RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 59 (1978), p. 11-15

<http://www.numdam.org/item?id=RSMUP_1978__59__11_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 59 (1978)

Normal Subgroups and Projectivities of Finite Groups.

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Let G be a group; by a projectivity $\sigma: G \to G^\sigma$ we shall mean an isomorphism of the lattice $\mathfrak{L}(G)$ of subgroups of G onto the lattice $\mathfrak{L}(G^\sigma)$ of subgroups of the group G^σ. If H is a normal subgroup of G, then H^σ needs not be normal in G^σ; H^σ is a Dedekind subgroup of G^σ, i.e. it satisfies the modular identities $(H^\sigma \vee U) \wedge V = H^\sigma \vee (U \wedge V)$ for every pair U, V of subgroups of G^σ such that $H^\sigma \leqslant V$, and $(U \vee H^\sigma) \wedge V = U \vee (H^\sigma \wedge V)$ for every pair U, V of subgroups of G^σ such that $U \leqslant V$. Assuming that G is finite, R. Schmidt has shown in [3] that if $S^\sigma = (H^\sigma)^{G^\sigma}$ is the normal closure of H^σ in G^σ and $T^\sigma = (H^\sigma)_{G^\sigma}$ is the core of H^σ in G^σ, then S and T are normal subgroups of G and S/T is supersolubly embedded in G/T. In the above notation, we prove that if $|G|$ is odd, then H/T is abelian.

1. – LEMMA. Let D be a Dedekind subgroup of the group G. Then $D_G = \bigwedge \{D^x | x \in \mathfrak{J}(D)\}$, where $\mathfrak{J}(D) = \{x \in G \,|\, |\langle x\rangle/\langle x\rangle \wedge D|$ is either ∞ or a prime power$\}$.

PROOF. We have to show that, for every $g \in G$, $D^g \geqslant \bigwedge \{D^x | x \in \mathfrak{J}(D)\}$. If $g \in \mathfrak{J}(D)$ this is obvious; so, assume $|\langle g\rangle/\langle g\rangle \wedge D| = n$ is finite, $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ is the prime decomposition of n, $n_i = n/p_i^{\alpha_i}$, and r_i, s_i are integers such that $1 = p_i^{\alpha_i} r_i + n_i s_i$ $(i = 1, \dots, t)$. Since $g^{n_i s_i} \in \mathfrak{J}(D)$, we have

$$\bigwedge \{D^x | x \in \mathfrak{J}(D)\} \leqslant \bigwedge_{i=1}^{t} D^{g^{n_i s_i}} \leqslant \bigwedge_{i=1}^{t} (D \vee \langle g^{p_i^{\alpha_i}}\rangle)^{g^{n_i s_i}} = \bigwedge_{i=1}^{t} (D \vee \langle g^{p_i^{\alpha_i}}\rangle)^g = D^g ,$$

q.e.d.

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COROLLARY. If D is a Dedekind subgroup of G, then $D_G = \bigwedge_{x \in \mathfrak{J}(D)} D_{\langle D, x\rangle}$.

THEOREM. Let G be a finite p-group, $p \neq 2$, $\sigma: G \to G^\sigma$ an index-preserving projectivity, H a normal subgroup of G such that G/H is cyclic and $H^\sigma_{G^\sigma} = 1$. Then H is abelian.

REMARK. The proof does not work for $p = 2$; but no actual counterexample is known to the author.

PROOF. We proceed by induction on $|H|$. If $|\Omega_1(H)| = p$, then H is cyclic; if $\Omega_1(H) = H$, then for every $x_1 \in G^\sigma$ $H^{\sigma x_1 \sigma^{-1}} \wedge H$ is quasinormal, hence normal in H with cyclic factor group: $H' \leqslant \bigwedge_{x_1 \in G^\sigma} (H^{\sigma x_1 \sigma^{-1}} \wedge H) = 1$ follows. So, we may assume that $|\Omega_1(H)| > p$ and that the exponent p^r of H is greater than p; assume also that $G = \langle a\rangle H$. We have $\langle a\rangle \wedge H = 1$; by [1] $\Omega_1(G) = \Omega_1(\langle a\rangle)\,\Omega_1(H)$ and $H^\sigma \Omega_1(G^\sigma)/\Omega_1(G^\sigma)$ has trivial core in $G^\sigma/\Omega_1(G^\sigma)$; by induction $H/\Omega_1(H) \cong H\Omega_1(G)/\Omega_1(G)$ is abelian. We now choose $a_1 \in G^\sigma$ such that $\langle a_1\rangle = \langle a\rangle^\sigma$; $H^\sigma \wedge \mathrm{C}_{G^\sigma}(a_1) = 1$, so the centralizer of a_1 in $\Omega_1(G^\sigma) = \Omega_1(\langle a_1\rangle) \times \Omega_1(H^\sigma)$ has order p; it follows that $|\mathrm{C}_{\Omega_1(H)}(a)| = p$, and there is a basis $e_0, e_1, \ldots, e_m$ (m is $\geqslant 3$) of $\Omega_1(G)$ such that

$$\langle e_0\rangle = \Omega_1(\langle a\rangle)\,, \qquad \langle e_1, \ldots, e_m\rangle = \Omega_1(H)\,,$$

$$[e_0, a] = [e_1, a] = 1\,, \qquad e_i^a = e_{i-1} e_i \qquad \text{for } i > 1\,.$$

In particular, every non trivial normal subgroup of G contained in $\Omega_1(H)$ contains $\langle e_1\rangle$. Put $\langle f_i\rangle = \langle e_i\rangle^\sigma$: then $\langle f_0\rangle = \Omega_1(\langle a_1\rangle) = Z(G^\sigma) \wedge \Omega_1(G^\sigma)$; and we can choose the symbols so that $f_1^{a_1} = f_0 f_1$, $\langle f_0 f_1\rangle = \langle e_0 e_1\rangle^\sigma$. Then $H^{\sigma a_1} \wedge \langle f_0, f_1\rangle = \langle f_0 f_1\rangle$, $\langle e_1\rangle \nleqslant H^{\sigma a_1 \sigma^{-1}}$, and if we put $(H^{\sigma a_1 \sigma^{-1}})_G = K^{\sigma a_1 \sigma^{-1}}$ we get $K^{\sigma a_1 \sigma^{-1}} \wedge \Omega_1(H) = 1$; hence K is cyclic, $\Omega_1(K^{\sigma a_1 \sigma^{-1}}) = \langle e_0 e_1\rangle$, $\Omega_1(K) = \langle e_1\rangle$. By induction H/K is abelian, and so $H' \leqslant \langle e_1\rangle$, $H^{\sigma a_1 \sigma^{-1}}/\langle e_0 e_1\rangle$ is a modular p-group.

If now $Q = H \wedge H^{\sigma a_1 \sigma^{-1}}$, we have $Q \lhd H^{\sigma a_1 \sigma^{-1}}$, $Q \wedge \langle e_1\rangle = 1$ and Q is abelian. H is nilpotent of class $\leqslant 2$, so it is regular and $1 \neq \mathfrak{V}_{r-1}(H) = \{x^{p^{r-1}} \mid x \in H\}$ is a subgroup of $\Omega_1(H)$ which is normal in G: but then $e_1 \in \mathfrak{V}_{r-1}(H)$, i.e. $\langle e_1\rangle = \Omega_1(\langle h\rangle)$ with $h \in H$, $|h| = p^r$; put $\langle k\rangle = \langle h\rangle^{\sigma a_1 \sigma^{-1}}$. We have $\langle h\rangle \wedge Q = \langle k\rangle \wedge Q = 1$ and, since $|H| \leqslant p^r |Q|$, we get $H = \langle h\rangle Q$, $H^{\sigma a_1 \sigma^{-1}} = \langle k\rangle Q$; furthermore $\langle k\rangle$, which contains $\langle e_0 e_1\rangle$, is quasi-normal in $H^{\sigma a_1 \sigma^{-1}}$. For every $x \in Q$ $\langle x, k\rangle \wedge Q = \langle x\rangle \lhd \langle x, k\rangle$, i.e. k induces a power automorphism on the abelian group Q: $\langle k\rangle Q = H^{\sigma a_1 \sigma^{-1}}$ is modular, as well as H, H^σ, $H^{\sigma a_1}$.

Let Q have exponent p^s ($\leqslant p^r$); $\mathfrak{V}_{s-1}(Q^G)$ is a normal subgroup of G contained in $\Omega_1(H)$, hence $e_1 \in \mathfrak{V}_{s-1}(Q^G)$; we see that $Q^G = \Omega_s(H)$; since $Q \lhd H^{\sigma a_1 \sigma^{-1}}$ and $G = \langle a \rangle H^{\sigma a_1 \sigma^{-1}}$ we have $Q^G = Q^{\langle a \rangle}$ and, more, the interval $[H/Q]$ being a chain, $Q^G = QQ^a$. We shall now prove that $[k, a]$ centralizes Q^G. First of all, $\langle k \rangle \wedge H = 1$ and $\Omega_r(G) = \Omega_r(\langle a \rangle)H$ imply $\langle k \rangle H = \Omega_r(G)$, i.e. $k = a^\beta h^\nu x$ where $|a^\beta| = p^r$, $x \in Q$; a simple induction based on [1] proves that from $|h^\nu| < p^r$ would follow the contradiction $\Omega_1(\langle a^\beta h^\nu \rangle) = \Omega_1(\langle kx^{-1} \rangle) = \langle e_0 \rangle \leqslant H^{\sigma a_1 \sigma^{-1}}$; moreover, we have $\langle h \rangle \Omega_{r-1}(H) \lhd G$: if $p^s < p^r$ this is H, while if $p^s = p^r$ it is the unique subgroup of H which is cyclic modulo $\Omega_{r-1}(H)$ and whose $\mathfrak{V}_{r-1}$ is $\langle e_1 \rangle$. From this it follows that $\langle h \rangle^\sigma \Omega_{r-1}(G^\sigma)$ is quasinormal of order p in $G^\sigma/\Omega_{r-1}(G^\sigma)$, hence $\langle k \rangle^\sigma \Omega_{r-1}(G^\sigma) \in \langle a^\beta \rangle^\sigma \cdot \langle h \rangle^\sigma \Omega_{r-1}(G^\sigma)$: and we may assume $x \in \Omega_{r-1}(H)$. If now the exponent of Q is $< p^r$, then $\Omega_{r-1}(H)$ is abelian and $[k, a] = [h^\nu, a]^x [x, a] \in \Omega_{r-1}(H)$, i.e. $[k, a]$ centralizes $Q^G = \Omega_s(H) \leqslant \Omega_{r-1}(H)$; if the exponent of Q is p^r (and assuming that H is not abelian, since otherwise everything is trivial) $H/Z(H)$ has order p^2, the order of $H/\Omega_{r-1}(H)$ is at least p^2, $Z(H) \leqslant \leqslant \Omega_{r-1}(H)$ i.e. $\Omega_{r-1}(H) = Z(H)$ and $[a, k]$ in this case also centralizes $Q^G = H$. It follows that k and k^a induce the same automorphism on Q^G; k operates as a power on Q, k^a (hence k) gives the same power on Q^a, and finally k operates as a power on $QQ^a = Q^G$ (if Q^G is abelian this is obvious; if $Q^G = H$, one checks that in a modular p-group with derived group of order p ($p \neq 2$) powers congruent to 1 (mod. p) are indeed automorphisms), so that the subgroup of the elements of G which induce a power automorphism in $\Omega_s(H)$ contains $\langle k \rangle \Omega_{r-1}(H) = = \langle k \rangle \Omega_{r-1}(G)$; furthermore, we can determine s by the condition that $H/\Omega_s(H)$ is cyclic, but $H/\Omega_{s-1}(H)$ is not. We now look at $\langle h \rangle^{\sigma a_1^{-1} \sigma^{-1}} = \langle l \rangle$: we have $\langle l \rangle^\sigma = \langle h \rangle^{\sigma a_1^{-1}}$, $(\langle l \rangle \Omega_{r-1}(G))^\sigma \leqslant (\langle a^\beta \rangle \cdot \langle h \rangle \Omega_{r-1}(G))^\sigma$, $(\langle l \rangle \Omega_{r-1}(G))^\sigma$ is neither $(\langle h \rangle \Omega_{r-1}(G))^\sigma$ nor $(\langle k \rangle \Omega_{r-1}(G))^\sigma = (\langle h \rangle \Omega_{r-1}(G))^{\sigma a_1}$ because the centralizer of $\langle a_1 \rangle \Omega_{r-1}(G^\sigma)/\Omega_{r-1}(G^\sigma)$ in $\Omega_r(G^\sigma)/\Omega_{r-1}(G^\sigma)$ is $\langle a_1^\beta \rangle \Omega_{r-1}(G^\sigma)/\Omega_{r-1}(G^\sigma)$. The properties we proved above for k hold for l too, i.e. l is in the subgroup of the elements of G which induce a power on $\Omega_s(H)$; but then this subgroup contains $\langle a^\beta \rangle$ and $\langle h \rangle$: so $[h, Q] \leqslant \langle e_1 \rangle \wedge Q = 1$, and H is abelian, q.e.d.

Corollary. Let $\sigma: G \to G^\sigma$ be a projectivity, with G a finite group of odd order. If H is a normal subgroup of G and $T = (H^\sigma_{G^\sigma})^{\sigma^{-1}}$, then H/T is abelian.

Proof. Let G be a counterexample of minimum order; we shall show that G satisfies the hypotheses of the theorem. Obviously $T = 1$,

and the lemma and corollary imply that $G = H\langle a\rangle$ where $|\langle a\rangle : \langle a\rangle \wedge H|$ is a prime power. G is not a P-group ([4], p. 11), for proper normal subgroups of P-groups are abelian. H^σ is a Dedekind subgroup of G^σ and the interval $[G/H]$ is a chain: by [2] G^σ is a p-group. Since G is neither a P-group, nor cyclic, σ must be index-preserving, q.e.d.

2. – The hypotheses of the theorem do not imply that H^σ is abelian, as the following example shows.

Let E, F be elementary abelian p-groups with bases $e_0, e_1, \ldots, e_p$; $f_0, f_1, \ldots, f_p$ respectively (p is an odd prime), and consider the groups

$$K = \langle E, u \mid u^{p^2} = e_0,\ e_i^u = e_i e_{i-1}\ (i = 2, \ldots, p),\ e_1^u = e_1\rangle$$

and

$$K_1 = \langle F, x \mid x^{p^2} = f_0,\ f_i^x = f_i f_{i-1}\ (i = 1, \ldots, p)\rangle\,.$$

K, K_1 have both order p^{p+3}; the map $\alpha\colon K \to K_1$ such that

$$\left(\prod_{i=1}^{p} e_i^{r_i}\right) u^r \mapsto \left(\prod_{i=1}^{p} f_i^{r_i}\right) x^r\,,$$

when restricted to E, is an isomorphism $E \to F$; moreover, it induces a map $\bar{\alpha}\colon K/\langle e_0\rangle \to K_1/\langle f_0\rangle$ which is also an isomorphism. Since every subgroup of K, not contained in E, contains $\langle e_0\rangle$, and similarly for K_1, α induces a projectivity $K \to K_1$ [3]. Now extend K to $G = \langle K, v \mid [v, K] = 1,\ v^p = e_1^{-1}\rangle$ and K_1 to

$$G_1 = \langle K_1, y \mid x^y = x^{1+p},\ f_i^y = f_i\ (i = 1, \ldots, p-1),\ f_p^y = f_p f_1^{-1},\ y^p = f_1^{-1}\rangle\,;$$

G, G_1 have order p^{p+4}; G_1 is, apart from slight changes of notation, Thompson's example [5]. $\Omega_2(G) = \langle E, u^p, v\rangle$ is abelian, while $\Omega_2(G_1) = \langle F, x^p, y\rangle$ is modular non-abelian, for f_p operates on the abelian group $\langle x^p, y f_p^{-1}, f_2, \ldots, f_{p-1}\rangle$ as the power $1 + p$. For every $a \in K_1$ $\langle (ay)^p\rangle = \langle a^p y^p\rangle$: if $a \in \Omega_2(G_1)$ $\langle a, y\rangle$ is modular, hence regular, with derived subgroup of order p at most, so $(ay)^p = a^p y^p$; if $|a| = p^3$ we have $\langle a^p, f_1\rangle = \langle x^p, f_1\rangle = [K_1, y]$, $\langle a^p\rangle \lhd \langle a, y\rangle$, $\langle a, y\rangle/\langle a^p\rangle$ is modular, so $\langle a, y\rangle$ is again modular, $\mho_1(\langle a, y\rangle') = \langle f_0\rangle = \langle a^{p^2}\rangle = \langle (ay)^{p^2}\rangle$, and $a^p y^p = (ay)^{p+mp^2}$ for some integer m. Hence, for every $k \in K$, $\langle k^\alpha y\rangle \wedge K_1 = (\langle kv\rangle \wedge K)^\alpha$. Every subgroup L of G, $L \not\leqslant K$, can be written as $L = (L \wedge K)\langle kv\rangle$ for some $k \in K$, and similarly for G_1; we

try to define a projectivity $\sigma: G \to G_1$ by the stipulation $L^\sigma = L^\alpha$ if $L \leqslant K$, $L^\sigma = (L \wedge K)^\alpha \langle k^\alpha y \rangle$ for $L = (L \wedge K)\langle kv \rangle \not\leqslant K$. The above discussion shows that this definition makes sense for cyclic subgroups; and for the intervals $[G/\langle u^p, e_1 \rangle]$, $[G_1/\langle x^p, f_1 \rangle]$ it is the map induced by the isomorphism

$$\beta: G/\langle u^p, e_1 \rangle \to G_1/\langle x^p, f_1 \rangle$$

such that $\bar{e}_i^\beta = \bar{f}_i$, $\bar{u}^\beta = \bar{x}$, $\bar{v}^\beta = \bar{y}$ (with the usual meaning of $^-$). Let now $L = (L \wedge K)\langle kv \rangle$ be a non-cyclic subgroup of G, not containing $\langle u^p, e_1 \rangle$; it is easily seen that $L \leqslant \Omega_2(G)$, so that $(L \wedge K)^\alpha$ and $\langle k^\alpha y \rangle$ are contained in $\Omega_2(G_1)$, which is modular, and $(L \wedge K)^\alpha \langle k^\alpha y \rangle$ is indeed a subgroup. Again with $L \leqslant \Omega_2(G)$, let k, $k' \in K$ be such that

$$L = (L \wedge K)\langle kv \rangle = (L \wedge K)\langle k'v \rangle ;$$

then $k' = tk$ with $t \in L \wedge K$, and one directly checks that $k'^\alpha = (t^\alpha)^{1+np} k^\alpha$ for some integer n, so that

$$(L \wedge K)^\alpha \langle k^\alpha y \rangle = (L \wedge K)^\alpha \langle k'^\alpha y \rangle .$$

This remark shows that σ is indeed a projectivity. If now $H = \langle v, e_2, \ldots, e_p \rangle$, we have $H \lhd G$, $H_{G^\sigma}^\sigma = 1$, and H^σ is not abelian.

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Manoscritto pervenuto in redazione il 3 novembre 1977.