1,2i}$ with $H_{1,2i+1}$ over $G_{1,2i} \cong G_{1,2i+1}$ and so on. We end up with an amalgam $H_{k,0}$ which may be viewed as a strong amalgam of all H_{0i} , $i = 0 \dots 2^k - 1$ over the G_{0i} . In H_{0i} the formula $\theta(\bar{h}_i, \bar{a})$ holds and, as θ is existential, it also holds in $H_{k,0}$, $i = 0 \dots 2^k - 1$. As all the $\bar{h}_i$ are different, this yields a contradiction to the fact that $1 \leq k < 2^k$ is the degree of $\theta(\bar{x}, \bar{a})$ over G . Therefore, all solutions to an algebraic formula have to lie inside the divisible group G . $\square$

It may be worth noting that analogous results as 4.1-4.3 hold true in the case of $\mathfrak{N}_2^+$. This follows easily from Satz 1 in [8] which gives conditions for the existence of amalgams in $\mathfrak{N}_2^+$ that are more simple than those in 3.1 here. In the class $\mathfrak{N}_1^+$ of torsion-free abelian groups the analogous statements are fairly obvious. Let us close with the remark that the classes of (strong) amalgamation bases are not first-order axiomatizable, neither in $\mathfrak{N}_3^+$ nor in $\mathfrak{N}_2^+$. This is in contrast to $\mathfrak{N}_1^+$, where they both are axiomatizable.

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Manoscritto pervenuto in redazione il 5 Agosto 1984.