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A Characterization of Discrete Linearly Compact Rings by Means of a Duality.

A. ORSATTI - V. ROSELLI (*)

1. Introduction.

All rings considered in this paper have a non zero identity and all modules are unitary.

A ring A is said to have a right Morita duality if there exists a faithfully balanced bimodule ${}_R K_A$ such that ${}_R K$ and K_A are injective cogenerators of $R\text{-Mod}$ and $\text{Mod-}A$ respectively. This means that the subcategories of $R\text{-Mod}$ and $\text{Mod-}A$ consisting of K -reflexive modules are both finitely closed and contain all finitely generated modules.

It is well known (see Müller [4]) that if A has a right Morita duality then A is right linearly compact (in the discrete topology). The converse of this result is false for non commutative rings (see Sandomierski [5]) while for commutative rings the question is still open and seems to be hard to solve (see Müller [4], Vamos [6], [7]).

The purpose of this paper is to show that a ring A is right linearly compact if and only if A has a *good duality*.

This means that there exists a faithfully balanced bimodule ${}_R K_A$ such that K_A is a cogenerator of $\text{Mod-}A$ and ${}_R K$ is strongly quasi-injective. This means also that there exists a duality between $\text{Mod-}A$ and the category of K -compact left R -modules (see section 2 below).

In particular it is shown that, if A is linearly compact, then such a duality may be induced by the minimal cogenerator of $\text{Mod-}A$.

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Furthermore we prove that if a ring A is Morita equivalent to a right linearly compact ring then A is such.

Finally we give a description of the basic ring of a linearly compact ring (which is semiperfect) by means of a representation property.

2. Strongly quasi-injective modules and good dualities.

In this section we recall some known facts which will be useful later.

2.1. Let A, R be two rings and ${}_R K_A$ a faithfully balanced bimodule (right on A and left on R). This means that $A \cong \text{End}({}_R K)$ and $R \cong \text{End}(K_A)$ canonically.

Denote by $\text{Mod-}A$ the category of right A -modules and by $R\text{-}LT$ the category of linearly topologized Hausdorff left R -modules over the ring R endowed with the ${}_R K$ -topology. This ring topology on R is obtained by taking as a basis of neighbourhoods of zero in R the annihilators of the finite subsets of K .

In the following ${}_R K$ will have the discrete topology. Then ${}_R K \in R\text{-}LT$.

Let M be a module belonging to $\text{Mod-}A$ (to $R\text{-}LT$). A *character* of M is a morphism of M in K_A (a continuous morphism of M in ${}_R K$).

Let $M \in \text{Mod-}A$; we define the *character module* M^* of M as the left R -module $\text{Hom}_A(M, K_A)$ endowed with the *finite topology*. This topology has as a basis of neighbourhoods of 0 in M^* the submodules

$$W(F) = \{\xi \in \text{Hom}_A(M, K_A) : \xi(F) = 0\}$$

where F is a finite subset of M . Then $M^* \in R\text{-}LT$ and it is *K-compact*.

Recall that a module $M \in R\text{-}LT$ is *K-compact* if it is topologically isomorphic to a closed submodule of a topological product of copies of ${}_R K$. Let $\mathcal{C}({}_R K)$ be the subcategory of $R\text{-}LT$ consisting of *K-compact* modules. Clearly M is *K-compact* if and only if M is complete and its topology coincides with the weak topology of characters.

Let $M \in R\text{-}LT$. The character module M^* of M is simply the abstract A -module $\text{Chom}_R(M, {}_R K)$. M^* is *K-discrete*. Recall that a right A -module is *K-discrete* if it is isomorphic to a submodule of a product of copies of K_A . Denote by $\mathcal{D}(K_A)$ the category of *K-discrete* modules. A module $M \in \text{Mod-}A$ is *K-discrete* if and only if $\text{Hom}_A(M, K_A)$ separates points of M .

Let $A_1: \mathcal{D}(K_A) \rightarrow \mathcal{C}({}_R K)$ be the contravariant functor that asso-

ciates to each K -discrete module M its character module M^* and to each morphism in $\mathcal{D}(K_A)$ its transposed morphism. The functor $\Delta_2: \mathcal{C}({}_R K) \rightarrow \mathcal{D}(K_A)$ is defined in a similar way.

We say that $\Delta_K = (\Delta_1, \Delta_2)$ is a *good duality* if:

- 1) Δ_K is a duality in the sense that for every K -discrete and every K -compact module M the canonical morphism $\omega_M: M \rightarrow M^{**}$ is an isomorphism in the corresponding category.
- 2) The category $\mathcal{C}({}_R K)$ has the extension property of characters, *i.e.* for every topological submodule L of a module $M \in \mathcal{C}({}_R K)$, any character of L extends to a character of M .

If Δ_K is a duality and $\mathcal{D}(K_A) = \text{Mod-}A$ then Δ_K is necessarily good (cf. [3], Prop. 1.11).

Looking for conditions in order that Δ_K be a good duality, leads us to consider strongly quasi-injective modules.

The module $M \in R\text{-Mod}$ is said *strongly quasi-injective* (s.q.i. for short) if for every submodule $L \leq_R M$ and every $x_0 \in M \setminus L$, any morphism $\xi: L \rightarrow {}_R M$ extends to an endomorphism $\tilde{\xi}$ of ${}_R M$ such that $(x_0)\tilde{\xi} \neq 0$. In particular ${}_R M$ is quasi-injective.

Recall that a module $M \in R\text{-Mod}$ is a *selfcogenerator* if for every $n \in \mathbb{N}$, given a submodule L of M^n and an element $x_0 \in M^n \setminus L$, there exists $f \in \text{Hom}_R(M^n, M)$ such that $(L)f = 0$, $(x_0)f \neq 0$.

2.2 PROPOSITION ([2], Lemmata 2.1 and 2.5). *A module $M \in R\text{-Mod}$ is strongly quasi-injective if and only if M is a quasi-injective selfcogenerator.*

Let $\mathcal{F}$ be the filter of open left ideals in the ${}_R K$ -topology of R . Put

$$\mathcal{C}_{\mathcal{F}} = \{M \in R\text{-Mod} : \text{Ann}_R(x) \in \mathcal{F}, \forall x \in M\}.$$

The modules belonging to $\mathcal{C}_{\mathcal{F}}$ will be called $\mathcal{F}$ -torsion modules. The $\mathcal{F}$ -torsion submodule of a module $M \in R\text{-Mod}$ will be denoted by $t_{\mathcal{F}}(M)$. For every $M \in R\text{-Mod}$ $E(M)$ is the injective envelope of M .

Let $(S_{\lambda})_{\lambda \in A}$ be a system of representatives of left $\mathcal{F}$ -torsion simple modules and set $S_{\mathcal{F}} = \bigoplus_{\lambda \in A} S_{\lambda}$.

2.3. THEOREM ([3], Theorem 6.7).

Let ${}_R K_A$ be a faithfully balanced bimodule. The following statements are equivalent.

- (a) Δ_K is a good duality between $\mathcal{D}(K_A)$ and $\mathcal{C}({}_R K)$.
- (b) ${}_R K$ is strongly quasi-injective.
- (c) ${}_R K$ is quasi-injective and contains a copy of $S_{\mathcal{F}}$.
- (d) ${}_R K$ is quasi-injective and contains a copy of $\bigoplus_{\lambda \in A} t_{\mathcal{F}}(E(S_{\lambda}))$.
- (e) ${}_R K$ is an injective cogenerator of $\mathcal{C}_{\mathcal{F}}$.
- (f) For every $M \in R\text{-LT}$, for every closed submodule L of M and for every $x_0 \in M \setminus L$, any character ξ of L extends to a character $\tilde{\xi}$ of M such that $(x_0)\tilde{\xi} \neq 0$.

Recall that the socle $\text{Soc}({}_R M)$ of the module ${}_R M$ is the sum of the simple submodules of ${}_R M$.

Observe that $\text{Soc}({}_R K)$ is the sum of the annihilators in ${}_R K$ of the maximal left ideals of R . Then $\text{Soc}({}_R K)$, being fully invariant, is a submodule of K_A .

2.4. PROPOSITION ([3], Proposition 6.10).

Let ${}_R K$ be a s.q.i. left R -module and let $A = \text{End}({}_R K)$. Then

- a) $\text{Soc}({}_R K) = \text{Soc}(K_A)$.
- b) $\text{Soc}(K_A)$ is an essential submodule of K_A .

2.5. PROPOSITION ([3], Corollary 7.4).

Let ${}_R K$ be a selfcogenerator and $A = \text{End}({}_R K)$. Then $\text{End}(K_A)$ is naturally isomorphic to the Hausdorff completion of R in its ${}_R K$ -topology.

2.6. REMARK. The theory of s.q.i. modules may be developed in the more general setting ${}_R K \in R\text{-Mod}$ and $A = \text{End}({}_R K)$.

Let $\tilde{R}$ be the Hausdorff completion of R in the ${}_R K$ -topology. Then ${}_R K$ is in a natural way a left $\tilde{R}$ -module and the R -submodules of ${}_R K$ are $\tilde{R}$ -submodules. Moreover $A = \text{End}({}_{\tilde{R}} K)$ and ${}_R K$ is s.q.i. iff ${}_{\tilde{R}} K$ is s.q.i. In this case $\text{End}(K_A) = \tilde{R}$ by Proposition 2.5 and thus ${}_{\tilde{R}} K_A$ is faithfully balanced.

Finally ${}_R K$ -compact modules and ${}_{\tilde{R}} K$ -compact modules are essentially the same.

For more information about s.q.i. modules and good dualities see [3].

3. Some useful results.

3.1. Let M be a linearly topologized Hausdorff left module over the discrete ring R . M is said to be *linearly compact* if any finitely solvable system of congruences $x \equiv x_i \pmod{X_i}$, where the X_i are closed submodules of M , is solvable.

R is left linearly compact if ${}_R R$ is such and multiplication is continuous.

We write d.l.c. for linearly compact in the discrete topology.

The following result is essentially due to Müller ([4], Lemma 4) and Sandomierski ([5], Corollary 2, pag. 342).

3.2. PROPOSITION. *Let ${}_R K$ be a selfcogenerator and let $A = \text{End}({}_R K)$. Then:*

K_A is injective if and only if ${}_R K$ is linearly compact in the discrete topology.

(For a proof see [3], Theorem 9.4).

3.3. LEMMA. *Let ${}_R K$ be a selfcogenerator and let $A = \text{End}({}_R K)$. Let L be a finitely generated submodule of a module $M \in \mathcal{D}(K_A)$. Then every morphism of L in K_A extends to a morphism of M in K_A .*

PROOF. Let $\{x_1, \dots, x_n\}$ be a set of generators of L and $f \in \text{Hom}_A(L, K_A)$. Consider the subset B of K^n defined by:

$$B = \{(g(x_1), \dots, g(x_n)) : g \in \text{Hom}_A(M, K_A)\}.$$

Since $\text{Hom}_A(M, K_A)$ is a left R -module, B is a submodule of ${}_R K^n$. Put $y = (f(x_1), \dots, f(x_n))$.

We claim that $y \in B$. Suppose $y \notin B$.

Then there exists $\alpha \in \text{Hom}_R(K^n, K)$ such that

$$B\alpha = 0, \quad y\alpha \neq 0.$$

Then $\alpha = (a_1, \dots, a_n)$ where $a_i \in A$, $i = 1, \dots, n$.

For every $g \in \text{Hom}_A(M, K_A)$ we have:

$$\sum_{i=1}^n g(x_i) a_i = \sum_{i=1}^n g(x_i a_i) = g\left(\sum_{i=1}^n x_i a_i\right) = 0,$$

thus $\sum_{i=1}^n x_i a_i = 0$ since $M \in \mathcal{D}(K_A)$.

Therefore $y\alpha = \sum_{i=1}^n f(x_i) a_i = f\left(\sum_{i=1}^n x_i a_i\right) = 0$, contradiction.

3.4. PROPOSITION. *Let ${}_R K_A$ be a faithfully balanced bimodule.*

a) If ${}_R K$ is a selfcogenerator and R is linearly compact in the ${}_R K$ -topology, then K_A is quasi-injective.

b) If ${}_R K$ is a cogenerator, then ${}_R R$ is linearly compact in the discrete topology if and only if K_A is quasi-injective.

PROOF. *a)* Let L be a submodule of K_A and $g \in \text{Hom}_A(L, K_A)$. We have to show that g coincides with the left multiplication by an element of R .

Let $(L_i)_{i \in I}$ be the family of all finitely generated submodules of L . By Lemma 3.3 $g|_{L_i}$ coincides with the left multiplication by an element $r_i \in R$. Consider the following system of congruences

$$(1) \quad r \equiv r_i \pmod{\text{Ann}_R(L_i)}.$$

Obviously $\text{Ann}_R(L_i)$ are closed left ideals in the ${}_R K$ -topology of R and (1) is finitely solvable. Let r be a solution of (1). Then for every $i \in I$ and $x \in L_i$ we have $rx = r_i x = g(x)$.

b) Suppose that ${}_R R$ is linearly compact in the discrete topology. Then ${}_R R$ is linearly compact in any Hausdorff linear topology. Therefore K_A is quasi-injective. Suppose that K_A is quasi-injective and consider the finitely solvable system of congruences

$$(2) \quad r \equiv r_i \pmod{J_i} \quad i \in I$$

where the J_i , $i \in I$, are left ideals of R . $L = \sum_{i \in I} \text{Ann}_K(J_i)$ is a submodule of K_A . Define the A -morphism $g: L \rightarrow K_A$ by putting $g\left(\sum_{i \in I} x_i\right) =$

$= \sum_{i \in F} r_i x_i$ where F is a finite subset of I and, for every $i \in F$, $x_i \in \text{Ann}_K(J_i)$.

Since (2) is finitely solvable, g is well defined. Indeed suppose $\sum_{i \in F} x_i = \sum_{i \in F} x'_i$. Then there exists $u \in R$ such that $r_i - u \in J_i$, $i \in F$. $\sum_{i \in F} (r_i - u)x_i = 0$ thus $\sum_{i \in F} r_i x_i = u \left(\sum_{i \in F} x_i \right)$ and similarly $\sum_{i \in F} r_i x'_i = u \sum_{i \in F} x'_i$.

Since K_A is quasi-injective g extends to an endomorphism $\bar{g}$ of K_A . $\bar{g}$ is the left multiplication by an element $r \in R$ so that we have for every $i \in I$ and $x \in \text{Ann}_K(J_i)$, $g(x) = rx = r_i x$.

Therefore $r - r_i \in \text{Ann}_R \text{Ann}_K(J_i) = J_i$ since ${}_R K$ is a cogenerator.

REMARK. The proof of the above proposition closely follows the methods of Müller [4].

4. The main theorem.

4.1. We say that a ring A has a (right) good duality if there exists a faithfully balanced bimodule ${}_R K_A$ such that K_A is a cogenerator of $\text{Mod-}A$ and ${}_R K$ is strongly quasi-injective. This means that Δ_K is a good duality between $\text{Mod-}A$ and $\mathcal{C}({}_R K)$.

We will prove that A is right d.l.c. if and only if A has a good duality.

By Proposition 3.4 b) we get the following

4.2. LEMMA. *If A has a good duality then A is right d.l.c.*

When ${}_R K$ is s.q.i. Proposition 3.2 may be sharpened in the following way.

4.3. PROPOSITION. *Let ${}_R K$ be a s.q.i. module and let $A = \text{End}({}_R K)$. Let $\mathcal{F}$ be the filter of open left ideals in the ${}_R K$ -topology of R . Then the following conditions are equivalent.*

- (a) ${}_R K$ is linearly compact in the discrete topology and $\text{Soc}({}_R K)$ is essential in ${}_R K$.
- (b) K_A is an injective cogenerator of $\text{Mod-}A$.

If these conditions are fulfilled then:

- 1) ${}_R K$ is a finite direct sum ${}_R K = \bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i))$ where S_i are $\mathcal{F}$ -torsion simple left modules.

2) A_A is linearly compact in the discrete topology.

PROOF. By Remark 2.6 we may suppose that the bimodule ${}_R K_A$ is faithfully balanced.

(a) $\Rightarrow$ (b). K_A is injective by Propositions 2.2 and 3.2. Let S be a simple module in the category $\text{Mod-}A$ and let us prove that $\text{Hom}_A(S, K_A) \neq 0$. Consider the exact sequence

$$0 \rightarrow P \xrightarrow{i} A \rightarrow S \rightarrow 0$$

where P is a right maximal ideal of A and i is the canonical inclusion. Since K_A is injective we have the exact sequence

$$0 \rightarrow \text{Hom}_A(S, K_A) \rightarrow {}_R K \xrightarrow{i^*} \text{Hom}_A(P, K_A) \rightarrow 0.$$

Suppose $\text{Hom}_A(S, K_A) = 0$. Then i^* is a continuous isomorphism of the K -compact module ${}_R K$ onto the K -compact module P^* . Since ${}_R K$ is linearly compact $\text{Soc}({}_R K)$ is a direct sum of a finite number of simple modules and moreover $\text{Soc}({}_R K)$ is essential in ${}_R K$. It is well known, and easily checked, that in this case the unique Hausdorff linear topology on ${}_R K$ (which is algebraically isomorphic to P^*) is the discrete one. Thus i^* is a topological isomorphism. Since the functor $\Delta_1: \mathcal{D}(K_A) \rightarrow \mathcal{C}({}_R K)$ is a good duality, i is an isomorphism. Contradiction.

(b) $\Rightarrow$ (a). Since ${}_R K$ and K_A are both s.q.i. and by Propositions 2.4 and 3.2 the conclusion is reached.

Suppose now that conditions (a) and (b) are fulfilled.

1) $\text{Soc}({}_R K)$ is d.l.c. thus it is the direct sum of a finite family $\{S_1, \dots, S_n\}$ of left $\mathcal{F}$ -torsion simple modules. For every $i = 1, \dots, n$ ${}_R K$ contains a copy of $t_{\mathcal{F}}(E(S_i))$ since ${}_R K$ is an injective object in $\mathcal{C}_{\mathcal{F}}$ (see Theorem 2.3). Thus ${}_R K$ contains a copy of $\bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i))$. Put $K_0 = \bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i))$ and let $E(K_0)$ be the injective envelope of K_0 . Then $E(K_0) = \bigoplus_{i=1}^n E(S_i)$. The identity map on K_0 extends to a morphism $j: {}_R K \rightarrow E(K_0)$. Since ${}_R K$ is $\mathcal{F}$ -torsion, $j({}_R K) \leq t_{\mathcal{F}}(E(K_0))$ and since $t_{\mathcal{F}}(E(K_0)) = K_0$, $j({}_R K) = K_0$. Thus K_0 is a direct summand of ${}_R K$ and contains the socle of ${}_R K$. Hence ${}_R K = K_0$.

Statement 2) follows from Proposition 3.4 b) since K_A is a cogenerator and ${}_R K$ is quasi-injective.

4.4. PROPOSITION. *Let A be a right d.l.c. ring, $J(A)$ the Jacobson radical of A , U_A the minimal cogenerator of $\text{Mod-}A$, $R = \text{End}(U_A)$. Then:*

- a) *$A/J(A)$ is a semisimple artinian ring, and thus $\text{Mod-}A$ has only a finite number of non isomorphic simple modules, so that U_A is injective.*
- b) *The bimodule ${}_R U_A$ is faithfully balanced.*

PROOF. a) By a well known result of Zelinski (cf. [8]), $A/J(A)$ is semisimple artinian, so that A has only a finite number of right maximal ideals. Since U_A is the direct sum of one copy of the injective envelope of each simple module, U_A is the direct sum of a finite number of injective modules, thus U_A is injective.

b) Since U_A is a selfcogenerator the endomorphism ring of ${}_R U$ is the Hausdorff completion of A in the K_A -topology by Proposition 2.5. On the other hand A is right d.l.c. so that A is complete in any right linear Hausdorff topology. Thus $A = \text{End}({}_R U)$.

Recall that a module M is *finitely embedded* if M is an essential submodule of a finite direct sum of injective envelopes of simple modules.

4.5. LEMMA ([6], Lemma 1.3; [4], Lemma 2). *Let $\{M_i\}_{i \in I}$ and H be submodules of a d.l.c. module M . Suppose that $\bigcap_{i \in I} M_i \leq H$ and that M/H is finitely embedded. Then there exists a finite subset F of I such that $\bigcap_{i \in F} M_i \leq H$.*

4.6. LEMMA. *Let ${}_R K$ be quasi-injective and $A = \text{End}({}_R K)$. Then ${}_R K$ is s.q.i. if and only if for every submodule L of ${}_R K$ it is $\text{Ann}_K \text{Ann}_A(L) = L$.*

4.7. THEOREM. *Let A be a ring, U_A the minimal cogenerator of $\text{Mod-}A$, $R = \text{End}(U_A)$. The following conditions are equivalent:*

- (a) *A is right linearly compact in the discrete topology.*
- (b) *${}_R U$ is strongly quasi-injective and $\text{End}({}_R U) = A$.*
- (c) *Δ_U is a good duality between $\text{Mod-}A$ and $C({}_R U)$.*
- (d) *A has a good duality on the right.*

- (e) For every faithfully balanced bimodule ${}_TK_A$, if K_A is a cogenerator then ${}_TK$ is quasi-injective.
- (f) $A = \text{End}({}_TK)$ where ${}_TK$ is a discrete linearly compact and strongly quasi-injective module with essential socle.

Moreover:

- 1) If condition (a) is fulfilled, then A is semiperfect, U_A is an injective cogenerator and ${}_RU$ is discrete linearly compact with essential socle.
- 2) If condition (f) is fulfilled, then K_A is an injective cogenerator of $\text{Mod-}A$ and ${}_TK$ is a finite direct sum of modules of the form $t_{\mathcal{F}}(E(S))$ where S is an $\mathcal{F}$ -torsion simple T -module.

PROOF. (a) $\Rightarrow$ (b). $A = \text{End}({}_RU)$ and U_A is an injective cogenerator by Proposition 4.4. Thus, by Proposition 3.4 b), ${}_RU$ is quasi-injective. Let L be a submodule of ${}_RU$ and let us show that $\text{Ann}_V \text{Ann}_A(L) = L$, from which it will follow that ${}_RU$ is s.q.i., by Lemma 4.6.

First of all observe that we have a good duality

$$\mathcal{D}({}_RU) \xrightleftharpoons[\Delta_2]{\Delta_1} \mathcal{C}(U_A)$$

by Theorem 2.3 and since U_A is s.q.i.

Being L an ${}_RU$ -discrete module, $\Delta_1(L)$ is $\text{Hom}_A(L, {}_RU)$ endowed with the finite topology and $\Delta_2\Delta_1(L) = \text{Chom}_A(\Delta_1(L), U_A)$. We claim that

$$\text{Chom}_A(\Delta_1(L), U_A) = \text{Hom}_A(\Delta_1(L), U_A).$$

For every $a \in A$ denote by v_a the right multiplication by a in U . Since ${}_RU$ is quasi-injective every character of L is of the form $v_a|_L$.

Let $\varphi \in \text{Hom}_A(\Delta_1(L), U_A)$ and consider the diagram

$$\begin{array}{ccc} \text{Hom}_R(L, {}_RU) & \xrightarrow{\varphi} & U_A \\ & \nwarrow j \quad \nearrow \varphi \circ j & \\ & A & \end{array}$$

where $j(a) = v_a|_L$.

Obviously $\text{Ann}_A(L) \leq \text{Ker}(\varphi \circ j)$, thus

$$\bigcap_{y \in L} \text{Ann}_A(y) \leq \text{Ker}(\varphi \circ j) \leq A.$$

Now A is linearly compact discrete, $A/\text{Ker}(\varphi \circ j)$ is a submodule of U_A and U_A is finitely embedded. Therefore it follows from Lemma 4.6 that there exists a finite subset F of L such that

$$\bigcap_{x \in F} \text{Ann}_A(x) \leq \text{Ker}(\varphi \circ j).$$

Put $W(F) = \{\xi \in \text{Hom}_R(L, {}_R U) : F\xi = 0\}$.

Note that $W(F) \leq \text{Ker} \varphi$. Indeed if $F\xi = 0$ there exists $a \in A$ such that $\xi = v_{a|L}$ and $(\varphi \circ j)(a) = 0$. Therefore $\varphi(v_{a|L}) = 0$. Since $W(F)$ is an open submodule of $\Delta_1(L)$, φ is continuous.

Therefore

$$\Delta_2 \Delta_1(L) = \text{Hom}_A(\Delta_1(L), U_A) = \text{Hom}_A(\text{Hom}_R(L, {}_R U), U_A).$$

Since ${}_R U$ is quasi-injective there exists the natural isomorphism

$$\psi: A/\text{Ann}_A(L) \rightarrow \text{Hom}_R(L, {}_R U)$$

given by $\psi(a + \text{Ann}_A(L)) = v_{a|L}$.

Using ψ we have the natural isomorphisms

$$\Delta_2 \Delta_1(L) \xrightarrow{f_1} \text{Hom}_A\left(\frac{A}{\text{Ann}_A(L)}, U_A\right) \xrightarrow{f_2} \text{Ann}_v \text{Ann}_A(L).$$

Putting $f = f_2 \circ f_1$, f works as follows:

for every $\xi \in \Delta_2 \Delta_1(L)$, $f(\xi) = (\xi \circ \psi \circ \pi)(1)$ where $\pi: A \rightarrow A/\text{Ann}_A(L)$ is the canonical mapping.

Let us show that the diagram

$$\begin{array}{ccc} L & \xhookrightarrow{i} & \text{Ann}_v \text{Ann}_A(L) \\ & \searrow \omega_L & \nearrow f \\ & \Delta_2 \Delta_1(L) & \end{array}$$

is commutative, where i is the inclusion and ω_L is the natural morphism. Since Δ_U is a duality, ω_L is an isomorphism.

Let $x \in L$. It is

$$\begin{aligned}(f \circ \omega_L)(x) &= (\omega_L(x) \circ \psi \circ \pi)(1) = \omega_L(x)[(\psi \circ \pi)(1)] = \\ &= [(\psi \circ \pi)(1)](x) = [\psi(1 + \text{Ann}_A(L))](x) = v_{1|L}(x) = i(x).\end{aligned}$$

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a) follows from Lemma 4.2.

(b) $\Rightarrow$ (f) We know that $A = \text{End}({}_R U)$ and that ${}_R U$ is s.q.i.

Since (a) $\Leftrightarrow$ (b), it follows from Proposition 4.4 that U_A is an injective cogenerator of $\text{Mod-}A$. Then by Proposition 4.3 ${}_R U$ is d.l.c. with essential socle.

(f) $\Rightarrow$ (a) By Proposition 4.3.

(e) $\Leftrightarrow$ (a) follows by Proposition 3.4 b).

1) Recall that A is semiperfect if $A/J(A)$ is semisimple artinian and the idempotents of $A/J(A)$ can be lifted in A .

If A_A is d.l.c. then $A/J(A)$ is semisimple artinian by Proposition 4.4. On the other hand, by (f), A is the endomorphism ring of a quasi-injective module, thus by a well known result the idempotents of $A/J(A)$ can be lifted in A .

2) Follows from Proposition 4.3.

REMARK. The equivalence between conditions (a) and (f) has been found by Sandomierski ([5], Theorem 3.10 pg. 344). Moreover it is well known that a d.l.c. ring is semiperfect.

5. Further results.

5.1. PROPOSITION. *Let B and A be two Morita equivalent rings and suppose that B_B is discrete linearly compact. Then A_A is discrete linearly compact.*

PROOF. This proposition may be obtained using some results of Sandomierski ([5], Corollary 1, pg. 336).

We give here a simple direct proof by means of good dualities.

Let $\text{Mod-}A \xrightleftharpoons[F]{G} \text{Mod-}B$ an equivalence and ${}_A P_B$ a faithfully balanced bimodule such that P_B and ${}_A P$ are both progenerators and $F = - \otimes_A P_B$, $G = \text{Hom}_B(P_B, -)$ (see [1], Theorem 22.2).

Let U_B be the minimal cogenerator of $\text{Mod-}B$ and $R = \text{End}(U_B)$. Then by Theorem 4.7, Δ_U is a good duality. Consider the diagram

$$\begin{array}{ccc}
 \text{Mod-}A & & \\
 \downarrow F & \nearrow D_1 & \\
 \text{Mod-}B & \xrightarrow{\Delta_1} & \mathcal{C}({}_R U) \\
 \uparrow G & \xleftarrow{\Delta_2} & \\
 \end{array}$$

where $D_1 = \Delta_1 \circ F$.

Clearly D_1 is a duality and for every $M \in \text{Mod-}A$

$$\begin{aligned}
 D_1(M) &= \Delta_1(M \otimes_A P_B) = \text{Hom}_B(M \otimes_A P_B, U_B) \cong \\
 &\cong \text{Hom}_A(M, \text{Hom}_B(P_B, U_B)) \cong \text{Hom}_A(M, G(U_B)),
 \end{aligned}$$

the isomorphisms being canonical and topological. Put $G(U_B) = K_A$. Since U_B is an injective cogenerator of $\text{Mod-}B$, K_A is an injective cogenerator of $\text{Mod-}A$ (see [1], Proposition 21.6). Clearly $\text{End}(K_A) = R$. Let us show that $\mathcal{C}({}_R K) = \mathcal{C}({}_R U)$ and that ${}_R K_A$ is faithfully balanced. It is $\Delta_1(P_B) = \text{Hom}_B(P_B, U_B) \cong {}_R G(U_B) = {}_R K$ so that $\mathcal{C}({}_R K) \subseteq \mathcal{C}({}_R U)$. Moreover $A = \text{End}({}_R K)$. In fact

$$\text{End}({}_R K) = \text{End}(\Delta_1(P_B)) \cong \text{End}(P_B) = A, \quad \text{canonically}.$$

Thus ${}_R K_A$ is faithfully balanced.

Since P_B is projective and finitely generated, B is a direct summand of P_B^m where m is a positive integer. Then ${}_R U = \text{Hom}_B(B, U_B)$ is a direct summand of $\text{Hom}_B(P_B^m, U_B) = {}_R K^m$, therefore $\mathcal{C}({}_R U) \subseteq \mathcal{C}({}_R K)$.

Thus $\mathcal{C}({}_R K) = \mathcal{C}({}_R U)$.

We know that $D_1 = \text{Hom}_A(-, K_A)$ endowed with the finite topology. On the other hand since K_A is a cogenerator, $\mathcal{D}(K_A) = \text{Mod-}A$. Therefore, by 2.1, D_1 gives a good duality between $\text{Mod-}A$ and $\mathcal{C}({}_R K)$. Thus, by Theorem 4.7, A_A is d.l.c.

5.2. Recall that a semiperfect ring B is a basic ring if $B/J(B)$ is a ring direct sum of division rings. It is well known (see [1], Proposition 27.14) that any semiperfect ring is Morita equivalent to a basic ring, which is unique up to isomorphisms. Our aim is to give a description of the basic ring of a right d.l.c. ring by means of a representation of it as endomorphism ring.

5.3. PROPOSITION. *Let A be a right d.l.c. ring, U_A the minimal cogenerator of $\text{Mod-}A$, $R = \text{End}(U_A)$, $\mathcal{F}$ the filter of open left ideals in the ${}_R U$ -topology of R . Then the basic ring B of A is isomorphic to the endomorphism ring of the minimal cogenerator of $\mathcal{E}_{\mathcal{F}}$, i.e.*

$$B \cong \text{End}_R \left(\bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i)) \right)$$

where $(S_i)_{i=1, \dots, n}$ is a system of representatives of the non isomorphic simple $\mathcal{F}$ -torsion left R -modules.

PROOF. Let $\{S_1, \dots, S_n\}$ be a system of representatives as above. Then by Theorem 4.7 and Proposition 4.3, A is the endomorphism ring of the left R -module

$${}_R U = \bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i))^{m_i}$$

where m_i are suitable positive integers (in general > 1 , as may be showed by examples).

$$\text{Put } {}_R K = \bigoplus_{i=1}^n t_{\mathcal{F}}(E(S_i)), \quad B = \text{End}({}_R K).$$

It is clear that ${}_R K$ is strongly quasi-injective, discrete linearly compact with essential socle.

Thus K_B is an injective cogenerator of $\text{Mod-}B$ by Proposition 4.3. Note that ${}_R K_B$ is faithfully balanced since the ${}_R K$ -topology of R coincides with the ${}_R U$ -topology and using Proposition 2.5.

Moreover it is obvious that $C({}_R K) = C({}_R U)$. Since Δ_U is a good duality between $\text{Mod-}A$ and $C({}_R U) = C({}_R K)$ and Δ_K is a good duality between $\text{Mod-}B$ and $C({}_R K)$, it follows that A and B are Morita equivalent so that B_B is d.l.c., hence semiperfect.

To conclude it is enough to show that $B/J(B)$ is a ring direct sum of division rings (see [1], Propositions 27.14 and 27.15).

For every $i = 1, \dots, n$ put $P_i = \text{Ann}_B(S_i)$ and consider the exact sequence

$$0 \rightarrow S_i \rightarrow {}_R K \rightarrow {}_R K/S_i \rightarrow 0.$$

Since ${}_R K$ is quasi-injective and S_i is fully invariant in ${}_R K$, applying $\text{Hom}_R(-, {}_R K)$ we get the exact sequence

$$0 \rightarrow P_i \rightarrow B \rightarrow \text{End}_R(S_i) \rightarrow 0.$$

Then $D_i = B/P_i \cong \text{End}_R(S_i)$ is a division ring and P_i is a maximal ideal of B .

We claim that $J(B) = \bigcap_{i=1}^n P_i$. It is clear that $J(B) \subseteq \bigcap_{i=1}^n P_i$. On the other hand let $b \in J(B)$. Since ${}_R K$ is quasi-injective, $\text{Ker}(b)$ is essential in ${}_R K$, thus $\text{ker}(b)$ contains $\bigoplus_{i=1}^n S_i$ which is the essential socle of ${}_R K$. Therefore $b \in \bigcap_{i=1}^n P_i$. Then $B/J(B)$ is the ring direct sum of the division rings D_i .

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