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DONNA BEERS
FRED RICHMAN
ELBERT A. WALKER

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# Group Algebras of Abelian Groups.

DONNA BEERS - FRED RICHMAN - ELBERT A. WALKER (*)

## 1. Introduction.

Let $R$ be a ring and $G$ a multiplicative group. The group ring $RG$ is the $R$-algebra of all formal finite sums $\sum r_g g$, where $g \in G$ and $r_g \in R$, with component-wise addition and with multiplication defined using the group multiplication of $G$. A problem of some interest is that of deducing information about $G$ from the group algebra $RG$. Of particular interest are conditions under which $RG$ determines $G$, or equivalently, conditions under which the isomorphism of the $R$-algebras $RG$ and $RH$ implies that of the groups $G$ and $H$. Our concern is with the case when $R$ is commutative and $G$ is abelian, and we assume this throughout the rest of this paper.

Let $Z$ be the ring of integers, $G_t$ the torsion subgroup of $G$, and $G_p$ the $p$-primary part of $G$. The principal known results, which may be found in [MAY1] or [BERM], are these:

1) $ZG$ determines $G$;
2) If $F$ is a field, then $FG$ determines $G/G_t$;
3) If $F$ is a field of characteristic $p$, then $F(G/G_p)$, the divisible part of $G_p$, and the Ulm $p$-invariants of $G$ are determined by $FG$.

In particular, if $F$ is a field of characteristic $p$ and $G$ is a countable $p$-group or, more generally, a totally projective $p$-group, then $G$ is determined by $FG$.

(*) Indirizzo degli AA.: Wellesley College and New Mexico State University.
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There are some sobering negative results. For example, there exist countable groups $G$ and $H$ which are not isomorphic, yet $FG$ and $FH$ are isomorphic for any field $F$ [MAY3, Example 2]. If $G$ and $H$ are any countable infinite $p$-groups and $F$ is algebraically closed of characteristic different from $p$, then $FG$ is isomorphic to $FH$ [MAY3, Example 1, BERM].

The positive results above, except for $FG$ determining $G/G_t$, follow immediately from our main theorem (Theorem 3.1). In addition there are a number of large classes of abelian groups $G$ for which the $p$-socle $G[p] = \{g \in G: g^p = 1\}$, together with the $p$-heights of the elements of $G[p]$ as computed in $G$, determine $G$. This is expressed by saying that the valuated vector space $G[p]$ determines $G$. For such classes, $FG$ determines $G$ if it determines $G[p]$ as a valuated subgroup of $G$. Our main theorem addresses this issue as well.

Two significant classes of abelian groups, both containing the class of totally projective $p$-groups, and admitting complete sets of numerical invariants, are Warfield groups and $S$-groups. For such groups $G$, if the pertinent numerical invariants can be extracted from $FG$, then so can $G$. We do this for Warfield groups (Theorem 4.1) and for $S$-groups (Theorem 4.3). A byproduct of this is a new characterization of the invariants for $S$-groups (Theorem 4.2).

## 2. Valuations and other preliminaries.

A fundamental notion in our treatment is that of a *valuation* [RW]. Let $S$ be a commutative semigroup and $p$ a prime. Define $S^p = \{s^p: s \in S\}$, set $S_0 = S$, and for any ordinal $\alpha$ set $S_\alpha = \bigcap_{\beta<\alpha} (S_\beta)^p$, setting $S_\infty = \bigcap_\alpha S_\alpha$. The elements of $S_\beta \setminus S_{\beta+1}$ are the elements of *p-height* $\beta$.

The $S_\alpha$ satisfy:

(1) if $\alpha < \beta$, then $S_\alpha \supset S_\beta$;

(2) if $\alpha$ is a limit ordinal, then $S_\alpha = \bigcap_{\beta<\alpha} S_\beta$;

(3) $(S_\alpha)^p \subset S_{\alpha+1}$.

Any family of subsets of $S$, indexed by the ordinals and $\infty$, that satisfies (1), (2) and (3) is called a *p-filtration* and gives rise to a function $v$ from $S$ to the ordinals and $\infty$ by setting $v(s)$ equal to the largest $\alpha$ such that $s \in S_\alpha$. The function $v$ is called a *p-valuation* on $S$. If $v$

is a $p$-valuation on $S$, then the restriction of $v$ to any subsemigroup $T$ of $S$ is a $p$-valuation on $T$. If $v$ is a $p$-valuation on $S$, and $w$ is a $p$-valuation on a subsemigroup $T$ of $S$, then $T$ is *$p$-isotype* in $S$ if $v$ agrees with $w$ on $T$. Note that $p$-height is a $p$-valuation; in fact it is the smallest $p$-valuation.

We will use these notions when $S$ is a multiplicative abelian group $G$ and when $S = RG$ where $R$ is a ring of prime characteristic $p$. The resulting filtrations on $RG$ and $G$ are denoted by $\{(RG)_\alpha : \alpha\}$ and $\{G_\alpha : \alpha\}$, respectively. If $G$ is additive we often write $p^\alpha G$ for $G_\alpha$. In pertinent situations, one is able to recover from $RG$ not just certain subgroups of $G$, but those subgroups along with the $p$-valuations on them that are inherited from $G$. Since in several significant cases the socle $G[p]$, together with its $p$-valuation as inherited from $G$, determines $G$, it is of interest to recover this more complicated structure from $RG$.

If $S$ is an $R$-algebra, then the $R$-algebra maps from $RG$ to $S$ are in one-to-one correspondence with the group maps from $G$ to the group of units of $S$. The *augmentation map* $\varepsilon: RG \to R$ is given by setting $\varepsilon(g) = 1$. If $\varphi: RG \to R$ is an $R$-algebra map, then there is an automorphism $\alpha$ of $RG$ given by $\alpha(\sum r_g)g = \sum r_g(\varphi(g)g)$ such that $\varphi = \varepsilon\alpha$. Thus any $R$-algebra map from $RG$ to $R$ may serve in lieu of $\varepsilon$, so we may assume that $\varepsilon$ can be retrieved from $RG$. We let $M$ denote the kernel of $\varepsilon$ (the augmentation ideal).

The group of units $U$ of $RG$ is the direct product of the group $R^*$ of units of $R$, and $U' = \{u \in U: \varepsilon u = 1\}$. The group $G$ is embedded as a subgroup of $U'$, an element $g$ of $G$ corresponding to the sum $\sum r_h h$ where $r_g = 1$ and $r_h = 0$ for $h \neq g$.

## 3. The main theorem and its corollaries.

Our main theorem is an isomorphism between tensor products of $R$ with certain subgroups of $G$, and entities which can often be retrieved from $RG$.

THEOREM 3.1. If $\lambda_R: R \to R$ is an injective ring homomorphism and $\lambda_G: G \to G$ is a group homomorphism, then $\lambda: RG \to RG$ defined by

$$\lambda(\sum r_g g) = \sum \lambda_R(r_g)\,\lambda_G(g)$$

is a ring homomorphism, and

$$R \otimes \operatorname{Ker} \lambda_G \cong \operatorname{Ker} \lambda / (M \operatorname{Ker} \lambda)$$

as $R$-modules. If $R$ is a perfect field of characteristic $p$, then the isomorphism above induces, for each $\alpha$, an isomorphism

$$R \otimes (\operatorname{Ker} \lambda_G)_\alpha \cong \big((\operatorname{Ker} \lambda)_\alpha + M \operatorname{Ker} \lambda\big)/(M \operatorname{Ker} \lambda) .$$

PROOF. Clearly $\lambda$ is a ring homomorphism. Consider the map $\varphi$ from $R \times \operatorname{Ker} \lambda_G$ to the $R$-module $\operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$ given by taking $(r, g)$ to $r(g-1) + M \operatorname{Ker} \lambda$. Clearly $\varphi$ is linear in $R$, and

$$\varphi(r, gh) = r(gh-1) + M \operatorname{Ker} \lambda = r(g-1) + r(h-1) + \\ + r(g-1)(h-1) + M \operatorname{Ker} \lambda .$$

Thus $\varphi$ is a bilinear map, and hence induces a homomorphism $\varphi$ from $R \otimes \operatorname{Ker} \lambda_G$ to $\operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$ given by $\varphi(r \otimes g) = r(g-1) + M \operatorname{Ker} \lambda$.

Let $\sum r_g g$ be in $\operatorname{Ker} \lambda$. Let $C$ be the set of cosets of $\operatorname{Ker} \lambda_G$ and $g_c \in c$ for each $c \in C$. Then

$$\sum r_g g = \sum_{c \in C} \sum_{g \in c} r_g g ,$$

and since $\lambda_R$ is one-to-one, $\sum_{g \in c} r_g = 0$ for each $c \in C$. Define $\psi$: $\operatorname{Ker} \lambda \to R \otimes \operatorname{Ker} \lambda_G$ by sending $\sum r_g g$ to $\sum_{c \in C} \sum_{g \in c} r_g \otimes (g - g_c)$. Since $\sum_{g \in c} r_g = 0$, this is independent of the choice of $g_c$'s. Now

$$\psi\big((h-1) \sum r_g g\big) = \psi\big(\sum r_g hg\big) - \psi\big(\sum r_g g\big) = \\ = \sum_{c \in C} \sum_{g \in c} r_g \otimes \big(h + g - (h + g_c)\big) - \sum_{c \in C} \sum_{g \in c} r_g \otimes (g - g_c) = 0 .$$

Thus $\psi$ induces an $R$-homomorphism from $M/(M \operatorname{Ker} \lambda)$ to $R \otimes \operatorname{Ker} \lambda_G$, and it is easy to check that $\varphi$ and $\psi$ are inverses of each other.

Let $R$ be a perfect field of characteristic $p$. It is clear that $\varphi$ induces a map from $R \otimes (\operatorname{Ker} \lambda_G)_\alpha$ into $\big((\operatorname{Ker} \lambda)_\alpha + M \operatorname{Ker} \lambda\big)/(M \operatorname{Ker} \lambda)$. An element of $(\operatorname{Ker} \lambda)_\alpha$ is a sum of elements of the form $\sum_{g \in c} r_g g$ such that $\sum_{g \in c} r_g = 0$, and $g \in G_\alpha$ whenever $r_g \neq 0$. Noting that

$$\sum r_g (g g_c^{-1} - 1) - \sum r_g g = \big(\sum r_g g\big)(g_c^{-1} - 1) \in M \operatorname{Ker} \lambda ,$$

we conclude that $\varphi\big(\sum r_g (g g_c^{-1})\big) = \sum r_g g + M \operatorname{Ker} \lambda$. □

This theorem will be applied by choosing $\lambda = \varepsilon$ or $\lambda x = x^p$. Throughout, $F_p$ will denote the field with $p$ elements.

COROLLARY 3.2. The $R$-modules $R \otimes G$ and $M/M^2$ are isomorphic. Thus the group $G$ can be retrieved from $ZG$, and if $pR = 0$, the group $G/G^p$ can be retrieved from $RG$.

PROOF. Apply Theorem 3.1 with $\lambda = \varepsilon$ and use the fact that $Z \otimes G \cong G$, and the $R \otimes G \cong R \otimes G/G^p$ is a free $R$-module of the same rank as $G/G^p$ if $pR = 0$. □

COROLLARY 3.3. If $R$ is an integral domain with $pR = 0$, then the vector space $G[p]$ can be retrieved from $RG$.

PROOF. Application of Theorem 3.1 with $\lambda x = x^p$ yields $R \otimes G[p] \cong \operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$. Then note that the rank of the free $R$-module $R \otimes G[p]$ is the dimension of the vector space $G[p]$. □

LEMMA 3.4. Let $F$ be a perfect field of characteristic $p$ and $G$ an abelian group. Then $(FG)_\beta = FG_\beta$, that is, $G$ is $p$-isotype in $FG$.

PROOF. We proceed by induction on $\beta$. The proof follows from the equations:

$$(FG)_\beta = \bigcap_{\alpha<\beta} \big((FG)_\alpha\big)^p = \bigcap_{\alpha<\beta} (FG_\alpha)^p = \bigcap_{\alpha<\beta} (FG_{\alpha+1}) = FG_\beta \,.$$

The first equality follows from the definition; the second is the induction hypothesis; the third uses perfectness of the field. □

COROLLARY 3.5. If $F$ is a perfect field of characteristic $p$, then $FG_\beta$ and $F(G/G_\beta)$ can be retrieved from $FG$.

PROOF. Apply Lemma 3.4 to get $(FG)_\beta = FG_\beta$. Then use the fact [PASS, Lemma 1.8] that $FG(M \cap FG_\beta)$ is the kernel of $FG \to F(G/G_\beta)$ to conclude that $FG/FG(M \cap FG_\beta) \cong F(G/G_\beta)$.

COROLLARY 3.6. If $F$ is a field of characteristic $p$, then $(G_p)_\infty$ can be retrieved from $FG$. If $G_q = 0$ when $q \neq p$, then $G_\infty$ can be retrieved from $FG$.

PROOF. We may assume that $F$ is perfect. In Corollary 3.5, we showed that $FG_\infty$ can be obtained from $FG$. Next, application of Theorem 3.1 with $\alpha = \infty$ and $\lambda x = x^p$ shows that $F \otimes G_\infty[p]$ may be

gotten from $FG_\infty$. But $F \otimes G_\infty[p]$ is a vector space over $F$ of dimension equal to the dimension of the vector space $G_\infty[p]$. Thus we may retrieve $(G_p)_\infty$ from $FG$. If $G_q = 0$ for $q \neq p$, we use the fact that $G_\infty/(G_\infty)_t$ can be obtained from $FG_\infty$ together with the identity $G_\infty \cong (G_p)_\infty \oplus G_\infty/(G_\infty)_t$ to finish the proof. □

COROLLARY 3.7. If $F$ is a perfect field of characteristic $p$, then $F \otimes G[p]$, as a filtered vector space with filtration given by the spaces $F \otimes G_\alpha[p]$, can be obtained from $FG$. Thus the Ulm $p$-invariants for $G$ may be gotten from $FG$, and if $F = F_p$, then $G[p]$ itself can be obtained as a valuated vector space.

PROOF. By Theorem 3.1, the filtered vector space $F \otimes G[p]$ is isomorphic to the vector space $\operatorname{Ker} \lambda/(M \operatorname{Ker} \lambda)$ filtered by the subspaces $((\operatorname{Ker} \lambda)_\alpha + M \operatorname{Ker} \lambda)/(M \operatorname{Ker} \lambda)$. We get the Ulm $p$-invariants for $G$ by observing that $F \otimes (G_\alpha[p]/G_{\alpha+1}[p])$ is isomorphic to $(F \otimes G_\alpha[p])/(F \otimes G_{\alpha+1}[p])$, and that $F \otimes (G_\alpha[p]/G_{\alpha+1}[p])$ has $F$-rank equal to the dimension of the vector space $G_\alpha[p]/G_{\alpha+1}[p]$. Finally, we use the fact that $F_p \otimes G_\alpha[p] \cong G_\alpha[p]$ to obtain $G[p]$ as a valuated vector space from $F_pG$. □

This last corollary raises the question as to how much information is lost in passing from the filtered $F_p$-space $G[p]$ to the filtered $F$-space $F \otimes G[p]$. The associated graded spaces are essentially the same, so we don't lose the Ulm invariants. But can we, for example, look at $F \otimes G[p]$ and tell whether or not $G[p]$ is complete?

We now look at specific classes of groups and fields for which the group algebra $FG$ determines $G$.

THEOREM 3.8. If $F$ is a field of characteristic $p$, and if $G$ is a totally projective $p$-group, then $FG$ determines $G$.

PROOF. We may assume that $F$ is perfect. By Corollary 3.7, the Ulm $p$-invariants for $G$ may be obtained from $FG$. Since totally projective $p$-groups are characterized by their Ulm $p$-invariants, the proof is complete. □

THEOREM 3.9. If $G$ belongs to a class of groups which are distinguished by their valuated $p$-socles, then $F_pG$ determines $G$. In particular, $F_pG$ determines $G$ when $G$ belongs to any one of the following classes of groups:

(i) the class of direct sums of torsion-complete $p$-groups;

(ii) the class of $p$-groups $G$ such that $p^{\omega+1}G = 0$ and $G/p^{\omega}G$ is torsion-complete;

(iii) the class of $p^{\omega+1}$-projective $p$-groups.

PROOF. From Corollary 3.7, $G[p]$ may be recovered as a valuated vector space from $F_pG$. Within each class stated in the theorem, two groups are distinguished by the valuated structure of their socles [HILL], [RICH, Corollary 1], [FUCHS2, Theorem 3]. □

A group is *p-local* if it is a module over the ring of integers localized at $p$.

THEOREM 3.10. The group algebra $F_pG$ determines $G$ when $G$ is a direct sum of $p$-local algebraically compact groups.

PROOF. Let $G$ be a direct sum of $p$-local algebraically compact groups. By Corollary 3.5 and Corollary 3.6 we may assume that $G$ is reduced. Then $G$ may be written as $A \oplus F$, where $A$ is a direct sum $\sum A_i$ of adjusted algebraically compact groups and $F$ is torsion-free. From Corollary 3.8, $F_pG$ determines the structure of $G[p]$, and so of $A[p]$, as a valuated vector space. The problem here is to show that $A$ is determined by the structure of $A[p]$ as a valuated vector space. Let $T = A_t$ and $T_i = (A_i)_t$. Then $A_i = \mathrm{Ext}\,(Q/Z, T_i) = \tilde{T}_i$ and each $T_i$ is torsion-complete. It is clear that $T = \sum T_i$. We can recover $T$ from $A[p]$ [HILL]. Suppose $T = \sum S_j$, where the $S_j$'s are torsion-complete. The two decompositions $\sum T_i$ and $\sum S_j$ have isomorphic refinements, with $T_i = \sum T_{ik}$ and $S_j = \sum S_{jm}$. For each $i$, there exists $n(i)$ such that $p^{n(i)}T_{ij} = 0$, for all but finitely many $j$. Therefore, $\tilde{T}_i = \sum_k \tilde{T}_{ik}$ and similarly for the $S_{jm}$'s. Thus $A = \sum \tilde{T}_{ik} = \sum \tilde{T}_i = \sum \tilde{S}_j = \sum \tilde{S}_{jm}$. But there exists a one-to-one correspondence between the $T_{ik}$'s and the $S_{jm}$'s such that corresponding $T_{ik}$'s and $S_{jm}$'s are isomorphic. Therefore, $A = \sum \tilde{S}_j$.

Furthermore, since $G/G_t \cong A/G_t \oplus F$ may be gotten from $F_pG$, and since $A/G_t$ is the maximal divisible subgroup of $G/G_t$, the group algebra $F_pG$ determines $G$. □

Note that the fact that $F$ is algebraically compact was not used, only that $F$ was torsion-free reduced. Thus the group algebra $F_pG$ determines $G$ when $G = A \oplus F$, with $A$ a direct sum of algebraically compact adjusted $p$-local groups and $F$ torsion-free reduced.

## 4. Warfield groups.

Local Warfield groups were introduced by Warfield in [WARF1], where he gave a complete set of invariants for them. A complete and leisurely account of their theory can be found in [HRW]. These groups are distinguished by their Ulm invariants and their Warfield invariants. For our purposes, it is enough to know that if $B$ is an isotype subgroup of $A$, and $A/B$ is torsion, then $A$ and $B$ have the same Warfield invariants.

THEOREM 4.1. If $F$ is a field of characteristic $p$ and $G$ is a $p$-local group, then $FG$ determines the Warfield invariants of $G$. Hence, if $G$ is a $p$-local Warfield group, then $FG$ determines $G$.

PROOF. We may assume $F$ is perfect. Let $U(FG)$ be the group of units of $FG$, and let $H$ be a torsion-free subgroup of $G$ such that $G/H$ is torsion. If $x = \sum r_g g$ is in $U(FG)$, then there exists a positive integer $q = p^n$ such that $x^q \in U(FH)$. Now $U(FG) = F^* \oplus U'$, where $U' = \{u \in U(FG) : \varepsilon u = 1\}$. Since $H$ is torsion-free, the units of $FH$ are trivial, so $U(FH) \cong F^* \oplus H$. Therefore $U'/H$, and hence $U'/G$, is torsion. But $G$ is isotype in $FG$ by Lemma 3.4, and so is isotype in $U'$. Thus $G$ and $U'$ have the same Warfield invariants. Corollary 3.7 guarantees that the Ulm $p$-invariants for $G$ may be gotten from $FG$, and since $U'$ may be gotten from $FG$, then $FG$ determines $G$ when $G$ is a $p$-local Warfield group. □

Torsion subgroups of Warfield groups are called *S-groups*. An $S$-group $G$ is characterized by its Ulm invariants and an invariant $k(\lambda, G)$ defined for each limit ordinal $\lambda$ not cofinal with $\omega$ [WARF2, page 158]. To show that we can recover an $S$-group $G$ from its group ring $F_p G$ we first need a new characterization of $k(\lambda, G)$.

THEOREM 4.2. If $G$ is an $S$-group and $\lambda$ is a limit ordinal that is not cofinal with $\omega$, then $k(\lambda, G)$ is the codimension of $(G/p^\lambda G)[p]$ in its $\lambda$-completion.

PROOF. As taking $\lambda$-completions commutes with direct sums, it suffices to show that the codimension of the socle of a $\lambda$-elementary $S$-group $H$ in its $\lambda$-completion is 1. Let $H = K_t$ where $K$ is a $\lambda$-elementary balanced projective. For details, see [WARF2]. Then $K/p^\lambda K$

is simply presented, so $P = (K/p^\lambda K)[p]$ is $\lambda$-complete. But $H[p] = K[p]$ is $\lambda$-dense in $P$ and $P/K[p] \cong p^\lambda K/p^{\lambda+1}K$ has dimension 1. □

THEOREM 4.3. If $G$ is an $S$-group, then $k(\lambda, G)$ can be recovered from $F_pG$. Hence, if $G$ is an $S$-group, then $FG$ determines $G$.

PROOF. By Theorem 4.2 it suffices to recover the valuated vector space $(G/p^\lambda G)[p]$ for limit ordinals $\lambda$ that are not cofinal with $\omega$. □

## 5. Splitting.

We close on a note about the splitting out of $G$ as a summand of the group of units of $FG$. A subsemigroup $T$ of a semigroup $S$ is *nice* if the set $\{v(st): t \in T\}$ has a maximum element for each $s$ in $S$.

LEMMA 5.1. If $F$ is a perfect field of characteristic $p$, then $G$ is nice in $FG$.

PROOF. Let $x = \sum r_g g$. We need $\{x \cdot h: h \in G\}$ to have an element of maximum height. If the height of $h$ is greater than the minimium of the heights of the $g$'s, then $xh$ has the same height as $x$. Therefore if one of the $g$'s equals 1, then the height of $x$ cannot be raised by multiplication by an element of $G$. Hence multiplying $x = \sum r_g g$ by any $g^{-1}$ yields an element of maximum height in the set $\{x \cdot h: h \in G\}$, and thus $G$ is nice in $FG$. □

THEOREM 5.2. Let $F$ be a perfect field of characteristic $p$. If $U'/G$ is a totally projective $p$-group, then $G$ is a summand of $U$, In particular, if $G$ is a countable $p$-group and $FG$ is countable, then $G$ is a summand of $U$.

PROOF. The group $G$ is nice and isotype in $FG$, and hence in $U$ By [FUCHS1, Theorem 81.9], $G$ is a summand of $U'$, and hence of $U$. □

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