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On a question of Josef Novák about convergence spaces

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On a question of Josef Novák about convergence spaces.

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Summary: In this paper we construct an example which answers to a question posed by Josef Novák about the validity of a statement in a convergence space.

Sommario: Viene costruito un esempio che risponde ad una domanda di Josef Novák relativamente alla validità di una proposizione per spazi di convergenza.

1. Introduction.

In a convergence (sequential) space (L, λ), Novák (see [5]) considered the following statement:

(+) If $A_n \subseteq L$ and $z \in L - \cup \lambda A_n$ is a point each neighbourhood of which contains points of A_n for nearly all n, then there is a sequence of $x_n \in A_n$ converging to z.

He asked if there exists a convergence space such that its convergence is the star convergence and that (+) is not true. (Problem 1.b).

In this paper we give an example which solves the above question in the affirmative and we add some considerations about cross properties in convergence spaces.

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This work is selfcontained. The notations and terms are summarized in § 2, there we give the tools necessary to the understanding of the text.

2. Preliminary.

Convergence structure, according Mario Dolcher, (see [2]), is a pair (L, λ), where L is a non void set and λ is a law which associates to each point x of L, a set $\mathfrak{J}_x$ of sequences of points of L, λ satisfying to suitable axioms. If $S = (s_n)_n \in \mathfrak{J}_x$, we will write $S \rightarrow x$ and read : « S converges to x ».

The axioms required by Dolcher for λ are the following:

1) $(x) \rightarrow x$, for every x in L (where (x) is the constant sequence $x, x, \ldots, x, \ldots$).
2) If a sequence S converges to x, then every subsequence S' of S, converges to x.
3) If a sequence S does not converge to x, then there exists a subsequence S' of S, no subsequences of which converge to x.

(Novák call such a structure, a multivalued convergence space, in [4]). Moreover, if the convergence is onevalued, (that is with uniqueness of limit) so that axiom

0) $$S \rightarrow x\,,\ S \rightarrow y \Rightarrow x = y\,,$$

holds, λ turns be a *star convergence* on L, in the sense of Novák.

If λ is a convergence in a some sense and A is a subset of L, by λA (or $\hat{A}$, according to Dolcher [2]), we will denote the set of all the limit points of sequences in A. If λ satisfies axioms 1) and 2), then λ can be thought like a closure operator (in the sense of Čech [1]), then a subset U of L is said to be a neighbourhood (*λ-neighbourhood*) of a point x, if $x \in L - \lambda(L - U)$. In other words, ([5]) U is a λ-neighbourhood of x if and only if $(x_n)_n \rightarrow x$ implies that $x_n \in U$ for nearly all n. The pair (L, λ) is also said to be a *convergence space* (see [4], [5]).

We remark that a convergence λ on L, given by the system $(\mathfrak{J}_x)_{x \in L}$ satisfying the axioms from 1) to 3), can be determinated by a smaller system $(\mathfrak{B}_x)_{x\in L}$, where $\mathfrak{B}_x \subseteq \mathfrak{J}_x \; \forall_x$; $(\mathfrak{B}_x)_x$ is called by Dolcher *convergence base for* λ.

Precisely $(\mathfrak{B}_x)_x$ must be a system such that $(\mathfrak{J}_x)_x$ is the smallest system which contains $(\mathfrak{B}_x)_x$ and which satisfies the prescribed axioms. If we introduce the following operations δ and ξ acting on sets of sequences :

(δ) $S \in \delta\mathfrak{C}$ iff $\exists\; R \in \mathfrak{C}$ and S is subsequence of R.

(ξ) $S \in \xi\mathfrak{C}$ iff for every subsequence S' of S, exists a subsequence S'' of S' such that $S'' \in \mathfrak{C}$,

then we immediately observe that δ and ξ are idempotent and $\xi(\delta\mathfrak{C})$ is the smallest set of sequences which contains $\mathfrak{C}$ and which is closed with respect to δ and ξ. Since δ and ξ replace axioms 2) and 3), we have that $\mathfrak{J}_x = \xi(\delta\mathfrak{B}_x)$, provided that $(x) \in \mathfrak{B}_x$ for every x. So, when we will give a convergence (in the sense of Dolcher) structure on a set L, it will suffice to assign to each point x, a set $\mathfrak{B}_x$ to which belong (x) and the other sequences that we would like to converge to x, and then will consider the convergence that is generated (through δ and ξ) by the system $(\mathfrak{B}_x)_x$. Notice that convergence is onevalued iff for $x \neq y$, is $\delta\,\mathfrak{B}_x \cap \delta\,\mathfrak{B}_y = \varnothing$.

3. Exibition of the example.

The aim of this chapter is to give an example of a convergence space with star convergence where statement (+) does not hold. For this purpose, it is necessary a previous lemma.

LEMMA. Let N be a countable set, then there exists a set $\mathfrak{F}^*$ of countable (1) subsets of N, such that :

i) *$\mathfrak{F}^*$ has the power of the continuum.*

ii) *$F_1, F_2 \in \mathfrak{F}^* \Rightarrow F_1 \cap F_2$ is a finite set* (2).

iii) *For each countable subset G of N, there exists an $F \in \mathfrak{F}^*$, such that $F \cap G$ is countable.*

(1) In this lemma, a set is said to be countable if it is infinite and countable.

(2) For conditions i) and ii), cfr. [3] (Ex. 6Q pag. 97).

PROOF. Let $\Psi = \{ \mathfrak{F} : \mathfrak{F}$ satisfies property i) and ii) of the Lemma$\}$. $\Psi \neq \emptyset$, in fact (see Gillmann-Jerison [3], Ex. 6Q p. 97) exists a set ξ which satisfies i) and ii). (ξ is obtained by one to one correspondence with a set of sequences of rational numbers such that each irrational number is the limit of exactly one of these sequences). Now, is easy to prove, using Zorn's Lemma, that Ψ possesses an element $\mathfrak{F}^*$, which is maximal in Ψ with respect to the inclusion order.

We have only to show that $\mathfrak{F}^*$ satisfies iii). If this does not happen, then there exists a countable subset H of N such that $H \cap F$ is finite for every F which belongs to $\mathfrak{F}^*$; so, it is $\mathfrak{F}^* \cup \{H\} \in \Psi$.

— A contradiction with the maximality of $\mathfrak{F}^*$ in Ψ.

q.e.d.

Now we can present the preannounced example:

EXAMPLE: *of a convergence space with star convergence where statement* $(+)$ *does not hold.*

Let L be the set $\{a_{r,s} : r, s = 1, 2, ...\} \cup \{z\}$.

It can be thought like an infinite matrix whose horizontal rows are sequences $A_r = (a_{r,s})_s$, together a point z.

Let N be the set of natural numbers, $\mathfrak{R}$ a set of subsets of N which, according to the Lemma, fulfils the conditions from i) to iii), $\mathfrak{S}$ the set of all sequences of natural numbers.

Let f be a one to one correspondence from $\mathfrak{S}$ onto $\mathfrak{R}$ and we define a map g from $\mathfrak{R}$ into $\mathfrak{S}$ which associates to $R = \{r_n\}_n$, the sequence $g(R) = (s_{r_n} + 1)_n$, where $(s_n)_n = f^{-1}(R)$.

Now we can assign (by a convergence base) a convergence on the set L.

Let λ be the following convergence:

$'$) The points $a_{r,s}$ are all isolated points (i. e. converge to $a_{r,s}$ only the sequences whose terms are nearly all equal to $a_{r,s}$).

$''$) Converge to z, the constant sequence (z) and the sequences $(a_{r_i,s_i})_i$ where $(r_i)_i$ is an increasing sequence of natural numbers (indices

of row) such that $\{r_i\}_i = R \in \mathfrak{R}$ and $(s_i)_i = S$ is a sequence greater or equal than $g(R)$ in the lexicographic order (³).
Converge to z exactly those sequences which can be deduced by the precedings (by δ and ξ).

It is immediate to verify that convergence λ above defined is onevalued ; so

(L, λ) is a convergence space where λ is a star convergence.

Observe that for each horizontal row A_r, is $\lambda A_r = A_r$ (in fact, no row converges to z and the points of the matrix are all isolated). So, we have that $z \in L - \cup \lambda A_r$.

We prove now that *each neighbourhood of z contains points of A_r for nearly all indices r.*

Proof. Let $U(z)$ be a λ-neighbourhood of z such that there is an increasing sequence $(r_i)_i$ of indices of row such that in $U(z)$ there are not points of the row A_{r_i}. From the property iii) of the set $\mathfrak{R}$ (see the Lemma) we know that there exists a sequence of row indices $(\tilde{r}_i)_i = \tilde{R} \in \mathfrak{R}$ which has a subsequence in common with the sequence $(r_i)_i$. From $''$) in the definition of λ, we can choose a suitable element $a_{\tilde{r}_i, \tilde{s}_i}$ in the row $A_{\tilde{r}_i}$, in such a way to obtain a sequence $(a_{\tilde{r}_i, \tilde{s}_i})_i$ which converges to z. Since every subsequence of the preceding one must converges to z, we conclude that there is a sequence of elements belonging to the rows A_{r_i}, for infinitely many i, which converges to z. Elements of this sequence are nearly all in $U(z)$ and so we contradict the initial assumption.

q.e.d.

At last, we prove that *there is no sequence of $x_r \in A_r$ which converges to z.*

Proof. If a sequence $(a_{n,s_n})_n$ converges to z, it must converge to z together with every its subsequence $(a_{r_i, s_{r_i}})_i$, while the sequence

(³) If $S = (s_n)_n$ and $S' = (s'_n)_n$ are sequences of natural numbers, we pose $S \leq S'$ if and only if for each index n, it is $s_n \leq s'_n$. The order so obtained is called *lexicographic*.

$(a_{\bar{r}_i, s_{\bar{r}_i}})_i$ where $(\bar{r}_i)_i = \bar{R} = f(S)$ and $S = (s_n)_n$, does not converge to z, thanks to the definition of λ. In fact, the sequence of column indices $(s_{\bar{r}_i})_i = S \circ f(S)$ is less (in the lexicographic order) than the sequence $(s_{\bar{r}_i} + l)_i$ which is the least sequence of indices of column such that $(a_{\bar{r}_i, o})_i$ converges to z. Neither the sequence $(a_{\bar{r}_i, \bar{s}_i})_i$ converges to z as a sequence deduced by δ and ξ from suitable other sequence of the base, converging to z. In fact, the assumption ii) on $\mathfrak{R}$ and the definition of the convergence λ exclude this eventuality.

q.e.d.

Our aim is so attained.

4. – Cross sequences in convergence spaces.

In a convergence space (L, λ), we say that *a matrix* $\big((x_{r,s})\big)_{r,s}$ *converges to a sequence* $(y_n)_n$ iff the r-th row $(x_{r,s})_s$ of the matrix converges to the r-th term y_r of the sequence.

We say that *cross property* (respectively *subcross property*) holds in (L, λ) iff for each matrix $\big((x_{r,s})\big)_{r,s}$, for each sequence $(y_r)_r$ and for each point z, such that the matrix converges to the sequence and this converges to the point, there exists a cross sequence $(x_{r,s_r})_r$ (respectively a cross subsequence $(x_{r_i,s_i})_i$) of the matrix, which converges to z.

Weak cross property (resp. *weak subcross property*) are defined in the same way only with the weaker assumption that $(y_r)_r$ is the constant sequence (z).

If we indicate with C and C' cross and subross property, and with C_0 and C'_0 the corresponding weaker conditions, we have immediately the following inferences:

$$\begin{array}{ccc} C & \Rightarrow & C_0 \\ \Downarrow & & \Downarrow \\ C' & \Rightarrow & C'_0 . \end{array}$$

By the comparision of the above four conditions, we notice that there exists an example (see [2], p. 87) of convergence space where C_0 holds and C' does not hold, while at the present status of our knowledges, we do not know whether $C_0' \Rightarrow C_0$ (respl. $C' \Rightarrow C$) is true or false. As a partial result, by a light modification of structure of the main example in the section 3, (modification only consists in imposing to each row of the matrix, to converge to z), we can present a structure where a matrix exists such that each its row converges to a point z, but no cross sequence converges to z, while every submatrix (that is a matrix obtained by the preceding one, catching infinitely many points from infinitely many rows) possesses a cross subsequence converging to z.

A property related with the preceding is the idempotency of the closure operator λ (see section 2), called by Dolcher in [2], Hedrick's condition. It is easy to prove that a sufficient condition for $\lambda\lambda = \lambda$, is the validity of C' (see [2]), moreover it can be proved (see [6], theorem 2, pag. 74) that C' holds if and only if C_0' and Hedrick's condition are both satisfied (this result can be proved also if λ is not onevalued). We conclude remarking that in a topological space first countable, with the common notion of convergence of sequences, all four cross properties always are satisfied.

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