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On Neumann's Problem for a Quasilinear Differential System of the Finite Elastostatics Type. Local Theorems of Existence and Uniqueness.

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1. Introduction.

This paper concerns the Neumann's boundary value problem for a quasilinear differential system of the type of finite elastostatics. More precisely, let Ω be a bounded open subset of $\mathbf{R}^n$, let ν be the unit outward normal to $\partial\Omega$ and let $a\colon \bar{\Omega}\times\mathbf{R}^{n^2}\to\mathbf{R}^{n^2}$, $f\colon \Omega\to\mathbf{R}^n$ and $g\colon \partial\Omega\to\mathbf{R}^n$ be given functions with $a(x, 1) = 0$, $\forall x\in\Omega$. Then we deal with the problem of finding $u\colon \bar{\Omega}\to\mathbf{R}^n$ such that (see sect. 3)

$$\text{(P)}\quad \operatorname{div} A(u) + \vartheta f = 0 \quad \text{in } \Omega\,, \qquad -A(u)\nu + \vartheta g = 0 \quad \text{on } \partial\Omega\,,$$

where $A(u)(x) = a\big(x, 1 + Du(x)\big)$, $\forall x\in\Omega$, $Du = (D_j u_i)_{i,j=1,\dots,n}$ and ϑ is a real parameter. When $n = 3$ this problem corresponds to the « dead traction problem » of finite elastostatics.

The main achievements we reach are local theorems of existence and uniqueness in Sobolev spaces and in Schauder spaces (see Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2). We obtain such results assuming that, if $c_1, \dots, c_n$ are the eigenvalues of the « astatic » matrix of (f, g) defined by (3.7), then $c_i + c_j \neq 0$ whenever $i \neq j$.

On the function a we only make hypotheses suggested by the physical problem from which our problem arises, thus avoiding artificial assumptions.

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Our results are essentially generalizations in various directions of a result of Stoppelli [11]. In fact, what we use throughout is a basic idea of [11], while it seems to us that the method previously devised by Stoppelli in [9] does not lead to a satisfactory uniqueness result (indeed, we do not see how the theorem stated at the end of section 10 in [9] can be derived from the existence and uniqueness theorem stated in section 9).

The starting point consists of a suitable modification of Problem (P) above which leads to another problem apt to be locally studied by iterated applications of the implicit function theorem. In effect, if we set

$$P(u,\vartheta) = \big(\operatorname{div} A(u) + \vartheta f,\ -A(u)\nu + \vartheta g\big),$$

we cannot directly apply the implicit function theorem to the equation $P(u,\vartheta) = 0$ in order to express u as a function of ϑ near $(0,0)$ when the symmetries (3.6) hold, as we suppose. Indeed, from (3.6) it follows that the partial differential $d_u P(0,0)$ takes « equilibrated » values (see section 4 and Remark A.3 of Appendix), while the values of P are not « equilibrated ».

We begin by replacing the operator P by the operator

$$N\colon (u,\vartheta) \mapsto \big(\operatorname{div} A(u) + E(u)h + \vartheta f,\ -A(u)\nu + \vartheta g\big),$$

where h is a suitable $\mathbf{R}^n$-valued function defined in Ω and E is an operator with values in the space of $n\times n$ skew-symmetric real matrices and is such that the pair $\big(\operatorname{div} A(u) + E(u)h + \vartheta f, -A(u)\nu + \vartheta g\big)$ is « equilibrated ».

Now, under our hypotheses, the kernel of the linear operator $d_u N(0,0)$ is the set of functions $r = (r_i)_{i=1,\dots,n}\colon \Omega \to \mathbf{R}^n$ such that $r_i(x) = c_i + s_{ij}x_j$, $\forall x\in\Omega$, where $c_i \in \mathbf{R}$ and $(s_{ij})_{i,j=1,\dots,n}$ is an $n\times n$ skew-symmetric real matrix (see sect. 5 and Remark A.2 of Appendix). Note that each $u \in \big(W^{1,1}(\Omega)\big)^n$ can be obviously written in a unique way in the form $u(x) = \big(v_i(x) + s_{ij}x_j\big)_{i=1,\dots,n}$, where $s = (s_{ij})_{i,j=1,\dots,n}$ is an $n\times n$ skew-symmetric real matrix and $v = (v_i)_{i=1,\dots,n} \in \big(W^{1,1}(\Omega)\big)^n$ and verifies the conditions

$$\int_\Omega (D_j v_i - D_i v_j)\, dx = 0\,, \qquad \forall i, j = 1, \dots, n\,.$$

By the above considerations it is convenient to regard u as the pair (v, s) and therefore to introduce the operator

$$M\colon (v, s, \vartheta) \mapsto \big(\operatorname{div} A(\psi(v, s)) + E(\psi(v, s))h + \vartheta f,\; -A(\psi(v, s))\nu + \vartheta g\big),$$

where $\psi(v, s)$ is the $\mathbf{R}^n$-valued function defined in Ω by $\psi(v, s)(x) = (v_i(x) + s_{ij}x_j)_{i=1,\dots,n}$.

The implicit function theorem applied to the equation $M(v, s, \vartheta) = 0$ gives v locally as a function, say $\hat{v}$, of s and ϑ.

At this point, we study the relation between the solutions of the (modified) equation $M(v, s, \vartheta) = 0$ and those of the (original) equation $P(\psi(v, s), \vartheta) = 0$, and we show that $\psi(\hat{v}(s, \vartheta), s)$ is a solution of Problem (P) for (s, ϑ) close enough to $(0, 0)$ if and only if $\tau(s, \vartheta) = 0$, where τ is a suitable $\mathbf{R}^{n^2}$-valued operator (see sect. 7). Then, an application of the implicit function theorem to the equation $\tau(s, \vartheta) = 0$ allows to express s as a function of ϑ near $(0, 0)$. Consequently, we locally obtain the solution u of Problem (P) as a function of ϑ, and thus we attain to Theorems 3.1 and 3.2 and to Corollaries 3.1 and 3.2.

The choice of spaces for solutions and data which are suitable for a local treatment of Problem (P) requires a study of problems of differentiability of operators of various types, and of isomorphism problems for a divergence type linear matrix differential operator, in Sobolev spaces and in Schauder spaces. (For the latter problems see appendix.) In proving the differentiability of the nonlinear operators we deal with, an important role is played by the fact that (under suitable assumptions on p and Ω) the Sobolev spaces $W^{m,p}(\Omega)$ and the Schauder spaces $C^{m,\lambda}(\Omega)$ are Banach algebras.

2. Notations and technical preliminaries.

Throughout this work Ω denotes a nonempty, bounded, open subset of $\mathbf{R}^n$, $(n \geqslant 1)$, such that $\int_\Omega x_i\,dx = 0$, $\forall i = 1, \dots, n$, m denotes a nonnegative integer, p denotes a real number > 1 and λ denotes a real number such that $0 < \lambda \leqslant 1$. $\bar{\Omega}$ is the closure of Ω, $\partial\Omega$ its boundary and ν is the unit outward normal to $\partial\Omega$ at any regular point of $\partial\Omega$. Unless explicitly stated otherwise, we use the summation convention, i.e., a summation from 1 to n must be understood when an index is repeated twice.

The gradient of a function $v = (v_i)_{i=1,\dots,n}: \Omega \to \mathbb{R}^n$ is denoted by Dv, i.e., we set

$$Dv = (D_j v_i)_{i,j=1,\dots,n} ,$$

where $D_j v_i = \partial v_i / \partial x_j$. The divergence of a function $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}: \Omega \to \mathbb{R}^{n^2}$ is denoted by $\operatorname{div} \sigma$, i.e., we set

$$\operatorname{div} \sigma = (D_j \sigma_{ij})_{i=1,\dots,n} .$$

The following notations are standard. $L^p(\Omega)$ is the space of (classes of) measurable functions $v: \Omega \to \mathbb{R}$ such that $|v|^p$ is Lebesgue-integrable, while $W^{m,p}(\Omega)$ is the (Banach) space of elements v of $L^p(\Omega)$ such that, for $|\alpha| \leqslant m$, the weak derivatives $D^\alpha v$ belong to $L^p(\Omega)$ equipped with the norm

$$\|v\|_{m,p} = \sum_{|\alpha| \leqslant m} \|D^\alpha v\|_{0,p} ,$$

where $\|\cdot\|_{0,p}$ is the usual norm of $L^p(\Omega)$. If Ω has the cone property (i.e., if there exist positive constants α, h such that for any $x \in \Omega$ one can construct a right spherical cone with vertex x, opening α, and height h such that it lies in Ω), and if $mp > n$, then $W^{m,p}(\Omega)$ is a Banach algebra, i.e.,

$$u, v \in W^{m,p}(\Omega) \Rightarrow uv \in W^{m,p}(\Omega) , \qquad \|uv\|_{m,p} \leqslant c_{m,p} \|u\|_{m,p} \|v\|_{m,p} ,$$

where $c_{m,p}$ is a positive number independent of u and v (see Adams [1], Th. 5.23). $C^{m,\lambda}(\bar{\Omega})$ denotes the (Banach) space of real functions of class C^m on $\bar{\Omega}$ such that, for $|\alpha| \leqslant m$, $D^\alpha v$ satisfies on $\bar{\Omega}$ a Holder condition of exponent λ, with the norm

$$\|v\|_{m,\lambda} = \sum_{|\alpha| \leqslant m} \sup_{x \in \bar{\Omega}} |D^\alpha v(x)| + \sum_{|\alpha| = m} \sup_{\substack{x', x'' \in \bar{\Omega} \\ x' \neq x''}} \frac{|D^\alpha v(x') - D^\alpha v(x'')|}{|x' - x''|^\lambda} .$$

We say that Ω is of class C^m [resp. $C^{m,\lambda}$], with $m \geqslant 1$, if $\bar{\Omega}$ is a submanifold of $\mathbb{R}^n$ with boundary of class C^m [resp. $C^{m,\lambda}$], i.e., if for each $x \in \partial\Omega$ there exists an open neighborhood U_x of x in $\mathbb{R}^n$ and a diffeomorphism t_x of class C^m [resp. $C^{m,\lambda}$] of $\bar{U}_x$ onto the ball $\{\xi \in \mathbb{R}^n: |\xi| \leqslant 1\}$ of $\mathbb{R}^n$, such that $t_x(\bar{\Omega} \cap U_x) = \{\xi \in \mathbb{R}^n: |\xi| < 1, \xi_n \geqslant 0\}$. It is easy to see that, if Ω is of class C^1, then it has the cone property.

If Ω is of class C^m and s is a real number such that $0 < s \leqslant m$, we can consider the spaces $W^{s,p}(\partial\Omega)$: for a definition of such spaces we refer to Lions and Magenes [8] and to Adams [1], p. 215.

If Ω is of class $C^{m,\lambda}$, we denote by $C^{m,\lambda}(\partial\Omega)$ the space of functions $g\colon \partial\Omega \to \mathbb{R}$ such that $g\circ t_x^{-1} \in C^{m,\lambda}(\sigma)$, $\forall x \in \partial\Omega$, where $\sigma = \{\xi \in \mathbb{R}^n\colon |\xi| \leqslant 1, \xi_n = 0\}$. On $C^{m,\lambda}(\partial\Omega)$ we consider the norm defined by

$$\|g\|_{m,\lambda,\partial\Omega} = \sup_{x\in\partial\Omega} \|g\circ t_x^{-1}\|_{C^{m,\lambda}(\sigma)}$$

for a fixed choice of the family $\{t_x\colon x \in \partial\Omega\}$. It is easy to check that different families give equivalent norms. One can prove (see [14], Osservazione 1) that, if Ω is connected and of class C^1, then $C^{m,\lambda}(\bar{\Omega})$ is a Banach algebra, i.e.,

$$u, v \in C_{m,\lambda}(\bar{\Omega}) \Rightarrow uv \in C_{m,\lambda}(\bar{\Omega}), \qquad \|uv\|_{m,\lambda} \leqslant c_{m,\lambda}\|u\|_{m,\lambda}\|v\|_{m,\lambda},$$

where $c_{m,\lambda}$ is a positive number independent of u and v.

If $v = (v_i)_{i=1,\dots,n}$ belongs to $(W^{m,p}(\Omega))^n$ and to $(C^{m,\lambda}(\bar{\Omega}))^n$ respectively, we set $\|v\|_{m,p} = \sum_{i=1}^{n} \|v_i\|_{m,p}$ and $\|v\|_{m,\lambda} = \sum_{i=1}^{n} \|v_i\|_{m,\lambda}$.

We denote by Q the set of $n\times n$ real orthogonal matrices, i.e., the set of real matrices $q = (q_{ij})_{i,j=1,\dots,n}$ such that $q^* q = 1$, where q^* is the transpose of q and 1 is the unit matrix.

We denote by $\mathcal{S}$ the set of $n\times n$ skew-symmetric real matrices. $\mathcal{S}$ will be regarded as a subspace of $\mathbb{R}^{n^2}$.

Moreover we denote by $\mathfrak{R}$ the set of the functions $r = (r_i)_{i=1,\dots,n}\colon \Omega \to \mathbb{R}^n$ such that $r_i(x) = c_i + s_{ij}x_j$, with $c_i \in \mathbb{R}$ and $(s_{ij})_{i,j=1,\dots,n} \in \mathcal{S}$.

In conclusion, we recall the statement of the implicit function theorem in Banach spaces, which we will use later.

IMPLICIT FUNCTION THEOREM. *Let X, Y, Z be Banach spaces, U an open subset of $X\times Y$ and let $f\colon U \to Z$ be a continuously differentiable function. Let $(x_0, y_0) \in U$ be such that $f(x_0, y_0) = 0$ and that the differential, $d_y f(x_0, y_0)$, of the function $y \mapsto f(x_0, y)$ at y_0 be a bijection of Y onto Z. Then there exists an open neighborhood U_0 of (x_0, y_0) in $X\times Y$ contained in U, an open neighborhood V_0 of x_0 in X and a continuously differentiable function $g\colon V_0 \to Y$ such that $\{(x, y) \in U_0\colon f(x, y) = 0\} = \{(x, y)\colon x \in V_0, y = g(x)\}$. Moreover, the differential, $dg(x_0)$, of g at x_0*

is given by

$$dg(x_0) = -\left(d_y f(x_0, y_0)\right)^{-1} \circ d_x f(x_0, y_0) .$$

where $d_x f(x_0, y_0)$ *is the differential at* x_0 *of the function* $x \mapsto f(x, y_0)$.

3. Formulation of the Problem and statements of the main results.

Let $(x, y) \mapsto a(x, y) = \left(a_{ij}(x, y)\right)_{i,j=1,\dots,n}$ be a function from $\bar{\Omega} \times \mathbf{R}^{n^2}$ to $\mathbf{R}^{n^2}$ such that

$$a(x, 1) = 0 , \quad \forall x \in \bar{\Omega} . \tag{3.1}$$

Moreover, let $f = (f_i)_{i=1,\dots,n}$ be a function from Ω to $\mathbf{R}^n$ and let $g = (g_i)_{i=1,\dots,n}$ be a function from $\partial\Omega$ to $\mathbf{R}^n$. If the functions a and $u \colon \bar{\Omega} \to \mathbf{R}^n$ are suitably smooth, we set, for any $x \in \bar{\Omega}$,

$$A(u)(x) = a\left(x, D(I + u)(x)\right) \left[= a\left(x, 1 + Du(x)\right)\right] ,$$

where I is the identity of $\bar{\Omega}$ into itself. Let ϑ be a real parameter. We consider the problem of finding $u \colon \bar{\Omega} \to \mathbf{R}^n$ such that

$$\text{(P)} \qquad \begin{cases} \operatorname{div} A(u) + \vartheta f = 0 , \\ -A(u)\nu + \vartheta g = 0 \end{cases}$$

where $A(u)\nu$ is the function of $\partial\Omega$ into $\mathbf{R}^n$ defined by

$$\left(A(u)\nu\right)(x) = \left(a_{ij}\left(x, 1 + Du(x)\right)\nu_j(x)\right)_{i=1,\dots,n} .$$

We remark that in the case $n = 3$, Problem (P) corresponds to the « dead traction problem » of non linear elastostatics. In the physical context, Ω represents a fixed reference configuration of an elastic body, u is the displacement from Ω, $I + u$ is the deformation corresponding to u, a defines the response of the material, in the sense that $a\left(x, 1 + Du(x)\right)$ is the first Piola-Kirchoof stress at the point $x \in \Omega$ when the body is deformed by $I + u$, f is the body force per unit volume, and g is the surface traction per unit surface area in the reference configuration Ω.

On the function a we consider the further hypotheses (3.2) (3.3) and (3.4), suggested by the physical problem from which our problem arises.

$$(3.2) \qquad a(x, qy) = qa(x, y) \quad \text{for any} \quad (x, y) \in \bar{\Omega} \times \mathbf{R}^{n^2} \text{ and } q \in Q\,,$$

$$(3.3) \qquad a(x, y)y^* = ya^*(x, y) \quad \text{for any} \quad (x, y) \in \bar{\Omega} \times \mathbf{R}^{n^2}\,,$$

$$(3.4) \qquad \frac{\partial a_{ij}}{\partial y_{hk}}(x, 1)\sigma_{ij}\sigma_{hk} > 0 \quad \text{for any} \quad x \in \bar{\Omega} \text{ and}$$
$$\sigma = (\sigma_{ij})_{i,j=1,\ldots,n} \in \mathbf{R}^{n^2} \setminus \{0\} \text{ such that } \sigma_{ij} = \sigma_{ji}\,.$$

Here y^* is the transpose of the matrix y and $a^*(x, y)$ is the transpose of the matrix $a(x, y)$.

In the physical context, the condition (3.1) expresses that the reference configuration is that of a « natural state », while hypothesis (3.2) derives from the principle of material frame-indifference, and (3.3) derives from the symmetry of the Cauchy-stress. From (3.1) and (3.3), it immediately follows

$$(3.5) \qquad \frac{\partial a_{ij}}{\partial y_{hk}}(x, 1) = \frac{\partial a_{ji}}{\partial y_{hk}}(x, 1)\,, \qquad \forall x \in \bar{\Omega}\,,$$

while combining (3.1) and (3.2) we obtain (see Gurtin [7]).

$$(3.6) \qquad \frac{\partial a_{ij}}{\partial y_{hk}}(x, 1) = \frac{\partial a_{ij}}{\partial y_{kh}}(x, 1)\,, \qquad \forall x \in \bar{\Omega}\,.$$

As far as f and g are concerned, we suppose that the pair (f, g) is *equilibrated*, in the sense that we have

$$\begin{cases} \int_{\Omega} f(x)\, dx + \int_{\partial\Omega} g(x)\, d\sigma = 0\,, \\ \int_{\Omega} (x_i f_j(x) - x_j f_i(x))\, dx + \int_{\partial\Omega} (x_i g_j(x) - x_j g_i(x))\, d\sigma = 0\,, \quad i, j = 1, \ldots, n\,. \end{cases}$$

This implies the symmetry of the astatic matrix of (f, g), namely, of the matrix $c = (c_{ij})_{i,j=1,\dots,n}$ where

$$c_{ij} = \int_\Omega x_i f_j(x)\,dx + \int_{\partial\Omega} x_i g_j(x)\,d\sigma\,. \tag{3.7}$$

We assume that, if $c_1, \dots, c_n$ are the eigenvalues of the matrix c, then $c_i + c_j \neq 0$ whenever $i \neq j$.

It is easy to see that, in the case $n = 3$, such hypothesis is equivalent to the condition $\det\big(c_{ij} - \delta_{ij}(c_{11} + c_{22} + c_{33})\big)_{i,j=1,2,3} \neq 0$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The mechanical interpretation of such a condition is that the «load» (f, g) does not possess an axis of equilibrium (see Stoppelli [9]).

In order to obtain local results of existence and uniqueness for Problem (P), a natural choice of the space for (u, f, g) might seem to be the following $\big(W^{1,2}(\Omega)\big)^n \times \big(W^{-1,2}(\Omega)\big)^n \times \big(W^{-\frac{1}{2},2}(\Omega)\big)^n$, where $W^{-1,2}(\Omega)$ is the strong normed dual of $W_0^{1,2}(\Omega)$. But because of the arguments exposed in [13] and [15], this choice is unfortunately not fruitful for our purposes; while suitable choices of the space for (u, f, g) are (as we shall see) the following: $\big(W^{m+2,p}(\Omega)\big)^n \times \big(W^{m,p}(\Omega)\big)^n \times \big(W^{m+1-1/p,p}(\partial\Omega)\big)^n$ with $p(m+1) > n$, and $\big(C^{m+2,\lambda}(\bar\Omega)\big)^n \times \big(C^{m,\lambda}(\bar\Omega)\big)^n \times \big(C^{m+1,\lambda}(\partial\Omega)\big)^n$.

Let us set

$$\begin{cases} V_{m,p} = \Big\{v \in \big(W^{m+2,p}(\Omega)\big)^n \colon \displaystyle\int_\Omega v\,dx = 0,\ \int_\Omega (D_j v_i - D_i v_j)\,dx = 0, \quad i, j = 1, \dots, n\Big\} \\ F_{m,p} = \big\{(f, g) \in \big(W^{m,p}(\Omega)\big)^n \times \big(W^{m+1-1/p,p}(\partial\Omega)\big)^n \colon (f, g) \text{ is equilibrated}\big\}, \end{cases}$$

and

$$\begin{cases} V_{m,\lambda} = \Big\{v \in \big(C^{m+2,\lambda}(\bar\Omega)\big)^n \colon \displaystyle\int_\Omega v\,dx = 0,\ \int_\Omega (D_j v_i - D_i v_j)\,dx = 0, \quad i, j = 1, \dots, n\Big\} \\ F_{m,\lambda} = \big\{(f, g) \in \big(C^{m,\lambda}(\bar\Omega)\big)^n \times \big(C^{m+1,\lambda}(\partial\Omega)\big)^n \colon (f, g) \text{ is equilibrated}\big\}. \end{cases}$$

We now give the statements of the main results.

THEOREM 3.1. *Assume that Ω is of class C^{m+2}, that $p(m+1) > n$, that $a \in \big(C^{m+2}(\bar\Omega \times \mathbb{R}^{n^2})\big)^{n^2}$ and that* (3.1), (3.3), (3.4) *and* (3.6) *apply. Moreover, let $(f, g) \in F_{m,p}$ be such that, if $c_1, \dots, c_n$ are the eigenvalues of the matrix c defined by* (3.7), *then $c_i + c_j \neq 0$ whenever $i \neq j$.*

Then there exist two positive numbers r *and* ϱ *such that there is one and only one family* $(u_\vartheta)_{\vartheta\in[-r,r]\setminus\{0\}}$, *with* $u_\vartheta\in\left(W^{m+2,p}(\Omega)\right)^n$, *satisfying the conditions* $\int_\Omega u_\vartheta\,dx=0$, $\|u_\vartheta\|_{m+2,p}\leqslant\varrho$ *and*

$$\text{(3.8)}\qquad\begin{cases}\operatorname{div}A(u_\vartheta)+\vartheta f=0\\[2mm] -A(u_\vartheta)\nu+\vartheta g=0\end{cases}$$

for every $\vartheta\in[-r,r]\setminus\{0\}$. *Furthermore, if we set* $u_0=0$, *then the map* $\vartheta\mapsto u_\vartheta$ *of* $[-r,r]$ *into* $\left(W^{m+2,p}(\Omega)\right)^n$ *is continuously differentiable.*

COROLLARY 3.1. *Under the hypotheses of the previous theorem, there exist two positive numbers* r *and* ϱ *such that, if* $0<|\vartheta|\leqslant r$, *Problem* (P) *has one and only one solution* $u\in\left(W^{m+2,p}(\Omega)\right)^n$ *such that* $\int_\Omega u\,dx=0$ *and* $\|u\|_{m+2,p}\leqslant\varrho$.

THEOREM 3.2. *Assume that* Ω *is connected and of class* C^{m+3}, *that* $a\in\left(C^{m+3}(\bar\Omega\times\mathbf{R}^{n^2})\right)^{n^2}$ *and that* (3.1), (3.3), (3.4) *and* (3.6) *apply. Moreover, let* $(f,g)\in F_{m,\lambda}$ *be such that, if* $c_1,\ldots,c_n$ *are the eigenvalues of the matrix* c *defined by* (3.7), *then* $c_i+c_j\neq0$ *whenever* $i\neq j$.

Then there exist two positive numbers r *and* ϱ *such that there is one and only one family* $(u_\vartheta)_{\vartheta\in[-r,r]\setminus\{0\}}$, *with* $u_\vartheta\in\left(C^{m+2,\lambda}(\bar\Omega)\right)^n$, *satisfying the conditions* $\int_\Omega u_\vartheta\,dx=0$, $\|u_\vartheta\|_{m+2,\lambda}\leqslant\varrho$, *and* (3.8) *for every* $\vartheta\in[-r,r]\setminus\{0\}$. *Furthermore, if we set* $u_0=0$, *then the map* $\vartheta\mapsto u_\vartheta$ *of* $[-r,r]$ *into* $\left(C^{m+2,\lambda}(\bar\Omega)\right)^n$ *is continuously differentiable.*

COROLLARY 3.2. *Under the hypotheses of the previous theorem, there exist two positive numbers* r *and* ϱ *such that, if* $0<|\vartheta|\leqslant r$, *Problem* (P) *has one and only one solution* $u\in\left(C^{m+2,\lambda}(\bar\Omega)\right)^n$ *such that* $\int_\Omega u\,dx=0$ *and* $\|u\|_{m+2,\lambda}\leqslant\varrho$.

Corollaries 3.1 and 3.2 are straightforward consequences of Theorems 3.1 and 3.2 respectively. The proof of Theorem 3.1 is given in section 7. Theorem 3.2 can be proved in a quite analogous way (using Theorems 5.2 and 6.1); therefore we will not give its proof.

4. Preliminary lemmas.

If the functions $a:\bar\Omega\times\mathbf{R}^{n^2}\to\mathbf{R}^{n^2}$ and $u:\bar\Omega\to\mathbf{R}^n$ are suitably smooth, we consider the functions $A_{ij}(u):\bar\Omega\to\mathbf{R}$ and $A_{ij,hk}(u):\bar\Omega\to\mathbf{R}$, $(i,j,$

$h, k = 1, \ldots, n$), defined by

$$A_{ij}(u)(x) = a_{ij}(x, 1 + Du(x)) \,, \qquad A_{ij,hk}(u)(x) = \frac{\partial a_{ij}}{\partial y_{hk}}(x, 1 + Du(x)) \,.$$

By considering that (see Adams [1], Th. 5.4) $W^{m+2,p}(\Omega)$ can be continuously imbedded into $C^1_B(\Omega)$ when Ω has the cone property and $p(m+1) > n$, it is easy to prove the following.

LEMMA 4.1. *Assume that Ω has the cone property and that $a \in (C^1(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$. Then $u \mapsto \left(\int_\Omega (A_{ij}(u)\, dx)\right)_{i,j=1,\ldots,n}$ is a continuously differentiable operator of $(W^{m+2,p}(\Omega))^n$ into $\mathbf{R}^{n^2}$ when $p(m+1) > n$* [in particular, of $(C^{m+2,\lambda}(\bar{\Omega}))^n$ to $\mathbf{R}^{n^2}$] *and its differential at any point u is the operator*

$$v \mapsto \left(\int_\Omega A_{ij,hk}(u)\, D_k v_h\, dx\right)_{i,j=1,\ldots,n}$$

We now recall the statements of two differentiability results holding for Sobolev and Schauder spaces (see [15], Cor. 5.1 and Cor. 5.2).

LEMMA 4.2. *Assume that Ω has the cone property, that $p(m+1) > n$ and that $a \in (C^{m+2}(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$* [resp. *that Ω is connected and of class C^1 and that $a \in (C^{m+3}(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$*]. *Then $u \mapsto \operatorname{div} A(u)$ is a continuously differentiable operator of $(W^{m+2,p}(\Omega))^n$ into $(W^{m,p}(\Omega))^n$* [resp. *of $(C^{m+2,\lambda}(\bar{\Omega}))^n$ into $(C^{m,\lambda}(\bar{\Omega}))^n$*] *and its differential at any point is the operator $v \mapsto \left(D_j(D_k v_h A_{ij,hk}(u))\right)_{i=1,\ldots,n}$.*

LEMMA 4.3. *Assume that Ω is of class C^{m+2}, that $p(m+1) > n$ and that $a \in (C^{m+2}(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$* [resp. *that Ω is connected and of class $C^{m+2,\lambda}$ and that $a \in (C^{m+3}(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$*]. *Then $u \mapsto A(u)\nu$ is a continuously differentiable operator of $(W^{m+2,p}(\Omega))^n$ into $(W^{m+1-1/p,p}(\partial\Omega))^n$* [resp. *of $(C^{m+2,\lambda}(\bar{\Omega}))^n$ into $(C^{m+1,\lambda}(\partial\Omega))^n$*] *and its differential at 0 is the operator $v \mapsto (\nu_j D_k v_h A_{ij,hk}(0))_{i=1,\ldots,n}$.*

PROOF. In [14] (see Theorems 1 and 2) it has been shown that, under our assumptions, $u \mapsto Au$ is a continuously differentiable operator of $(W^{m+2,p}(\Omega))^n$ into $(W^{m+1,p}(\Omega))^{n^2}$ [resp. of $(C^{m+2,\lambda}(\bar{\Omega}))^n$ into $(C^{m+1,\lambda}(\bar{\Omega}))^{n^2}$] and its differential at any point u is the operator $v \mapsto (D_k v_h A_{ij,hk}(u))_{i,j=1,\ldots,n}$. Thus, it suffices to show that the linear operator $\tau_j\colon w \mapsto w|_{\partial\Omega}\nu_j$, $(j = 1, \ldots, n)$, where $w \in W^{m+1,p}(\Omega)$ [resp. $w \in C^{m+1,\lambda}(\bar{\Omega})$], maps $W^{m+1,p}(\Omega)$ into $W^{m+1-1/p,p}(\partial\Omega)$ [resp. $C^{m+1,\lambda}(\bar{\Omega})$ into $C^{m+1,\lambda}(\partial\Omega)$] and is continuous. For this purpose we first consider the case of the Sobolev spaces. Let $N\colon \mathbf{R}^n \to \mathbf{R}^n$ be a function of class C^{m+1} extending ν. Evidently, if $w \in W^{m+1,p}(\Omega)$, then $wN_j \in W^{m+1,p}(\Omega)$ and there exists a positive number c_j independent of w such that $\|wN_j\|_{m+1,p} \leqslant c_j\|w\|_{m+1,p}$. Since $wN_j \in W^{m+1,p}(\Omega)$, then, by a well-known trace theorem (see Lions and Magenes [8], Theorem 5.1), $(wN_j)|_{\partial\Omega} \in W^{m+1-1/p,p}(\partial\Omega)$ and there exists a positive number c_j' independent of w such that $\|(wN_j)|_{\partial\Omega}\|_{m+1-1/p,p,\partial\Omega} \leqslant c_j'\|wN_j\|_{m+1,p}$, where $\|\cdot\|_{m+1-1/p,p,\partial\Omega}$ denotes a norm on $W^{m+1-1/p,p}(\partial\Omega)$ defining its topology. Hence, observing that $wN_j|_{\partial\Omega} = w|_{\partial\Omega}\nu_j$, we have $\|w|_{\partial\Omega}\nu_j\|_{m+1-1/p,p,\partial\Omega} \leqslant c_j c_j'\|w\|_{m+1,p}$. Thus τ_j is continuous from $W^{m+1,p}(\Omega)$ to $W^{m+1-1/p,p}(\partial\Omega)$. We now prove that the operator τ_j is continuous from $C^{m+1,\lambda}(\bar{\Omega})$ to $C^{m+1,\lambda}(\partial\Omega)$. Accordingly, for any $x \in \partial\Omega$ let U_x and t_x be respectively a neighborhood of x in $\mathbf{R}^n$ and a differomorphism of class $C^{m+1,\lambda}$ of $\bar{U}_x$ onto the ball $\{\xi \in \mathbf{R}^n\colon |\xi| \leqslant 1\}$ such that $t_x(\bar{\Omega} \cap U_x) = \{\xi \in \mathbf{R}^n\colon |\xi| < 1,\ \xi_n \geqslant 0\}$; obviously $w \circ t_x^{-1} \in C^{m+1,\lambda}(\sigma)$, where $\sigma = \{\xi \in \mathbf{R}^n\colon |\xi| \leqslant 1,\ \xi_n = 0\}$. Now, $\nu_j \in C^{m+1,\lambda}(\partial\Omega)$ and therefore $\nu_j \circ t_x^{-1}$ is an element of $C^{m+1,\lambda}(\sigma)$. Since $C^{m+1,\lambda}(\sigma)$ is a Banach algebra (see section 2), we have $(w|_{\partial\Omega}\nu_j)\circ t_x^{-1} \in C^{m+1,\lambda}(\sigma)$ and $\|(w|_{\partial\Omega}\nu_j)\circ t_x^{-1}\|_{m+1,\lambda,\sigma} \leqslant c_{m+1,\lambda}\|w \circ t_x^{-1}\|_{m+1,\lambda,\sigma}\|\nu_j \circ t_x^{-1}\|_{m+1,\lambda,\sigma}$, where $c_{m+1,\lambda}$ is a positive number independent of w. Then

$$\|w|_{\partial\Omega}\nu_j\|_{m+1,\lambda,\sigma} = \sup_{x\in\partial\Omega} \|(w|_{\partial\Omega}\nu_j)\circ t_x^{-1}\|_{m+1,\lambda,\sigma} \leqslant$$

$$\leqslant \sup_{x\in\partial\Omega} \|w \circ t_x^{-1}\|_{m+1,\lambda,\sigma} \sup_{x\in\partial\Omega} \|\nu_j \circ t_x^{-1}\|_{m+1,\lambda,\sigma} = \|w|_{\partial\Omega}\|_{m+1,\lambda,\partial\Omega}\|\nu_j\|_{m+1,\lambda,\partial\Omega}\,.$$

So the continuity of τ_j from $C^{m+1,\lambda}(\bar{\Omega})$ to $C^{m+1,\lambda}(\partial\Omega)$ is proved. □

5. Local theorems of existence and uniqueness for a modified boundary value problem.

For technical reasons we always assume (without loss of generality) that the astatic matrix $c = (c_{ij})_{i,j=1,\ldots,n}$ of (f, g) defined by (3.7) is diagonal.

Let $h = (h_i)_{i=1,\ldots,n}\colon \Omega \to \mathbf{R}^n$ be a C^∞-function verifying the following conditions:

$$\int_\Omega h_i(x) = 0\,, \qquad (i = 1, \ldots, n)$$

$$\int_\Omega x_i h_j(x)\, dx = 0\,, \qquad (i, j = 1, \ldots, n;\ i \neq j)$$

$$\int_\Omega \big(x_i h_i(x) + x_j h_j(x)\big)\, dx \neq 0\,, \qquad (i, j, = 1, \ldots, n;\ i \neq j)\,,$$

where no summation is understood.

Such a function certainly exists.

PROPOSITION 5.1. *Assume that Ω is of class C^1 and that $a \in \big(C^1(\bar{\Omega} \times \times \mathbb{R}^{n^2})\big)^{n^2}$. There exists one and only one operator $E = (E_{ij})_{i,j=1,\ldots,n}$ of $\big(W^{m+2,p}(\Omega)\big)^n$ with $p(m+1) > n$ into $\mathbb{S}$ such that the pair $\big(\operatorname{div} A(u) + E(u)h, -A(u)v\big)$ is equilibrated* (see sect. 3). *Here*

$$E(u)h = \big(E_{ij}(u)h_j\big)_{i=1,\ldots,n}\,.$$

Operator E is defined by

$$E_{ij}(u) = \frac{1}{\int_\Omega \big(x_i h_i(x) + x_j h_j(x)\big)\, dx} \cdot \int_\Omega \Big(a_{ij}\big(x, 1 + Du(x)\big) - a_{ji}\big(x, 1 + Du(x)\big)\Big)\, dx\,, \tag{5.1}$$

(where no summation with respect to i and j is understood). *Moreover, $u \mapsto E(u)$ is a continuously differentiable operator from $\big(W^{m+2,p}(\Omega)\big)^n$ to $\mathbb{S}$ when $p(m+1) > n$, and consequently from $\big(C^{m+2,\lambda}(\bar{\Omega})\big)^n$ to $\mathbb{S}$.*

PROOF. Fix arbitrarily $u \in \big(W^{m+2,p}(\Omega)\big)^n$ with $p(m+1) > n$. Note that, being $p(m+1) > n$, $W^{m+2,p}(\Omega)$ can be continuously imbedded into $C^1(\bar{\Omega})$ (see Adams [1], Th. 5.4), and therefore it is easily seen that $Au \in \big(W^{1,p}(\Omega)\big)^{n^2}$. Let now $s = (s_{ij})_{i,j=1,\ldots,n} \in \mathbb{S}$ and set $sh = (s_{ij}h_j)_{i=1,\ldots,n}$. Imposing that the pair $\big(\operatorname{div} A(u) + sh, -A(u)v\big)$ be equilibrated and recalling the properties of h, we deduce with ease that s_{ij} is exactly the right hand-side of (5.1).

The second part of the statement follows immediately from Lemma 4.1. □

If $v \in V_{m,p}$ (in particular if $v \in V_{m,\lambda}$) and $s = (s_{ij})_{i,j=1,\dots,n} \in \mathbb{S}$, we will denote by $\psi_i(v, s)$, $(i = 1, \dots, n)$, the real function defined in Ω by

$$\psi_i(v, s)(x) = v_i(x) + s_{ij}x_j\,. \tag{5.2}$$

THEOREM 5.1. *Assume that Ω is of class C^{m+2}, that $p(m+1) > n$, that $a \in \big(C^{m+2}(\bar{\Omega}\times\mathbf{R}^{n^2})\big)^{n^2}$ and that a verifies* (3.4), (3.5) *and* (3.6). *If $(f, g) \in F_{m,p}$, then there exist an open neighborhood W of 0 in $V_{m,p}\times \mathbb{S}\times\mathbf{R}$, an open neighborhood S of 0 in $\mathbb{S}$, an open neighborhood R of 0 in $\mathbf{R}$ and a C^1 map $\hat{v}$ of $S\times R$ into $V_{m,p}$ such that*

$$\begin{aligned}\{(v, s, \vartheta) \in W\colon \big(\operatorname{div} A(\psi(v, s)) + E(\psi(v, s))h + \vartheta f,\\ -A(\psi(v, s))\nu + \vartheta g\big) = (0, 0)\} = \{(v, s, \vartheta)\colon (s, \vartheta) \in S\times R,\ v = \hat{v}(s, \vartheta)\}\,,\end{aligned} \tag{5.3}$$

where $\psi = (\psi_i)_{i=1,\dots,n}$, and ψ_i is defined by (5.2).

PROOF. Let $(f, g) \in F_{m,p}$: We begin by observing that, from Lemmas 4.1, 4.2 and 4.3 it follows that the operator

$$\begin{aligned}(v, s, \vartheta) \mapsto \big(\operatorname{div} A(\psi(v, s)) + E(\psi(v, s))h + \vartheta f,\\ -A(\psi(v, s))\nu + \vartheta g\big)\end{aligned} \tag{5.4}$$

maps $V_{m,p}\times\mathbb{S}\times\mathbf{R}$ into $F_{m,p}$, is continuously differentiable and the differential at zero of the operator $v \mapsto \big(\operatorname{div} A(\psi(v, 0)) + E(\psi(v, 0))h + 0f, -A(\psi(v, 0))\nu + 0g\big)$ is (by the hypothesis (3.5)) the operator

$$v \mapsto \Big(\big(D_j(D_k v_h A_{ij,hk}(0))\big)_{i=1,\dots,n}\,,\ \big(-\nu_j D_k v_h A_{ij,hk}(0)\big)_{i=1,\dots,n}\Big)\,. \tag{5.5}$$

Note that $A_{ij,hk}(0) \in C^{m+1}(\bar{\Omega})$. Then from Theorem A.1 of the appendix it follows that the operator (5.5) is an isomorphism of $V_{m,p}$ onto $F_{m,p}$, because of the hypoteses (3.4), (3.5) and (3.6). Furthermore operator (5.4) vanishes at $(0, 0, 0)$, and hence we can apply the implicit function theorem to the equation $\big(\operatorname{div} A(\psi(v, s)) + E(\psi(v, s))h + \vartheta f, -A\psi(v, s)\nu + \vartheta g\big) = (0, 0)$ near the point $(0, 0, 0)$, so obtaining the desired result. □

Analogously, using Lemmas 4.1, 4.2 and 4.3 and Theorem A.2 of the appendix, we can prove the following

THEOREM 5,2. *Assume that* Ω *is connected and of class* $C^{m+2,\lambda}$, *that* $a \in (C^{m+3}(\bar{\Omega} \times \mathbb{R}^{n^2}))^{n^2}$ *and that* a *verifies* (3.4), (3.5) *and* (3.6). *If* $(f, g) \in F_{m,\lambda}$, *then there exist an open neighborhood* W *of* 0 *in* $V_{m,\lambda} \times \mathbb{S} \times \mathbb{R}$, *an open neighborhood* S *of* 0 *in* $\mathbb{S}$, *an open neighborhood* R *of* 0 *in* $\mathbb{R}$ *and a* C^1 *map* $\hat{v}$ *of* $S \times R$ *into* $V_{m,\lambda}$ *such that* (5.3) *holds.*

6. A relation between the set of solutions of Problem (P) and the set of solutions of the modified problem.

LEMMA 6.1. *Assume that* Ω *is of class* C^1 *and that* $a \in (C^1(\bar{\Omega} \times \mathbb{R}^{n^2}))^{n^2}$. *Moreover let* $u \in (W^{m+2,p}(\Omega))^n$, *with* $p(m+1) > n$, *and* $(f, g) \in F_{m,p}$ (in particular, $u \in (C^{m+2,\lambda}(\bar{\Omega}))^n$ and $(f, g) \in F_{m,\lambda}$). *Then from*

$$(6.1) \qquad \big(\operatorname{div} A(u) + \vartheta f, -A(u)\nu + \vartheta g\big) = (0, 0)$$

it follows that

$$(6.2) \qquad \int_\Omega \big(a_{ij}(x, 1 + Du(x)) - a_{ji}(x, 1 + Du(x))\big)\, dx = 0\,, \qquad \forall i, j = 1, \dots, n$$

and therefore (see (5.1)) (6.1) *implies*

$$(6.3) \qquad \big(\operatorname{div} A(u) + E(u)h + \vartheta f, -A(u)\nu + \vartheta g\big) = (0, 0)\,.$$

Moreover, (6.3) *implies* (6.1) *if and only if* (6.2) *holds.*

PROOF. Using the divergence theorem, from (6.1) we easily deduce that

$$\vartheta \int_\Omega \big(x_i f_j(x) - x_j f_i(x)\big)\, dx + \vartheta \int_{\partial\Omega} \big(x_i g_j(x) - x_j g_i(x)\big)\, d\sigma = \int_\Omega \big(a_{ji}(x, 1 + Du(x)) - a_{ij}(x, 1 + Du(x))\big)\, dx\,.$$

Then (6.1) implies (6.2), because (f, g) is equilibrated. The second part of the statement is an immediate consequence of (5.1). □

THEOREM 6.1. *Let Ω be of class C^1 and $a \in (C^1(\bar{\Omega} \times \mathbf{R}^{n^2}))^{n^2}$ such that* (3.3) *applies. Assume that $(f, g) \in F_{m,p}$ with $p(m+1) > n$* [*respectively that $(f, g) \in F_{m,\lambda}$*].

Then there exist a neighborhood V' of 0 in $V_{m,p}$ [respectively *in $V_{m,\lambda}$*] *and a neighborhood S' of 0 in $\mathcal{S}$ such that, if $(v, s) \in V' \times S'$ and $\vartheta \neq 0$, then*

$$(6.4) \qquad \Big(\operatorname{div} A\big(\psi(v, s)\big) + E\big(\psi(v, s)\big)h + \vartheta f, -A\big(\psi(v, s)\big)\nu + \vartheta g\Big) = (0,0)$$

implies

$$(6.5) \qquad \Big(\operatorname{div} A\big(\psi(v, s)\big) + \vartheta f, -A\big(\psi(v, s)\big)\nu + \vartheta g\Big) = (0, 0)$$

if and only if the pair (v, s) verifies

$$(6.6) \qquad s_{ij}\Big(\int_\Omega \big(x_i f_i(x) + x_j f_j(x)\big)\, dx + \int_{\partial\Omega} \big(x_i g_i(x) + x_j g_j(x)\big)\, d\sigma\Big) + \\ + \int_\Omega \big(v_i(x) f_j(x) - v_j(x) f_i(x)\big)\, dx + \int_{\partial\Omega} \big(v_i(x) g_j(x) - v_j(x) g_i(x)\big)\, d\sigma = 0\,, \\ \forall i, j = 1, \ldots, n\,,$$

where no summation with respect to the indices i and j is understood. *Sufficiency of condition* (6.6) *holds in the case $\vartheta = 0$ too.*

PROOF. We will prove the theorem for Sobolev spaces. In the case of Schauder spaces, the procedure is exactly the same.

First of all we remark that (6.6) is equivalent to

$$(6.7) \qquad \int_\Omega \Big(\big(x_i + v_i(x) + s_{il}x_l\big) f_j(x) - \big(x_j + v_j(x) + s_{jl}x_l\big) f_i(x)\Big)\, dx + \\ + \int_{\partial\Omega} \Big(\big(x_i + v_i(x) + s_{il}x_l\big) g_j(x) - \big(x_j + v_j(x) + s_{jl}x_l\big) g_i(x)\Big)\, d\sigma = 0\,, \\ \forall i, j = 1, \ldots, n\,.$$

To justify this equivalence it suffices to recall that we have assumed that (f, g) is equilibrated and that the astatic matrix of (f, g), defined by (3.7), is diagonal.

We now prove the existence of two neighborhoods V' and S' of 0 in $V_{m,p}$ and S respectively such that if $(v, s) \in V' \times S'$ then (6.4) implies (6.5). Let $(v, s, \vartheta) \in V_{m,p} \times \mathrm{S} \times \mathbb{R}$. Using the divergence theorem, it is not difficult to verify that from (6.4) it follows that

$$\vartheta \int_{\Omega} \Big((x_i + v_i(x) + s_{il} x_l) f_j(x) - (x_j + v_j(x) + s_{jl} x_l) f_i(x) \Big) \, dx +$$
$$+ \vartheta \int_{\partial\Omega} \Big((x_i + v_i(x) + s_{il} x_l) g_j(x) - (x_j + v_j(x) + s_{jl} x_l) g_i(x) \Big) \, d\sigma =$$
$$= \int_{\Omega} \Big((\delta_{ih} + D_h \psi_i(v, s)) A_{jh}(\psi(v, s)) - (\delta_{jh} + D_h \psi_j(v, s)) A_{ih}(\psi(v, s)) \Big) \, dx -$$
$$- \int_{\Omega} \Big((x_i + v_i(x) + s_{il} x_l) E_{jk}(\psi(v, s)) h_k(x) -$$
$$- (x_j + v_j(x) + s_{jl} x_l) E_{ik}(\psi(v, s)) h_k(x) \Big) \, dx ,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The penultimate integral vanishes because of the hypothesis (3.3). Therefore, from (6.4) it follows that

$$\vartheta \int_{\Omega} \Big((x_i + v_i(x) + s_{il} x_l) f_j(x) - (x_j + v_j(x) + s_{jl} x_l) f_i(x) \Big) \, dx +$$
$$+ \vartheta \int_{\partial\Omega} \Big((x_i + v_i(x) + s_{il} x_l) g_j(x) - (x_j + v_j(x) + s_{jl} x_l) g_i(x) \Big) \, d\sigma =$$
$$= - \int_{\Omega} \Big((x_i + v_i(x) + s_{il} x_l) E_{jk}(\psi(v, s)) h_k(x) -$$
$$- (x_j + v_j(x) + s_{jl} x_l) E_{ik}(\psi(v, s) h_k(x) \Big) \, dx .$$

Thus, if $(v, s, \vartheta) \in V_{m,p} \times \mathrm{S} \times \mathbb{R}$ and (6.4) applies, then (6.7), or (6.6) as well, gives

$$E_{jk}(\psi(v, s)) \varphi_{ik}(v, s) - E_{ik}(\psi(v, s)) \varphi_{jk}(v, s) = 0 , \quad (i, j = 1, \dots, n) , \tag{6.8}$$

where

$$\varphi_{ik}(v, s) = \int_{\Omega} (x_i + v_i(x) + s_{il} x_l) h_k(x) \, dx . \tag{6.9}$$

We shall consider for any $(v, s) \in V_{m,p} \times \mathbb{S}$ the linear operator $\pi(v, s)$ $\mathbb{S} \to \mathbb{S}$ defined by

$$\pi(v, s)\big((w_{ij})_{i,j=1,\dots,n}\big) = \big(w_{jk}\varphi_{ik}(v, s) - w_{ik}\varphi_{jk}(v, s)\big)_{i,j=1,\dots,n}.$$

The determinant of this operator is a continuous function in $V_{m,p} \times \mathbb{S}$, because the functions φ_{ik} defined by (6.9) are evidently continuous in $V_{m,p} \times \mathbb{S}$. Moreover, since it is easy to verify that

$$\det \pi(0, 0) = \prod_{i<j} \int_\Omega \big(h_i(x)x_i + h_j(x)x_j\big)\, dx\,,$$

then $\det \pi(0, 0) \neq 0$ (because of the hypotheses made on h at the beginning of section 5).

Therefore there exist V' and S', where V' is a neighborhood of 0 in $V_{m,p}$ and S' a neighborhood of 0 in $\mathbb{S}$, such that $\det \pi(v, s) \neq 0$, $\forall (v, s) \in V' \times S'$. Then, if $(v, s) \in V' \times S'$, the kernel of $\pi(v, s)$ is trivial. Hence by (6.8) we have $E_{ij}\big(\psi(v, s)\big) = 0$, $(i, j = 1, \dots, n)$, for any $(v, s) \in V' \times S'$. By Lemma 6.1 this is enough to conclude that condition (6.6) is sufficient in order that (6.4) imply (6.5) when $(v, s, \vartheta) \in V' \times S' \times \mathbb{R}$.

We now prove that condition (6.6) is necessary in order that (6.4) imply (6.5) at any $(v, s, \vartheta) \in V' \times S' \times \mathbb{R}$ with $\vartheta \neq 0$. We note that by calculations analogous to those developed at the beginning of the proof, from (6.5) it follows that

$$\vartheta \int_\Omega \Big(\big(x_i + v_i(x) + s_{il}x_l\big)f_j(x) - \big(x_j + v_j(x) + s_{jl}x_l\big)f_i(x)\Big)\, dx +$$

$$+ \vartheta \int_{\partial\Omega} \Big(\big(x_i + v_i(x) + s_{il}x_l\big)g_j(x) - \big(x_j + v_j(x) + s_{jl}x_l\big)g_i(x)\Big)\, d\sigma =$$

$$= \int_\Omega \Big(\big(\delta_{ik} + D_h\psi_i(v, s)\big)A_{jh}\big(\psi(v, s)\big) - \big(\delta_{jk} + D_h\psi_j(v, s)\big)\, A_{ih}\big(\psi(v, s)\big)\Big)\, dx\,,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

To complete the proof it sufficies to note that the last integral vanishes in virtue of hypothesis (3.3) on the function a. □

7. Proof of Theorem 3.1.

Let W, S, R and $\hat{v}$: $S\times R \to V_{m,p}$ be as in the statement of Theorem 5.1, and let V' and S' be as in the statement of Theorem 6.1. Without loss of generality we can assume that V' and S' are such that $S'\subseteq S$, $V'\times S'\times R'\subseteq W$ and $\hat{v}(S'\times R')\subseteq V'$, where R' is a suitable neighborhood of 0 in $\mathbb{R}$ with $R'\subseteq R$. Then, if we set $W' = V'\times S'\times R'$, from Theorem 5.1 we evidently derive that

$$\text{(7.1)}\qquad \{(v,s,\vartheta)\in W'\colon \big(\operatorname{div} A(\psi(v,s)) + E(\psi(v,s))h + \vartheta f,$$
$$-A(\psi(v,s))\nu + \vartheta g\big) = (0,0)\} = \{(v,s,\vartheta)\colon (s,\vartheta)\in S'\times R',\ v = \hat{v}(s,\vartheta)\}\,.$$

To simplify the notation we set $M(v,s,\vartheta) = \big(\operatorname{div} A(\psi(v,s)) + E(\psi(v,s))h + \vartheta f, -A(\psi(v,s))\nu + \vartheta g\big)$ for any $(v,s,\vartheta)\in V_{m,p}\times \mathbb{S}\times\mathbf{R}$.

We remember that we have assumed (see section 5) that the matrix c defined by (3.7) is diagonal.

We now set $b_{ij} = c_{ii} + c_{jj}$, $(i,j = 1,\dots,n)$, where no summation is to be understood, and consider the function τ: $S'\times R'\to \mathbb{S}$ defined by

$$\tau(s,\vartheta) = \Big(s_{ij}b_{ij} + \int_\Omega (\hat{v}_i(s,\vartheta)f_j - \hat{v}_j(s,\vartheta)f_i)\,dx + \int_{\partial\Omega}(\hat{v}_i(s,\vartheta)g_j - \hat{v}_j(s,\vartheta)g_i)\,d\sigma\Big)_{i,j=1,\dots,n},$$

where no summation is to be understood. We remark that condition (6.6) of Theorem (6.1) takes the form $\tau(s,\vartheta) = 0$. τ is of class C^1, being a composite of functions of class C^1; furthermore $\tau(0,0) = 0$.

We denote by $d_s\tau(0,0)$ the differential at 0 of the function $s\mapsto \tau(s,0)$ of S' into $\mathbb{S}$, by $d_s\hat{v}(0,0)$ the differential at 0 of the function $s\mapsto \hat{v}(s,0)$ of S' into $V_{m,p}$, by $d_vM(0,0,0)$ the differential at 0 of the function $v\mapsto M(v,0,0)$ of $V_{m,p}$ into $F_{m,p}$, and by $d_{(s,\vartheta)}M(0,0,0)$ the differential at $(0,0)$ of the function $(s,\vartheta)\mapsto M(0,s,\vartheta)$ of $\mathbb{S}\times\mathbf{R}$

into $F_{m,p}$. If $s = (s_{ij})_{i,j=1,\dots,n} \in \mathrm{S}$, we have

$$d_s\tau(0,0)(s) = \Big(s_{ij}b_{ij} + \int_\Omega \big((d_s\hat{v}(0,0)(s))_i f_j(x) - (d_s\hat{v}(0,0)(s))_j f_i(x)\big)\,dx + \int_{\partial\Omega}\big((d_s\hat{v}(0,0)(s))_i g_j(x) - (d_s\hat{v}(0,0)(s))_j g_i(x)\big)\,d\sigma\Big)_{i,j=1,\dots,n} = (s_{ij}b_{ij})_{i,j=1,\dots,n},$$

where no summation with respect to the indices i and j is to be understood. In fact, since the function $\hat{v}$ is of class C^1, we have (see the statement of the implicit function theorem in section 2)

$$d_s\hat{v}(0,0)(s) = d\hat{v}(0,0)(s,0) = -\Big(\big(d_v M(0,0,0)\big)^{-1}\circ d_{(s,\vartheta)}M(0,0,0)\Big)(s,0)$$

and hence $d_s\hat{v}(0,0)(s) = 0$, since $A_{ij,hk}(0) = A_{ij,kh}(0) = 0$. The last equality derives from the hypothesis (3.6).

We recall that we have supposed $b_{ij} \neq 0$ when $i \neq j$; hence the function $s \mapsto d_s\tau(0,0)(s)$ of S into S is an isomorphism. Consequently, by the implicit function theorem, there exist an open neighborhood R'' of 0 in $\mathbb{R}$, contained in R', an open neighborhood S'' of 0 in S contained in S', and a function of class C^1 $\vartheta \mapsto \hat{s}(\vartheta)$ from R'' to S' such that

$$\{(s,\vartheta) \in S'' \times R'' \colon \tau(s,\vartheta) = 0\} = \{(s,\vartheta) \colon \vartheta \in R'',\ s = \hat{s}(\vartheta)\}. \tag{7.2}$$

Then let us set $u_\vartheta = \psi\big(\hat{v}(\hat{s}(\vartheta),\vartheta),\hat{s}(\vartheta)\big)$. The function $\vartheta \mapsto u_\vartheta$ is continuously differentiable from R'' to $\big(W^{m+2,p}(\Omega)\big)^n$ since it is composite of functions of class C^1. Moreover $u_0 = 0$.

Fixed (arbitrarely for now) a positive number ϱ, let r be a positive number such that $[-r,r] \subseteq R''$ and $\|u_\vartheta\|_{m+2,p} \leq \varrho$, $\forall \vartheta \in [-r,r]$. Since $\hat{s}$ maps $R''(\subseteq R')$ into S', by the definition of u_ϑ we derive that $u_\vartheta = \psi(v_\vartheta, s_\vartheta)$, with $v_\vartheta = \hat{v}(s_\vartheta,\vartheta)$ and $(s_\vartheta,\vartheta) \in S' \times R'$. Then, from (7.1) we get $\big(\operatorname{div} A(u_\vartheta) + E(u_\vartheta)h + \vartheta f, -A(u_\vartheta)\nu + \vartheta g\big) = (0,0)$. Moreover from (7.2) it follows that $\tau\big(\hat{s}(\vartheta),\vartheta\big) = 0$ and therefore by Theorem 6.1, (3.8) holds. Furthermore we have $\int_\Omega u_\vartheta\,dx = 0$, because $\int_\Omega v_\vartheta\,dx = 0$ and $\int_\Omega x_i\,dx = 0$, $\forall i = 1,\dots,n$.

We now proceed to prove the uniqueness of $(u_\vartheta)_{\vartheta\in[-r,r]\setminus\{0\}}$, if ϱ is suitable.

Let π_j, $(j = 1, \ldots, n)$, be the functions of $\mathbb{R}^n$ into $\mathbb{R}$ defined by $\pi_j(x) = x_j$. Trivially $\pi_j \in W^{m+2,p}(\Omega)$.

Let $B\colon (W^{m+2,p}(\Omega))^n \times \mathbb{R} \to V_{m,p} \times \mathbb{S} \times \mathbb{R}$ be the function defined by setting

$$B(u, \vartheta) = \left(\left(u_i - \frac{1}{2 \operatorname{mis} \Omega} \pi_k \int_\Omega (D_k u_i - D_i u_k)\, dx\right)_{i=1,\ldots,n}, \right.$$
$$\left. \frac{1}{2 \operatorname{mis} \Omega}\left(\int_\Omega (D_k u_i - D_i u_k)\, dx\right)_{i,k=1,\ldots,n}, \vartheta\right).$$

Since this function is evidently continuous, then $B^{\leftarrow}(V' \times S'' \times R'')$ is an open neighborhood of the origin in $(W^{m+2,p}(\Omega))^n \times \mathbb{R}$. Hence there exists $\varrho > 0$ such that, if we set $\mathfrak{J} = \{(u, \vartheta) \in (W^{m+2,p}(\Omega))^n \times \mathbb{R}\colon \|u\|_{m+2,p} \leqslant \varrho,\ |\vartheta| \leqslant r\}$ where r is related to ϱ as above, then we have $\mathfrak{J} \subseteq B^{\leftarrow}(V' \times S'' \times R'')$. Therefore, if $(u_\vartheta)_{\vartheta \in [-r,r]}$ and $(u'_\vartheta)_{\vartheta \in [-r,r]}$ are such that $(u_\vartheta, \vartheta) \in \mathfrak{J}$, $(u'_\vartheta, \vartheta) \in \mathfrak{J}$ and such that (u_ϑ, ϑ) and $(u'_\vartheta, \vartheta)$ satisfy Problem (P), then by (7.1) and (7.2), it follows that $u_\vartheta = u'_\vartheta$, $\forall \vartheta \in \in [-r, r] \setminus \{0\}$. □

Appendix: isomorphism theorems for a linear matrix differential operator.

Let l_{ijhk}, $(i, j, h, k = 1, \ldots, n)$, be real functions defined in $\bar{\Omega}$. We consider the (linear) matrix differential operator $L = (L_{ih})_{i,h=1,\ldots,n}$, where

$$L_{ih} = -D_j(l_{ijhk} D_k)$$

and the boundary matrix operator $B = (B_{ih})_{i,h=1,\ldots,n}$, where

$$B_{ih} = l_{ijhk} \nu_j D_k\,.$$

Recall that ν is the unit outward normal to $\partial\Omega$ at any regular point of $\partial\Omega$. We set

$$T = (L, B) \tag{A.1}$$

and we remark that, putting $-(\partial a_{ij}/\partial y_{hk})(x, 1) = l_{ijhk}(x)$, the operator

(5.5) is exactly $v \mapsto Tv = \big((L_{ih}v_h)_{i=1,\dots,n}, (B_{ih}v_h)_{i=1,\dots,n}\big)$; so the conditions (3.4), (3.5) and (3.6) take the form

$$\text{(A.2)} \qquad \begin{cases} l_{ijhk} = l_{jihk} = l_{ijkh}\,, \\ l_{ijhk}(x)\,\sigma_{ij}\,\sigma_{hk} > 0 \quad \text{for every } x \in \bar{\Omega} \text{ and every } n\times n \text{ symmetric real matrix } \sigma = (\sigma_{ij})_{i,j=1,\dots,n}\,. \end{cases}$$

If we set $l^*_{ijhk} = l_{hkij}$, $L^*_{ih} = -D_j(l^*_{ijhk}D_k)$, $B^*_{ih} = l^*_{ijhk}\nu_j D_k$, and $T^* = (L^*, B^*)$, then T^* is the formal adjoint to T.

REMARK A.1. *Assume that Ω has the cone property. If the functions l_{ijhk} are continuous in $\bar{\Omega}$ and verify* (A.2), *then L is uniformly strongly elliptic and T has the complementing property* (as given by Agmon, Douglis and Nirenberg [2]).

PROOF. If the functions l_{ijhk} are continuous on $\bar{\Omega}$, from (A.2) it easily follows that there exists a positive number c_1 independent of x and ξ such that $l_{ijhk}(x)\sigma_{ij}\sigma_{hk} \geq c_1|\sigma|^2$ for every $x \in \bar{\Omega}$ and every $n\times n$ symmetric real matrix $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$. Clearly, this implies that L is uniformly strongly elliptic and that

$$\text{(A.3)} \qquad \int_\Omega l_{ijhk} D_j v_i D_k v_h\, dx \geq c_1 \sum_{i,j=1}^n \|D_j v_i + D_i v_j\|^2_{0,2}\,, \qquad \forall v \in \big(W^{1,2}(\Omega)\big)^n\,.$$

On the other hand, it is well-known (see Gobert [6]) that, if Ω has the cone property, then the following (Korn's) inequality holds:

$$\text{(A.4)} \qquad \|v\|^2_{1,2} \leq c_2\Big(\|v\|^2_{0,2} + \sum_{i,j=1}^n \|D_j v_i + D_i v_j\|^2_{0,2}\Big)\,, \qquad \forall v \in \big(W^{1,2}(\Omega)\big)^n\,,$$

where c_2 is a positive number independent of v. Combining (A.3) with (A.4) we get

$$\text{(A.5)} \qquad \int_\Omega l_{ijhk} D_j v_i D_k v_h\, dx + \|v\|^2_{0,2} \geq c_3 \|v\|^2_{1,2}\,, \qquad \forall v \in \big(W^{1,2}(\Omega)\big)^n\,,$$

where c_3 is a positive number independent of v. It is possible to prove (see Thompson [12], Theorem 12) that (A.5) implies that T has the complementing property. □

REMARK A.2. *Let Ω of class C^{m+2} and $l_{ijhk} \in C^{m+1}(\bar{\Omega})$* [resp. *$\Omega$ of class C^{m+3} and $l_{ijhk} \in C^{m+1,\lambda}(\bar{\Omega})$*]. *Then T and T^* are continuous operators from $(W^{m+2,p}(\Omega))^n$* [resp. $(C^{m+2,\lambda}(\bar{\Omega}))^n$] *to $(W^{m,p}(\Omega))^n \times (W^{m+1-1/p,p}(\partial\Omega))^n$* [resp. $(C^{m,\lambda}(\bar{\Omega}))^n \times (C^{m+1,\lambda}(\partial\Omega))^n$]. *Moreover, if* (A.2) *applies, then*

$$\text{Ker}\, T = \text{Ker}\, T^* = \mathfrak{R}\,. \tag{A.6}$$

PROOF. As far as the first part of the statement is concerned, we can refer to the proofs of Lemmas 4.2 and 4.3. Since $D_j r_i + D_i r_j = 0$, $\forall r \in \mathfrak{R}$, the symmetries $l_{ijhk} = l_{jihk}$ and $l_{ijhk} = l_{ijkh}$ imply, respectively, $\mathfrak{R} \subseteq \text{Ker}\, T$ and $\mathfrak{R} \subseteq \text{Ker}\, T^*$. In order to prove that $\text{Ker}\, T \subseteq \mathfrak{R}$ when (A.2) applies, we can suppose $p \geqslant 2$. Indeed, if $Tu = 0$ and $u \in (W^{m+2,p_1}(\Omega))^n$ for some $p_1 > 1$, then $u \in (W^{m+2,p}(\Omega))^n$, $\forall p > 1$, in virtue of a regularizing result of Browder (see [3], Theorem 1). Then let $p \geqslant 2$ and suppose that (A.2) applies. From $Tu = (f, g)$ it follows (by the divergence theorem) that

$$\int_\Omega f_i v_i \, dx + \int_{\partial\Omega} g_i v_i \, d\sigma = \int_\Omega l_{ijhk} D_k u_h D_j v_i \, dx\,, \qquad \forall v \in (W^{1,2}(\Omega))^n\,. \tag{A.7}$$

Combining (A.3) with (A.7) we get

$$\int_\Omega f_i u_i \, dx + \int_{\partial\Omega} g_i u_i \, d\sigma \geqslant c_1 \sum_{i,j=1}^{n} \|D_j u_i + D_i u_j\|_{0,2}^2$$

Hence, from $Tu = 0$ it follows that $D_j u_i + D_i u_j = 0$, $\forall i, j = 1, \ldots, n$, namely, that $u \in \mathfrak{R}$, because $\mathfrak{R}$ is the kernel of the operator $u \mapsto (D_j u_i + D_i u_j)_{i,j=1,\ldots,n}$ (see, e.g., Fichera [5]). Thus $\text{Ker}\, T \subseteq \mathfrak{R}$. Analogously one can prove that $\text{Ker}\, T^* \subseteq \mathfrak{R}$ when (A.2) applies. □

REMARK A.3. *Let Ω of class C^1 and $l_{ijhk} \in C^1(\bar{\Omega})$. The symmetries $l_{ijhk} = l_{jihk}$* [resp. $l_{ijhk} = l_{ijkh}$] *imply that Tu* [resp. T^*u] *is equilibrated* (see section 3) *for any $u \in (W^{2,p}(\Omega))^n$.*

PROOF. Let $l_{ijhk} = l_{jihk}$ and $u \in (W^{2,p}(\Omega))^n$. Then from (A.7) it follows that $\int_\Omega f_i r_i \, dx + \int_{\partial\Omega} g_i r_i \, d\sigma = 0$, $\forall r \in \mathfrak{R}$, where $(f, g) = Tu$, and this immediately yields that Tu is equilibrated. Analogously we can see that $l_{ijhk} = l_{ijkh}$ implies that T^*u is equilibrated for any $u \in (W^{2,p}(\Omega))^n$.

With Remarks A.1, A.2 and A.3 in mind, it is possible to prove the following Theorems A.1 and A.2 on the ground of well-known estimates for elliptic boundary value problems (see Agmon, Douglis and Nirenberg [2], Theorems 9.3 and 10.5). A proof of Theorems A.1 and A.2 can be developed, e.g., by the procedure of Browder [3].

THEOREM A.1. *Let* Ω *of class* C^{m+2} *and let* $l_{ijhk} \in C^{m+1}(\bar{\Omega})$ *be such that* (A.2) *applies. Then the operator* T *defined by* (A.1) *is an isomorphism* (for the topological vector structures) *of* $V_{m,p}$ *onto* $F_{m,p}$.

THEOREM A.2. *Let* Ω *of class* C^{m+2} *and let* $l_{ijhk} \in C^{m+1,\lambda}(\bar{\Omega})$ *be such that* (A.2) *applies. Then* T *is an isomorphism* (for the topological vector structures) *of* $V_{m,\lambda}$ *onto* $F_{m,\lambda}$.

For the definitions of $V_{m,p}$, $V_{m,\lambda}$, $F_{m,p}$ and $F_{m,\lambda}$ see sect. 3.

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