RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FABIO ZANOLIN

On forced periodic oscillations in dissipative Liénard systems

Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 51-62

<http://www.numdam.org/item?id=RSMUP_1983__69__51_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1983, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

On Forced Periodic Oscillations in Dissipative Liénard Systems.

FABIO ZANOLIN (*)

1. Introduction.

In this paper we study the problem of the existence of periodic solutions for second order differential systems of the Liénard type:

$$(L) \qquad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t) \qquad (' = d/dt)$$

where $h(t)$ is a periodic forcing term and φ and g are gradient functions. Our main result considers the case when the amplitude of the dissipative term φ, in the direction of the restoring force g, overcomes the L^1-norm of h.

Moreover, we suppose the field g satisfies a suitable geometric condition (see (k) below) which generalizes the usual assumptions required in the literature. Then, as a corollary, we obtain an extension to the systems of some classical and recent results like the theorems of Lefschetz [5], Reuter [14], Reissig [12], Mawhin [8], Bebernes-Martelli [1], Ward [17].

Our main tool in the proof is the following theorem by J. Mawhin [9], [10], recalled here in a simpler but less general form (adapted for second order systems) for the reader's convenience.

(*) Indirizzo dell'A.: Istituto Matematico, Università, Piazzale Europa 1, 34100 Trieste.

Work announced at the Conference on Differential Equations and Applications, Retzhof (Graz), 1-6 June 1981.

LEMMA 1 ([10, Theorem 1]). *Let* $f: \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^n \to \mathbf{R}^n$ *be continuous and* p*-periodic in* t.

(α) *Assume that for all possible* p*-periodic solutions* $x(t)$ *of class* C^2 *of*

$$(D_\lambda) \qquad x''(t) = \lambda f(t, x(t), x'(t)), \qquad \lambda \in (0, 1],$$

there exist two constants K_0 *and* K_1, *independent of* λ *and* $x(t)$ *such that*

$$\sup_{[0,p]} |x(t)| < K_0, \qquad \sup_{[0,p]} |x'(t)| < K_1.$$

(β) $\deg_B(S, \Omega, 0) \neq 0$ *where*

$$S: z \mapsto \frac{1}{p}\int_0^p f(t, z, 0)\, dt, \qquad z \in \mathbf{R}^n$$

and $\Omega = B(0, r_0)$ *for* r_0 *sufficiently great. Then the differential system* (D_1) $(\lambda = 1)$ *has at least one* p*-periodic solution of class* C^2.

2. Notations.

In what follows, $|\cdot| = (\cdot|\cdot)^{\frac{1}{2}}$ is the euclidean norm in $\mathbf{R}^n$ induced by the inner product $(\cdot|\cdot)$. If $u, v: \mathbf{R} \to \mathbf{R}^n$ are continuous and p-periodic functions $(p > 0)$, we define $|u|_\infty = \sup_{[0,p]} |u(t)|$, $(u, v)_2 = \int_0^p (u(s)|v(s))ds$ (the L^2-scalar product of u and v), $|v|_q = \left(\int_0^p |v(s)|^q\, ds\right)^{1/q}$, $q \geqslant 1$ (the L^q-norm of v). Moreover, $Qu = (1/p)\int_0^p u(s)\, ds$ is the mean value of $u(t)$ is a period. If $x \in \mathbf{R}^n$, $r > 0$, then $B(x, r)$, $\overline{B(x, r)}$ are the open and the closed ball of center x and radius r; the sphere S^{n-1} is the boundary of $B(0, 1)$. Finally we recall from Krasnosel'skii [4] the following definition: let $V: \mathbf{R}^n \to \mathbf{R}$ be a C^1-function; we say that V is nondegenerate with nonzero index, shortly $\gamma(V, \infty) \neq 0$, if there exists $r_0 > 0$ such that: grad $V(x) \neq 0$ for every $|x| \geqslant r_0 > 0$ and $\deg_B(\text{grad } V, B(0, r), 0) \neq 0$ for every $r \geqslant r_0$, where $\deg_B(\cdot, \cdot, \cdot)$ is the Brouwer degree (see [15], [6]).

3. The main result and its consequences.

Let us consider the differential system

$$(L) \qquad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t)$$

where it is assumed once for all:

(i) $\varphi = \operatorname{grad} F$, with $F: \mathbb{R}^n \to \mathbb{R}$ a C^2-function;

(ii) $g = \operatorname{grad} G$, with $G: \mathbb{R}^n \to \mathbb{R}$ a C^1-function

(ii) $h: \mathbb{R} \to \mathbb{R}^n$ continuous and p-periodic ($p > 0$).

For any $z \in S^{n-1}$, we consider $V_z = \{w \in \mathbb{R}^n: (g(w)|z) = 0\}$. Let $M > 0$ be a real number and define

$$W(M) = \{x \in \mathbb{R}^n: \overline{B(x, M)} \cap V_z \neq \emptyset, \text{ for every } z \in S^{n-1}\}\,.$$

Remark that $x \in W(M)$ if and only if, for every $z \in S^{n-1}$ there exist $|y_z| \leqslant M$ such that $(g(x + y_z)|z) = 0$.

Then the following result holds true.

THEOREM 1. *Let us assume* (i), (ii) *and* (iii). *Let* $a > 0$, $b \geqslant 0$, *exist such that*

$$(j) \qquad (\varphi(x)|g(x)) \geqslant a|g(x)| - b \qquad (x \in \mathbb{R}^n)$$

or

$$(j') \qquad (\varphi(x)|g(x)) \leqslant -a|g(x)| + b \qquad (x \in \mathbb{R}^n)$$

holds. Finally, let G *be nondegenerate with nonzero index and let us assume*

(*k*) *For any* $M > 0$, *either* $W(M)$ *is bounded, or*

$$\lim_{\substack{|x| \to +\infty \\ x \in W(M)}} |g(x)| = +\infty\,.$$

Then equation (*L*) *has a* p*-periodic solution, for any forcing term* h,

such that

$$Qh = 0 \quad \text{and} \quad \sup_t \left| \int_0^t h(s)\, ds \right| < a\,.$$

REMARK 1. It is easy to see that (k) is satisfied if

$$(s) \qquad \begin{aligned} &x_i g_i(x) > 0 \quad \text{if } |x_i| \geqslant R \text{ for } i = 1, \ldots, k \\ &x_i g_i(x) < 0 \quad \text{if } |x_i| \geqslant R \text{ for } i = k+1, \ldots, n \end{aligned}$$

holds, where $x = (x_1, \ldots, x_n)$, $g = (g_1, \ldots, g_n)$.

In fact, (s) implies that, for any M, $W(M)$ is bounded: it is sufficient to observe that the set $\{x: \overline{B(x, M)} \cap V_{e_i} \neq \emptyset,\ i = 1, \ldots, n\}$, with $\{e_1, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$, is bounded by $(R + M)n^{\frac{1}{2}}$. Moreover, from known results on topological degree, (s) also implies that $\gamma(G, \infty) = \deg_B(g, B(0, r), 0) \neq 0$ for $r > Rn^{\frac{1}{2}}$ $(g = \operatorname{grad} G)$ and hence G is nondegenerate with nonzero index.

Condition (s) was used in [11], [7], [19] in order to get periodic solutions in Liénard equations.

Let there exist $N: \mathbb{R}^n \to \mathbb{R}^n$ continuous and such that $\deg_B(N, B(0, r), 0) \neq 0$, for $r \geqslant r_0 > 0$. If

(w) *for any* $M > 0$, *there exists* $r_M > 0$ *such that* $(g(x + y)|N(x)) > 0$, *for every* $|x| \geqslant r_M$, $|y| \leqslant M$,

holds, then it is not difficult to see that $\gamma(G, \infty) \neq 0$ and (k) is satisfied. In fact, through a homothopy, $\gamma(G, \infty) = \deg_B(N, B(0, r), 0) \neq 0$ (for r sufficiently large) and $W(M)$ is bounded because it is contained in the ball $\overline{B(0, r_M)}$. The case $N(x) = x$ (or $= -x$) has been considered in [18] for Rayleigh equations.

Another simple condition on g which ensures (k) is

$$(m) \qquad \lim_{|x| \to +\infty} |g(x)| = +\infty\,.$$

Observe that if g is a homeomorphism, then (m) is trivially satisfied (g is proper) and G is also nondegenerate with index different from zero (see [6, Th. 3.3.3]).

Moreover, recall (Hadamard Theorem [3]) that (m), together with

(m') G *of class* C^2, $\det(\operatorname{Hess} G(x)) \neq 0$ *for all* $x \in \mathbb{R}^n$

implies that $g = \operatorname{grad} G$ is a homeomorphism (diffeomorphism). Finally, recall from Krasnosel'skii [4, Lemma 6.5] that (m) together with

$$(m'') \qquad \lim_{|x| \to +\infty} G(x) = +\infty \quad (-\infty),$$

implies that G is nondegenerate with nonzero index.

Actually, the degree condition on g holds true if G satisfies (m'') and has a finite number of critical points [2, Th. 2].

The following simple example shows how condition (k) can be easily proved (without make use of (s), (m), (w)).

EXAMPLE. Let $n = 2$, $g(x) = (g_1, g_2)(x) = (x_2 - q(x_1), x_1 - x_2)$, $x = (x_1, x_2)$; where q is a continuously differentiable function such that

$$(e) \qquad q'(0) \neq 1;$$

$$(ee) \qquad q(s) = s \quad \textit{iff } s = 0.$$

Under (e) *and* (ee), $g = \operatorname{grad} G$ *with* G *nondegenerate with nonzero index and* (k) *holds.*

PROOF. Obviously, g admits a potential G, moreover, by (ee), $g(x) = 0$ only for $x = 0$. Evaluating the jacobian of g at 0, we see that $\det g'(0) = q'(0) - 1$ and therefore (e) implies $\deg(g, B(0, r), 0) \neq 0$ for every $r > 0$ (see [6, p. 3]).

Take $z_1 = (1, 1)/\sqrt{2} \in S^1$. Then $(g(x)|z_1) = 0$ iff $x_1 = q(x_1)$, iff $x_1 = 0$ (for (ee)). Hence $V_{z_1} = \{(0, x_2): x_2 \in \mathbb{R}\}$ (x_2 — axis).

Take $z_2 = e_2 = (0, 1)$. Then $(g(x)|z_2) = 0$ iff $x_1 = x_2$ and $V_{z_2} = \{(t, t): t \in \mathbb{R}\}$ ($x_2 = x_1$ line).

Since $w \in W(M)$ implies $\overline{B(w, M)} \cap V_{z_i} \neq \emptyset$ for $i = 1, 2$, one easily realizes that, for any M, $W(M)$ is bounded.

Observe that, in this example, (s) fails and (m) holds only if

$$\lim_{|s| \to +\infty} |q(s) - s| = +\infty$$

(which is not assumed here). Condition (w) is not so easily reached.

From Theorem 1 we have, as an immediate consequence:

COROLLARY 1. *Let* (i), (ii), (iii) *hold and let* (s), *either* (w), *either* (m) *and* (m'), *or* (m) *and* (m'') *be satisfied. Finally, let us assume* $Qh = 0$

and

$$(\circ) \qquad \liminf_{|x|\to+\infty} (\varphi(x)|g(x))/|g(x)| \geqslant K > \sup_t \left| \int_0^t h(s)\,ds \right|$$

or

$$(\circ') \qquad \limsup_{|x|\to+\infty} (\varphi(x)|g(x))/|g(x)| \leqslant -K < -\sup_t \left| \int_0^t h(s)\,ds \right| .$$

Then equation (L) has a p-periodic solution.

PROOF OF COROLLARY 1. It is easy to see that $(\circ)$, (respectively $(\circ')$) implies (j) (respectively (j')) with $a = K - \varepsilon$, $\varepsilon > 0$ sufficiently small, and $b = \sup_{|x|\leqslant r_\varepsilon} a|g(x)| + \sup_{|x|\leqslant r_\varepsilon} |(\varphi(x)|g(x))|$, for a suitable r_ε depending on ε. Then, from Remark 1 and Theorem 1, the thesis follows. Q.E.D.

Theorem 1 also extends a theorem of Lefschetz [5] to the systems (see [13, Satz 5.3.4, p. 227]); in fact:

COROLLARY 2. *Assume* (i), (ii) *and* (iii). *Let G be nondegenerate with nonzero index and let*

$$(a') \qquad \lim_{|x|\to+\infty} |g(x)|/|x| = +\infty ,$$

$$(b') \qquad |\varphi(x) - C\cdot g(x)| \leqslant B|x| , \quad \text{for } |x| \geqslant R,$$

with C a positive (negative) definite matrix and B, R positive constants. Then equation (L) has a p-periodic solution.

PROOF OF COROLLARY 2. Set $\tilde{G}(x) = G(x) - (Qh|x)$ and $\tilde{h}(t) = h(t) - Qh$. Then equation (L) is equivalent to $x'' + (d/dt)\varphi(x) + \tilde{g}(x) = \tilde{h}$; $\tilde{g} = \operatorname{grad} \tilde{G}$. Clearly, $\tilde{G}$ is nondegenerate with index $\neq 0$ $(\gamma(\tilde{G}, \infty) = \gamma(G, \infty))$ and satisfies (m) of Remark 1. Moreover, from (b') we have:

$$-|C\cdot Qh||\tilde{g}(x)| - B|x||\tilde{g}(x)| + (C\cdot\tilde{g}(x)|\tilde{g}(x)) \leqslant (\varphi(x)|\tilde{g}(x)) \leqslant$$
$$\leqslant (C\cdot\tilde{g}(x)|\tilde{g}(x)) + B|x||\tilde{g}(x)| + |C\cdot Qh||\tilde{g}(x)| ; \quad \text{for } |x| \geqslant R$$

and hence, using (a'), we easily obtain $(\circ)$ or $(\circ')$ of Corollary 1, with $K = +\infty$, according to the fact that C be positive or negative definite. Then we apply Theorem 1 and the thesis follows. Q.E.D.

Remark that, for the scalar case, hypothesis (a) in [13, p. 227] implies (a') and also the condition on the index of G.

In the case of the scalar differential equation

$$(L') \qquad x''(t) + f(x(t))\cdot x'(t) + g(x(t)) = h(t)\,,$$

with f, g, $h\colon \mathbb{R} \to \mathbb{R}$ continuous functions and h, p-periodic in t, we immediately obtain:

COROLLARY 3 (scalar case). *Let us assume*

$$(s') \qquad g(x)\cdot \operatorname{sign} x > 0 \ (<0) \quad \textit{for } |x| \geqslant d > 0,$$

$$(s'') \qquad \Big(\int_0^x f(t)\,dt\Big)\cdot \operatorname{sign} x \geqslant K \ (\leqslant -K) \quad \textit{for } |x| \geqslant d > 0.$$

Then the scalar Liénard equation (L') has a p-periodic solution for any h such that

$$Qh = 0 \quad \textit{and} \quad \sup_t \left| \int_0^t h(s)\,ds \right| < K\,.$$

PROOF OF COROLLARY 3. Set $\varphi(x) = \int_0^x f(s)\,ds$ and observe that

$$\big(\varphi(x)|g(x)\big) = \varphi(x)\cdot g(x) = \big(\varphi(x)\operatorname{sign} x\big)\big(g(x)\operatorname{sign} x\big)$$

and $|g(x)| = g(x)\operatorname{sign} x$ (resp. $= -g(x)\operatorname{sign} x$) for $|x| \geqslant d$. Then, apply Corollary 1 with sign condition (s). Q.E.D.

REMARK 2. Using a standard perturbation argument based on Ascoli-Arzelà Theorem (see [11, Remark 1]) and the a-priori bounds reached in the proof of Theorem 1 (see the next section), it is not difficult to see that the thesis of Corollary 3 holds true even if in (s') the inequalities are not strict.

Let us observe also that, if

$$\lim_{|x|\to+\infty} \Big(\int_0^x f(t)\,dt\Big) \operatorname{sign} x = +\infty\,(-\infty)\,,$$

then no bound is required on $\left|\int_0^t h(s)\,ds\right|$. If, moreover,

$$\lim_{|x|\to+\infty} g(x)\cdot \operatorname{sign} x = +\infty\,(-\infty)\,,$$

then also the assumption $Qh = 0$ can be dropped. In fact, passing to the equivalent equation $x'' + f(x)x' + \tilde{g}(x) = \tilde{h}$, where $\tilde{g}(x) = g(x) - Qh$, $\tilde{h}(t) = h(t) - Qh$, we see that all the hypotheses in Corollary 3 are satisfied.

Therefore, we obtain as corollaries of our result: the theorem of Reuter [14] (see also [16, p. 509], [13, Satz 5.4.3]) in the part concerning existence of periodic solutions, as well as other classical results of the same type (see [16, p. 487-488], [13, Satz 5.3.7, p. 234; Satz 5.3.8-9, p. 236-9] and [12, Satz I]), a theorem by Mawhin [8, Th. 5.4, p. 378] when f and g are polinomials (with even and odd degree respectively) and theorems by Bebernes-Martelli [1, Th. 2, $c \neq 0$] and Ward [17].

4. Proofs.

In the proof of our main result, the following easily estabilished inequality is used.

LEMMA 2. *Let $y = (y_1, \ldots, y_n)\colon \mathbb{R} \to \mathbb{R}^n$ be a p-periodic C^1-function and let $t_1, \ldots, t_n \in [0, p]$. Then, for every $t \in \mathbb{R}$,*

$$|y(t) - w| \leqslant (p/2)^{\frac{1}{2}} |y'|_2\,, \qquad \textit{where } w = \operatorname{col}\,(y_i(t_i))\,.$$

Idea of the proof: use Jensen's Inequality on each component.

PROOF OF THEOREM 1. First of all, let us observe that

$$S(z) := \frac{1}{p}\int_0^p \left\{-\frac{d}{dt}\varphi(z) - g(z) + h(t)\right\} dt = -g(z)\,,$$

for all $z \in \mathbb{R}^n$. Thus, as G is nondegenerate with nonzero index, we immediately see that condition (β) of Mawhin's Lemma is satisfied.

Let $x(t)$ be a p-periodic solution of

$$(L_\lambda) \qquad x''(t) + \lambda \frac{d}{dt}\varphi(x(t)) + \lambda g(x(t)) = \lambda h(t)\,, \qquad \lambda \in (0, 1]\,.$$

Passing to the mean value in (L_λ) and recalling $Qh = 0$, we obtain (after a division by $\lambda > 0$)

$$\int_0^p g(x(t))\,dt = 0 \tag{1}$$

and hence it follows that

$$Z(x)(t) := \int_0^t g(x(s))\,ds\,, \qquad \text{is } p\text{-periodic.}$$

We take now the L^2-scalar product of (L_λ) by $Z(x)$ and observe:

$$(x'', Z(x))_2 = -(g(x), x')_2 = G(x(0)) - G(x(p)) = 0\,,$$

$$\left(\frac{d}{dt}\varphi(x), Z(x)\right)_2 = -(\varphi(x), g(x))_2\,,$$

$$(g(x), Z(x))_2 = \tfrac{1}{2}\,|Z(x)(p)|^2 - \tfrac{1}{2}\,|Z(x)(0)|^2 = 0\,,$$

$$(h, Z(x))_2 = -(H, g(x))_2\,, \qquad \text{where } H(t) := \int_0^t h(s)\,ds\,.$$

Thus we immediately obtain (Hölder inequality)

$$|(\varphi(x), g(x))_2| \leqslant |g(x)|_1 |H|_\infty\,. \tag{2}$$

Then, from (j) or (j') and (2), a bound for $|g(x)|_1$ is reached:

$$|g(x)|_1 \leqslant C_1 = bp/(a - |H|_\infty)\,. \tag{3}$$

Moreover, from (3), it easily follows

$$\min_{[0,p]} |g(x(t))| \leqslant C_1/p\,. \tag{4}$$

Let $z \in S^{n-1}$ and let us take the inner product of (1) by z. Then, thanks to the mean value theorem, we have that $\big(g(x(t_z))|z\big) = 0$ for some $t_z \in [0, p]$.

So, for every $z \in S^{n-1}$, there exists $x_z = x(t_z) \in \mathbb{R}^n$, such that

$$x_z \in V_z \,. \tag{5}$$

Let us set $u_z(t) = x(t) - x_z$ and take the L^2-scalar product of $(L_\lambda^{\sim})$ by u_z. Observe that

$$(x'', u_z)_2 = -|x'|_2^2 \,,$$

$$\left(\frac{d}{dt}\varphi(x), u_z\right)_2 = \left(\frac{d}{dt}\varphi(x), x\right)_2 = -(\varphi(x), x')_2 = F(x(0)) - F(x(p)) = 0$$

and so, with easy computations (using Hölder inequality and $\lambda \leqslant 1$) one has

$$|x'|_2^2 = |u_z'|_2^2 \leqslant |g(x)|_1 |u_z|_\infty + |h|_1 |u_z|_\infty \,. \tag{6}$$

Moreover, taking into account that $|u_z|_\infty^2 \leqslant (p/2)|u_z'|_2^2$ (Lemma 2) we obtain also (from (6))

$$|u_z|_\infty \leqslant (p/2)\big(|g(x)|_1 + |h|_1\big) \,. \tag{7}$$

Hence, from (7) and (3), we have a bound for $|u_z|_\infty$:

$$|u_z|_\infty = |x - x_z|_\infty \leqslant M \,, \tag{8}$$

with $M = (p/2)(C_1 + |h|_1)$.

Finally, from (8) and (6), also a bound for $|x'|_2$ (independent of λ and x) is given:

$$|x'|_2 \leqslant C_2 \,. \tag{9}$$

Observe that (5) and (8) mean that

$$x(t) \in W(M) \quad \text{for every } t \in [0, p] \,. \tag{10}$$

Then, condition (k), together with (4) and (10), implies that there ex-

ists a constant C_3 (independent of λ and x) such that

$$|x(t)| \leqslant C_3 \quad \text{for some } t \in [0, p]. \tag{11}$$

From (9) and (11) we easily obtain a bound for $|x|_\infty$:

$$|x|_\infty < K_0 . \tag{12}$$

From (9) and (12) and equation (L_λ) we see that $|x''|_2$ is also bounded (in virtue of the continuity of g and $\varphi' = \text{Hess } F$) and so, with easy inequality (Lemma 2), we obtain

$$|x'|_\infty < K_1 . \tag{13}$$

Therefore, condition (α) in Mawhin's Lemma holds true and the thesis follows. Q.E.D.

At last, we observe that (from the proof) it is clear that the assumption $|H|_\infty < a$ can be changed into $|\bar{H}|_\infty < a$, where $\bar{H}$ is any other primitive of h $(\bar{H}'(t) = h(t))$.

REFERENCES

[1] J. W. Bebernes - M. Martelli, *Periodic solutions for Liénard systems*, Boll. Un. Mat. Ital., (5), **16**-A (1979), pp. 398-405.

[2] A. Castro - A. C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., (IV), **70** (1979), pp. 113-137.

[3] W. B. Gordon, *On the diffeomorphisms of euclidean space*, Amer. Math. Monthly, **79** (1972), pp. 755-759.

[4] M. A. Krasnosel'skii, *The operator of translation along the trajectories of differential equations*, Amer. Math. Soc. Providence, R.I. (1968).

[5] S. Lefschetz, *Existence of periodic solutions for certain differential equations*, Proc. Nat. Acad. Sci. USA, **29** (1943), pp. 29-32.

[6] N. G. Lloyd, *Degree theory*, Cambridge University Press, Cambridge (1978).

[7] M. Martelli, *On forced nonlinear oscillations*, J. Math. Anal. Appl., **69** (1979), pp. 496-504.

[8] J. MAWHIN, *Degré topologique et solutions périodiques des systèmes différentielles non linéaires*, Bull. Soc. Roy. Sci. Liège, **38** (1969), pp. 308-398.

[9] J. MAWHIN, *Equations intégrales et solutions périodiques des systèmes différentielles non linéaires*, Bull. Acad. Roy. Belge Cl. Sci., (5), **55** (1969), pp. 934-947.

[10] J. MAWHIN, *Existence of periodic solutions for higher order differential systems that are not of class D*, J. Differential Equations, **8** (1970), pp. 523-530.

[11] J. MAWHIN, *An extension of a theorem of A. C. Lazer on forced nonlinear oscillations*, J. Math. Anal. Appl., **40** (1972), pp. 20-29.

[12] R. REISSIG, *Über eine nichtlineare Differentialgleichung 2. Ordnung*, Math. Nachrichten, **13** (1955), pp. 313-318.

[13] R. REISSIG - G. SANSONE - R. CONTI, *Qualitative theorie nichtlinearer differentialgleichungen*, Ed. Cremonese, Roma (1963).

[14] G. E. H. REUTER, *A boundedness theorem for nonlinear differential equations of the second order*, Proc. Cambridge Philos. Soc., **47** (1951), pp. 49-54.

[15] N. ROUCHE - J. MAWHIN, *Equations Differentielles Ordinaires* (2 vol.), Masson, Paris (1973).

[16] G. SANSONE - R. CONTI, *Equazioni differenziali non lineari*, Ed. Cremonese, Roma (1956).

[17] J. R. WARD, *Periodic solutions for a class of ordinary differential equations*, Proc. Amer. Math. Soc., **78** (1980), no. 3, pp. 350-352.

[18] F. ZANOLIN, *Periodic solutions for second order differential systems with damping*, Rend. Sem. Mat. Univ. Padova, **65** (1981) (in print).

[19] F. ZANOLIN, *Remarks on multiple periodic solutions for nonlinear ordinary differential systems of Liénard type*, Boll. Un. Mat. Ital. (6), **1**-B (1982). (in print).

Manoscritto pervenuto in redazione il 28 dicembre 1981.