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Torsion-Free Abelian Groups and Completely Decomposable p -adic Modules.

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Let $\mathcal{C}$ be the class of all torsion-free abelian groups G such that $J_p \otimes_{\mathbf{Z}} G$ is a completely decomposable J_p -module for every prime p where J_p is the ring of p -adic integers. This class has been studied by Procházka in [5], where it is proved that $\mathcal{C}$ is quite large. In fact, by ([5] Theorem 4*), $\mathcal{C}$ contains all torsion-free groups which belong to some Baer class Γ_σ .

In section 1 of this note we show that, if G is torsion-free, then $G \in \mathcal{C}$ if and only if, for every prime p , the group $G/p^\omega G$ is a p -pure subgroup with divisible cokernel of a free J_p -module whose rank coincides with that of G/pG .

In section 2 we give some examples of groups that are not in $\mathcal{C}$. Indeed, since $\mathcal{C}$ is not closed under direct products, very few reduced torsion-free separable or cotorsion groups belong to $\mathcal{C}$.

In section 3 we describe the behaviour of $\mathcal{C}$ with respect to pure subgroups and extensions.

I am indebted to Prof. L. Procházka for the advice that the original proof of theorem 1 was incorrect.

§ 1. All groups considered in the following are abelian groups. For all unexplained terminology and notation we refer to [1]. In particular, $\mathbf{P}$ is the set of prime numbers, $\mathbf{N}$ the set of natural numbers; $\mathbf{Z}$ and J_p are respectively the groups (or rings) of integers and p -adic

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integers. If G is a torsion-free group and X is a subset of G , then $\langle X \rangle_*$ denotes the pure subgroup of G generated by X . If H is a pure subgroup of G , then we write $H \leq G$. Throughout the paper, the symbol $\otimes$ always stands for $\otimes_{\mathbb{Z}}$. For every reduced torsion-free group G , we view G as a subgroup of its cotorsion completion $G^* = \text{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ and we identify G^* with $\prod_{p \in \mathbf{P}} G_p^*$ where $G_p^* = \text{Ext}(\mathbb{Z}(p^\infty), G)$ for all p . If G is torsion-free and p is a prime, then we regard $G_p = G/p^\infty G$ as subgroup of G_p^* .

We can now determine the groups that belong to $\mathcal{C}$.

THEOREM 1. *Let G be a torsion-free group and let $p \in \mathbf{P}$. The following statements are equivalent:*

- (i) $J_p \otimes G$ is a completely decomposable J_p -module.
- (ii) $G_p = G/p^\infty G$ is a p -pure subgroup with divisible cokernel of a free J_p -module.

PROOF. (i) $\Rightarrow$ (ii). Since $J_p \otimes p^\infty G$ is divisible, $J_p \otimes G$ is isomorphic to $(J_p \otimes p^\infty G) \oplus (J_p \otimes G/p^\infty G)$. Hence, by (i), $J_p \otimes G_p = J_p \otimes G/p^\infty G = F \oplus D$ where F is a free J_p -module and D is divisible. Let $f: J_p \times G_p \rightarrow G_p^*$ be the map defined by $f((\alpha, x)) = \alpha x$ for all $\alpha \in J_p$, $x \in G_p$. Then there is a homomorphism φ that makes the following diagram commute:

$$\begin{array}{ccc} J_p \times G_p & \xrightarrow{f} & G_p^* \\ & \searrow & \nearrow \varphi \\ & J_p \otimes G_p & \end{array}$$

Let $\pi: J_p \otimes G_p = F \oplus D \rightarrow F$ be the natural projection and identify G_p with the subgroup $\mathbb{Z} \otimes G_p$ of $J_p \otimes G_p$. Thus G_p is p -pure in $J_p \otimes G_p$ and $J_p \otimes G_p/G_p$ is divisible. Since $\text{Ker } \pi \leq \text{Ker } \varphi$, it is easy to check that $G_p \cap \text{Ker } \pi = 0$. Hence G_p is isomorphic to $\pi(G_p)$ and (ii) holds.

(ii) $\Rightarrow$ (i). By the remark made in the first part of the proof, it is not restrictive to assume $G \cong G_p$. Consequently, by (ii), there is a free J_p -module F such that G is a p -pure subgroup of F and $H = F/G$ is divisible; so $H = A \oplus T$ where A is torsion-free and T is a torsion-group with $t_p(T) = 0$. Since $J_p \otimes H \cong J_p \otimes F/J_p \otimes G$ and $J_p \otimes H \cong J_p \otimes A$ is torsion-free, we conclude that $J_p \otimes G$ is a pure subgroup of $J_p \otimes F$. Let $J_p \otimes F = D \oplus L$, $J_p \otimes G = D' \oplus R'$ with D, D' divisible, R' reduced and L a free J_p -module. Since $R' \leq J_p \otimes F =$

$= D \oplus L$, one obtains $R' \cap D = 0$. Therefore R' is isomorphic to a submodule of L under the canonical projection $D \oplus L \rightarrow L$. Thus R' is a free J_p -module ([1] Theorems 14.5 and 14.7) and $J_p \otimes G$ is a completely decomposable J_p -module. $\square$

The above characterization of the class $\mathcal{C}$ is the analogue of a result ([3] Theorem 4.1) concerning torsion-free modules over discrete valuation rings. Also note that, if $G \in \mathcal{C}$ and $p \in \mathbf{P}$, then the rank r of G and the p -rank r_p of G , i.e. the dimension of G/pG over the field with p elements, uniquely determine the structure of $J_p \otimes G$. Indeed let $J_p \otimes G = F \oplus D$ with F free and D divisible; then, as in ([3] Lemma 1.2), $J_p \otimes G$ and F are J_p -module of rank r and r_p respectively.

§ 2. We now look at the behaviour of $\mathcal{C}$ with respect to direct products.

PROPOSITION 2. *Let $G = \prod_{i \in I} G_i$ where $G_i \in \mathcal{C}$ for every $i \in I$ and let $p \in \mathbf{P}$. The following are equivalent:*

- (i) $pG_i = G_i$ for almost all $i \in I$.
- (ii) $J_p \otimes G$ is a completely decomposable J_p -module.

PROOF. (i) $\Rightarrow$ (ii). Let $I' = \{i \in I: pG_i \neq G_i\}$ and let $I'' = I \setminus I'$. Since $J_p \otimes G = (J_p \otimes \prod_{i \in I'} G_i) \oplus (J_p \otimes \prod_{i \in I''} G_i)$ where the second summand is divisible, I' is finite and $G_i \in \mathcal{C}$ for all $i \in I'$, evidently (ii) holds.

(ii) $\Rightarrow$ (i). As before, let $I' = \{i \in I: pG_i \neq G_i\}$. We want to prove that I' is finite. Assume the contrary. To see that this is impossible, choose, for every $i \in I'$, an element $x_i \in G_i \setminus pG_i$ and let $L_i = \langle x_i \rangle_* \leq G_i$. Let now $L = \bigoplus L_i$ and $H = \prod_{i \in I'} G_i$. Then, by (ii), $J_p \otimes H$ has a decomposition of the form $F \oplus D$ where F is a free J_p -module and D is divisible. Write $F = \bigoplus_{k \in K} F_k$ with $F_k \cong J_p$ for all $k \in K$. Let π and π_k , for every k , be the canonical projections of $J_p \otimes H$ onto F and F_k respectively. Observe now that there exists a commutative diagram

$$\begin{array}{ccc} J_p \otimes H & \xrightarrow{\iota} & H_p \\ & \searrow & \nearrow \varphi \\ & J_p \otimes H & \end{array}$$

where φ is a homomorphism and f sends (α, x) to $\alpha\bar{x} = \alpha(x + p^o H)$ for all $\alpha \in J_p$, $x \in H$. Hence the same arguments used in theorem 1 and the definition of L tell us that L is isomorphic to $\pi(L)$. Since L is p -pure in $J_p \otimes H$, it follows that $\pi(L)$ is p -pure in F ; consequently $L/pL \cong \pi(L)/p\pi(L) \cong \pi(L) + pF/pF$. From this remark and the hypothesis that L/pL is not finite, we deduce that the set $S = \{k \in K: \pi_k(L) \neq 0\}$ is not finite; thus we may assume $\mathbb{N} \subseteq S$. For every $n \in \mathbb{N}$, let $S_n = \{k \in K: \pi_k(x_n) \neq 0\}$ and let $T_n = \bigcup_{m=0}^n S_m$. Define by induction a sequence $\{y_n\}_{n \in \mathbb{N}}$ in L with the following properties: if $n = 0$, then $y_n = x_n$; if y_m is defined for all $m \leq n$ and $h_n = \max \{h(\pi_k(y_n)): k \in T_n\}$ where h is the p -height function on F , then $y_{n+1} = y_n + p^{h_n+n+1}x_{n+1}$. Obviously $\{y_n\}_{n \in \mathbb{N}}$ has a limit $y \in H$ with respect to the p -adic topology. On the other hand, the support of $\pi(y_n)$ is T_n for every $n \in \mathbb{N}$ and, by the choice of $\{y_n\}_{n \in \mathbb{N}}$, $\bigcup_{n \in \mathbb{N}} T_n$ is not finite. Therefore $\{\pi(y_n)\}_{n \in \mathbb{N}}$ cannot converge in F equipped with the p -adic topology. This contradiction shows that I' is finite, as claimed. $\square$

While completely decomposable torsion-free groups clearly belong to $\mathcal{C}$, we have the following

COROLLARY 3. *The class $\mathcal{C}$ does not contain the class of torsion-free separable groups.*

PROOF. Let $G = \mathbb{Z}^{\mathbb{N}}$; then, by ([1] Proposition 87.4), G is separable and, by proposition 2, $G \notin \mathcal{C}$. $\square$

We next prove that very few reduced cotorsion groups belong to $\mathcal{C}$.

COROLLARY 4. *Let G be a reduced torsion-free group and let $G^* = \prod_{p \in \mathbb{P}} G_p^*$ be its cotorsion completion. The following facts hold:*

(i) $G^* \in \mathcal{C}$ if and only if, for every prime p , G_p^* is a J_p -module of finite rank.

(ii) If $G^* \in \mathcal{C}$, then $G \in \mathcal{C}$.

PROOF. (i) Fix $p \in \mathbb{P}$. Then

$$J_p \otimes G^* = (J_p \otimes G_p^*) \oplus \left(J_p \otimes \prod_{q \neq p} G_q^* \right)$$

where the second summand is divisible. Let $\mathbb{Z}_p$ be the group of rational

numbers whose denominators are prime to p . Since the sequence

$$0 \rightarrow G_p^* \cong \mathbb{Z}_p \otimes G_p^* \rightarrow J_p \otimes G_p^* \rightarrow J_p \otimes G_p^*/G_p^* \rightarrow 0$$

is exact and $J_p \otimes G_p^*/G_p^*$ is torsion-free, we get

$$J_p \otimes G_p^* = G_p^* \otimes (J_p \otimes G_p^*)/G_p^* \quad \text{with} \quad J_p \otimes G_p^*/G_p^*$$

divisible. Consequently $J_p \oplus G_p^*$ is a completely decomposable J_p -module if and only if the same applies to G_p^* , i.e. if and only if G_p^* is a J_p -module of finite rank.

(ii) Assume $G \in \mathcal{C}$ and let p be a prime. Then, by (i), $G_p^* \cong J_p^n$ for some $n \in \mathbb{N}$, while, ([4] Ch. II § 5.5 Theorem 1), G_p is a p -pure subgroup with divisible cokernel of G_p^* . This remark and theorem 1 assure that $J_p \otimes G$ is a completely decomposable J_p -module. Thus $G \in \mathcal{C}$ and (ii) follows. $\square$

The above proof indicates that, if G is a reduced torsion-free cotorsion group and $p \in \mathbf{P}$, then $J_p \otimes G$ has a decomposition of the form $C \oplus D$ where $C \cong G_p^* \cong G/p^\omega G$ and D is divisible. Therefore $J_p \otimes G$ is generally very far from being completely decomposable.

As the following statement shows, also the larger class of locally cotorsion groups, introduced in [6], does not contain many groups of $\mathcal{C}$.

COROLLARY 5. *Let G be a reduced group and let $G \in \mathcal{C}$. Then G is locally cotorsion if and only if $G/p^\omega G$ is a J_p -module of finite rank for every $p \in \mathbf{P}$.*

PROOF. Recall that, if G is reduced and torsion-free, then, by ([6] Theorem 4.2), G is locally cotorsion if and only if $G/p^\omega G \cong G_p^*$ for all prime p . Hence our claim is an immediate consequence of corollary 4. $\square$

§ 3. The next result gives a closure property of the class $\mathcal{C}$.

PROPOSITION 6. *The class $\mathcal{C}$ is closed under pure subgroups.*

PROOF. Let $G \in \mathcal{C}$ and let H be a pure subgroup of G . To see that $H \in \mathcal{C}$, fix $p \in \mathbf{P}$. Since $H_p = H/p^\omega H = H/p^\omega G \cap H \cong H + p^\omega G/p^\omega G \leq G/p^\omega G = G_p^*$ and $G \in \mathcal{C}$, theorem 1 implies that H_p is a p -pure subgroup of a free J_p -module F . Let R be the submodule of F generated

by H_p ; then R is free and clearly R/H_p is divisible. Applying theorem 1, we conclude that $J_p \otimes H$ is a completely decomposable J_p -module. Therefore $H \in \mathcal{C}$, as required. $\square$

Let us note that $\mathcal{C}$ is the class of all torsion-free groups G such that, for every prime p , the $\mathbb{Z}_p$ -module $\mathbb{Z}_p \otimes G/p^\omega G$ admits $\mathbb{Q}_p$, the field of p -adic numbers, as a splitting field in the sense of [3]. Hence proposition 6 is similar to half of the first assertion of ([3] Corollary 2.2). Indeed, since free groups belong to $\mathcal{C}$, the class $\mathcal{C}$ is not closed with respect to torsion-free homomorphic images.

Observe now that there exists a group G such that $G \notin \mathcal{C}$, but $G \leq H$ where H is reduced, $H \in \mathcal{C}$ and H/G is a torsion group. For instance, let $G = \mathbb{Z}^{\mathbb{N}}$ and let $H = \langle G \rangle_* \leq \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. Then, by proposition 2, $G \notin \mathcal{C}$ and $\prod_{p \in \mathbb{P}} \mathbb{Z}_p \in \mathcal{C}$; thus, by proposition 6, $H \in \mathcal{C}$ and the rest is obvious.

Nevertheless, it is easy to find examples of reduced torsion-free groups which are not subgroups of a reduced group of $\mathcal{C}$. Indeed we can prove the following

PROPOSITION 7. *Let G be a reduced torsion-free cotorsion group and let R be a reduced group of $\mathcal{C}$. If $G \leq R$, then $G \in \mathcal{C}$.*

PROOF. Let G and R be as in the hypothesis and let $G \leq R$; we claim that $G \in \mathcal{C}$. By corollary 4, it is enough to prove that, if p is any prime and G is a J_p -module as in the hypotheses, then G is a J_p -module of finite rank. Indeed, by theorem 1, there is a free J_p -module F such that $R/p^\omega R \leq F$. Since $G \cap p^\omega R = 0$, we deduce that G is isomorphic to a submodule of F . Consequently G must be a J_p -module of finite rank and the proof is complete. $\square$

Recall that ([5] Lemma 4), if G is a torsion-free group such that $G/H \in \mathcal{C}$ where H is of finite rank, then $G \in \mathcal{C}$. We shall now see that the restriction on H cannot be omitted.

PROPOSITION 8. *The class $\mathcal{C}$ is not closed under extensions.*

PROOF. Let $G = J_p^{\mathbb{N}}$ and let B be a free J_p -module such that $B \leq G$ and G/B is divisible. Evidently $B, G/B \in \mathcal{C}$, while, by proposition 2, $G \notin \mathcal{C}$. $\square$

Let G be a torsion-free group and let $H \leq G$. Then the following are obvious consequences of the preceding results.

- (i) $H, G/H \in \mathcal{C} \not\Rightarrow G \in \mathcal{C}$;
- (ii) $H, G \in \mathcal{C} \not\Rightarrow G/H \in \mathcal{C}$;
- (iii) $G, G/H \in \mathcal{C} \Rightarrow H \in \mathcal{C}$.

A lot of slender groups belong to $\mathcal{C}$; however, we have the following

PROPOSITION 9. *There exists a slender group G such that $G \notin \mathcal{C}$.*

PROOF. Fix $p \in \mathbf{P}$ and let G be a $\mathbf{Z}_p$ -module of J_p such that $1 \in G$; $p^n J_p \not\leq G$ for every $n \in \mathbf{N}$ and G is not p -pure in J_p . Since G is slender ([4] Ch. V § 2.4 Theorem), it remains only to check that $G \notin \mathcal{C}$. Suppose the contrary. Then, by theorem 1, there is an embedding $\psi: G \rightarrow F$ where F is a free J_p -module and $\psi(G)$ is p -pure in F . If $\bar{\psi}$ is the extension of ψ to the cotorsion completions of G and F , evidently $\psi(g) = \bar{\psi}(g) = g\bar{\psi}(1) = g\psi(1)$ for all $g \in G$. Hence $\psi(G) \leq J_p \psi(1)$ and this enables us to assume $F = J_p^n$ for some $n \in \mathbf{N}$. Let now $\psi(1) = (\alpha_1, \dots, \alpha_n)$ where $\alpha_i \in J_p$ for every i . Then the hypothesis that $\psi(G)$ is p -pure in F guarantees that $\alpha_k \notin pJ_p$ for some k . Let $\tau: G \rightarrow J_p$ be the map that takes g to $\alpha_k g$ for all $g \in G$. To see that $\tau(G)$ is p -pure in J_p , suppose $p^n x = \tau(g)$ with $x \in J_p$, $n \in \mathbf{N}$ and $g \in G$. Since $\psi(G)$ is p -pure in F and $h(\alpha_i g) \geq h(\alpha_k g) \geq n$ for every i , there exists $g' \in G$ such that $\psi(g) = p^n \psi(g')$. Therefore $p^n x = \tau(g) = p^n \tau(g')$ and so $\tau(G)$ is p -pure in J_p . On the other hand, since G is not p -pure in J_p , we can choose $\alpha \in J_p$, $m \in \mathbf{N}$ such that $\alpha \notin G$ and $p^m \alpha \in G$. Consequently $\alpha_k \alpha \notin \tau(G)$, while $p^m \alpha_k \alpha \in \tau(G)$. This contradiction implies that $G \notin \mathcal{C}$, as claimed. $\square$

By the preceding result, $\mathcal{C}$ does not contain the class of all $\mathbf{Z}_p$ -modules that are subgroups with divisible cokernel of a free J_p -module.

Let p be a prime and let $R = K \cap J_p$ where, as in [3], K is a field such that $\mathbf{Q} < K < \mathbf{Q}_p$. If $G \in \mathcal{C}$, then it is natural to ask if $R \otimes G$ is a completely decomposable R -module. We shall prove that, if $R \otimes G$ is a completely decomposable R -module, then R contains the subgroup of J_p generated by the set S , where

$$S = \{\alpha \in J_p : \alpha x \in G_p = G/p^\omega G \text{ for some } x \in G_p \setminus pG_p\}.$$

Indeed, with the same arguments used in the first part of theorem 1, it is easy to show that there is an injective homomorphism $\psi: G_p \rightarrow F$ such that F is a free R -module and $\psi(G_p)$ is p -pure in F . Let now $\alpha x \in G_p$ where $\alpha \in J_p$ and $x \in G_p \setminus pG_p$. Since F has a decomposition

of the form $F = F' \oplus F''$ with $\psi(x) \in F'$ and $F' \cong R^n$ for some $n \in \mathbb{N}$, we may write $\varphi(x) = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in R$ for all i . Moreover, since $x \notin pG_p$ and $\psi(G_p)$ is p -pure in F , there exists some k such that $\alpha_k \notin pR$; hence α_k is a unit of R . Since $\alpha x \in G_p$ and $\psi(\alpha x) = \alpha\psi(x)$, it immediately follows that $\alpha \in R$. This completes the proof.

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