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The Weak Cauchy Problem for Abstract Differential Equations.

S. ZAIDMAN (*)

Introduction.

We consider the weak Cauchy problem in arbitrary Banach space for equations $(d/dt - A)u = 0$, as were defined by Kato-Tanabe. After proving some elementary relationships, we obtain a result which shows how uniqueness of Cauchy problem for strong solutions in the second dual space implies uniqueness of the weak Cauchy problem.

A simple result by Barbu-Zaidman (Notices A.M.S., April 1973, 73T-B120) gets then a new proof, and an uniqueness result for weakened solutions by Liubic-Krein gets a partial extension.

The last result is a certain extension of Barbu-Zaidman result to non-reflexive B -spaces, using as a main tool in the proof Phillips's theorem on dual semi-groups in B -spaces.

§ 1. — Let $\mathfrak{X}$ be a given Banach space, and $\mathfrak{X}^*$ be its dual space. If A is a linear closed operator with dense domain $\mathcal{D}(A) \subset \mathfrak{X}$, mapping $\mathcal{D}(A)$ into $\mathfrak{X}$, then, the dual operator A^* is defined on the set $\mathcal{D}(A^*) = \{x^* \in \mathfrak{X}^*, \text{ s.t. } \exists y^* \in \mathfrak{X}^*, \text{ satisfying}$

$$(1.1) \quad \langle x^*, Ax \rangle = \langle y^*, x \rangle \quad \forall x \in \mathcal{D}(A) \} .$$

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By definition $A^*x^* = y^*$, and A^* is a well-defined linear operator from $\mathcal{D}(A^*)$ into $\mathfrak{X}^*$.

Furthermore, the domain $\mathcal{D}(A^*)$ is a total set in $\mathfrak{X}^*$; this means that, given any element $x \in \mathfrak{X}$, $x \neq \theta$, $\exists x^* \in \mathcal{D}(A^*)$, such that $x^*(x) \neq 0$; hence, if $x^*(x) = 0$, $\forall x^* \in \mathcal{D}(A^*)$, then $x = \theta$.

Let now be given a finite interval $-\infty < a < b < +\infty$ on the real axis; a class of «test-functions» associated to the operator $\mathcal{D}(A^*)$ and to the given interval, denoted here by $K_{A^*}[a, b]$, consists of continuously differentiable functions $\phi^*(t)$, $a < t < b \rightarrow \mathfrak{X}^*$, which are $= \theta$ near b (that is $\phi^*(t) = \theta$ for $b - \delta < t < b$, where δ depends on ϕ^*); furthermore, $\phi^*(t)$ belong to $\mathcal{D}(A^*)$, $\forall t \in [a, b]$, and $(A^*\phi^*)(t)$ is $\mathfrak{X}^*$ -continuous on $[a, b]$.

Obviously, if $\varphi(t)$ is scalar-valued, $C^1[a, b]$ -function, and $\varphi = 0$ near b , and if ϕ^* is any element in $\mathcal{D}(A^*)$, then $\varphi(t)\phi^*$ belongs to $K_{A^*}[a, b]$.

Let us consider now the Bochner space $L_{\text{loc}}^p([a, b]; \mathfrak{X})$, where p is any real ≥ 1 , consisting of strongly measurable $\mathfrak{X}$ -valued functions f defined on $[a, b)$, such that $\int_a^c \|f(t)\|_{\mathfrak{X}}^p dt < \infty$ for any $c < b$.

The weak forward Cauchy problem is here defined as follows: given any element $u_a \in \mathfrak{X}$ and any function $f(t) \in L_{\text{loc}}^p([a, b]; \mathfrak{X})$, find a function $u(t) \in L_{\text{loc}}^p([a, b]; \mathfrak{X})$ verifying

$$(1.2) \quad -\langle \varphi^*(a), u_a \rangle - \int_a^b \left\langle \frac{d\varphi^*}{dt}, u(t) \right\rangle dt = \int_a^b \langle (A^*\varphi^*)(t), u(t) \rangle dt + \\ + \int_a^b \langle \varphi^*(t), f(t) \rangle dt, \quad \forall \varphi^* \in K_{A^*}[a, b].$$

In similar way define a weak backward Cauchy problem: The class $K_{A^*}(a, b]$ is defined like $K_{A^*}[a, b)$, with the only difference that the test-functions must be null near a , instead of being null near b .

There is also the space $L_{\text{loc}}^p((a, b]; \mathfrak{X})$ of $\mathfrak{X}$ -mesurable functions such that $\int_c^b \|f\|_{\mathfrak{X}}^p dt < \infty$, $\forall c > a$, $c < b$.

Then given any element $u_b \in \mathfrak{X}$, and again $f \in L_{\text{loc}}^p((a, b]; \mathfrak{X})$, find

$u(t) \in L_{loc}^2((a, b]; \mathfrak{X})$, satisfying

$$(1.3) \quad -\langle \varphi^*(b), u_b \rangle - \int_a^b \left\langle \frac{d\varphi^*}{dt}, u(t) \right\rangle dt = \int_a^b \langle (A^*\varphi^*)(t), u(t) \rangle dt + \\ + \int_a^b \langle \varphi^*(t), f(t) \rangle dt, \quad \forall \varphi^* \in K_{A^*}(a, b].$$

REMARK. This definitions are slightly more general than the weakened Cauchy problem as defined for example in S. G. Krein [3]; extending his definition from the interval $[0, T]$ to an arbitrary interval $[a, b]$, we say that $u(t)$, $a \leq t \leq b \rightarrow \mathfrak{X}$ is a weakened solution of

$$(1.4) \quad \dot{u}(t) = Au(t) + f(t), \quad u(a) = u_a \in \mathfrak{X},$$

where $f(t)$, $a \leq t \leq b \rightarrow \mathfrak{X}$ is $\mathfrak{X}$ -continuous, if: $u(t)$ is $\mathfrak{X}$ -continuous on the closed interval $[a, b]$; $u(t)$ is $\mathfrak{X}$ -differentiable with continuous derivative on the half-open interval $(a, b]$; $u(t) \in \mathcal{D}(A)$ on same $(a, b]$;

$$u'(t) = Au(t) + f(t), \quad a < t \leq b, \quad u(a) = u_0.$$

The following result holds:

PROPOSITION 1.1. *If $u(t)$ is a weakened solution of (1.4), then (1.2) is also verified.*

Consider the equality $u'(t) - Au(t) = f(t)$, valid on the half-open interval $a < t \leq b$. Then take any test-function $\phi^*(t) \in K_{A^*}[a, b)$ (ϕ^* is null near b). We get obviously

$$(1.5) \quad \langle \phi^*(t), u'(t) \rangle - \langle \phi^*(t), Au(t) \rangle = \langle \phi^*(t), f(t) \rangle, \quad a < t \leq b.$$

Also we see that

$$(1.6) \quad \frac{d}{dt} \langle \phi^*(t), u(t) \rangle = \left\langle \frac{d\phi^*}{dt}(t), u(t) \right\rangle + \left\langle \phi^*(t), \frac{du}{dt}(t) \right\rangle, \quad a < t \leq b.$$

If we integrate (1.6) between $a + \varepsilon$ and b , $\forall \varepsilon > 0$, we obtain

$$-\langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle = \int_{a + \varepsilon}^b \{ \langle \phi^*(t), u(t) \rangle + \langle \phi^*(t), u'(t) \rangle \} dt.$$

Let us integrate now (1.5) between $a + \varepsilon$ and b , and remark also that $\langle \phi^*(t), Au(t) \rangle = \langle A^* \phi^*(t), u(t) \rangle$, $a < t \leq b$. We get

$$\begin{aligned} \int_{a + \varepsilon}^b \langle \phi^*(t), u'(t) \rangle dt &= -\langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle - \int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle dt = \\ &= \int_{a + \varepsilon}^b \langle A^* \phi^*(t), u(t) \rangle dt + \int_{a + \varepsilon}^b \langle \phi^*(t), f(t) \rangle dt. \end{aligned}$$

By continuity of all functions here involved, one obtains, when $\varepsilon \rightarrow 0$

$$-\langle \phi^*(a), u(a) \rangle - \int_a^b \langle \phi^*(t), u(t) \rangle dt - \int_a^b \langle A^* \phi^*(t), u(t) \rangle dt = \int_a^b \langle \phi^*(t), f(t) \rangle dt.$$

A converse result is also given in the following.

PROPOSITION 1.2. *Let us assume: $f(t)$, $a < t \leq b \rightarrow \mathfrak{X}$, be strongly continuous; $u_a \in \mathfrak{X}$ be arbitrarily given. Then $u(t)$, $a < t \leq b \rightarrow \mathfrak{X}$ be a $\mathfrak{X}$ -continuous function, which is continuously differentiable for $a < t \leq b$, and belongs to $\mathfrak{D}(A)$ for $t \in (a, b]$. Let also (1.2) be satisfied. Then it follows that $u' - Au = f$ on $a < t \leq b$, and also $u(a) = u_a$.*

In order to prove this simple fact, we shall first introduce in (1.2) test-functions of the special form $\phi^*(t) = v(t)x^*$ where $x^* \in \mathfrak{D}(A^*)$ and $v(t)$ is scalar-valued continuously differentiable function which $= 0$ near a and near b . It results then, if $[a_1, b_1] \subset (a, b)$ contains $\text{supp } \phi^*$

$$(1.7) \quad - \int_{a_1}^{b_1} \langle \phi^*(t), u(t) \rangle dt = \int_{a_1}^{b_1} \langle (A^* \phi^*)(t), u(t) \rangle dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle dt.$$

As $u(t)$ is continuously differentiable on $[a_1, b_1]$, and $\phi^*(a_1) = \phi^*(b_1) = 0$ (intervals (a, a_1) , (b_1, b) are in the null set of ϕ^* : hence, by continuity,

$\phi^*(a_1) = \phi^*(b_1) = \theta$ also), it results

$$-\int_{a_1}^{b_1} \langle \phi^*(t), u(t) \rangle dt = \int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle dt.$$

Also because $u(t) \in \mathcal{D}(A)$ for $t \in [a_1, b_1]$, it is $\langle \phi^*(t), Au(t) \rangle = \langle A^* \phi^*(t), u(t) \rangle$. Hence, relation (1.7) becomes

$$(1.8) \quad \int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle dt = \int_{a_1}^{b_1} \langle \phi^*(t), (Au)(t) \rangle dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle dt$$

or, as $\phi^*(t)$ is here $= v(t)x^*$,

$$(1.9) \quad \int_{a_1}^{b_1} \langle x^*, u'(t) \rangle v(t) dt = \int_{a_1}^{b_1} \langle x^*, (Au)(t) \rangle v(t) dt + \int_{a_1}^{b_1} \langle x^*, f(t) \rangle v(t) dt$$

or

$$(1.10) \quad \int_{a_1}^{b_1} \langle x^*, u'(t) - Au(t) - f(t) \rangle v(t) dt = 0.$$

By continuity of the scalar function $\langle x^*, u'(t) - Au(t) - f(t) \rangle$ in $[a_1, b_1]$, letting $v(t)$ to vary, we get $\langle x^*, u'(t) - Au(t) - f(t) \rangle = 0$ in $[a_1, b_1]$, $\forall x^* \in \mathcal{D}(A^*)$.

(If $\int_{\alpha}^{\beta} \phi(t) v(t) dt = 0$, $\forall v \in C_0^1(\alpha, \beta)$, $\phi \in C[\alpha, \beta] \Rightarrow \phi = 0$ on (α, β) ; if not, $\exists \xi \in (\alpha, \beta)$, $\phi(\xi) > 0$ say; in $(\xi - \delta, \xi + \delta)$, $\phi > 0$; take $0 \leq v$, $v = 1$ on $(\xi - \delta/2, \xi + \delta/2)$, $= 0$ outside $(\xi - \delta, \xi + \delta)$, $\in C^1$; then

$$\int_{\alpha}^{\beta} \phi v dt = \int_{\xi - \delta}^{\xi + \delta} \phi v dt > \int_{\xi - \delta/2}^{\xi + \delta/2} \phi dt > 0, \quad \text{absurde.}$$

If $\phi = 0$ on (α, β) , $\Rightarrow \phi = 0$ on $[\alpha, \beta]$.)

Now, if we fix $t \in [a_1, b_1]$, and vary x^* over the total set $\mathcal{D}(A^*)$, we get $u'(t) = Au(t) + f(t)$.

This is true for any $t \in [a_1, b_1]$, hence for any $t \in (a, b)$ too. But $u'(t)$, $f(t)$, hence $Au(t)$ are continuous on $t = b$; so we obtain $u'(b) = Au(b) + f(b)$ also to be valid.

We still must prove that $u(a) = u_a$.

Consider again the relation (1.2), for general test-functions $\phi^*(t) \in K_A[a, b]$. Take an arbitrary small $\varepsilon > 0$, and get

$$\begin{aligned} -\langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \dot{\phi}^*(t), u(t) \rangle dt - \int_{a+\varepsilon}^b \langle \dot{\phi}^*(t), u(t) \rangle dt = \\ = \int_a^{a+\varepsilon} \langle (A^* \phi^*)(t), u(t) \rangle dt + \int_{a+\varepsilon}^b \langle (A^* \phi^*)(t), u(t) \rangle dt + \int_a^b \langle \phi^*(t), f(t) \rangle dt \end{aligned}$$

we have also,

$$\begin{aligned} \int_{a+\varepsilon}^b \langle \phi^*(t), u(t) \rangle dt = \int_{a+\varepsilon}^b \frac{d}{dt} \langle \phi^*(t), u(t) \rangle dt - \int_{a+\varepsilon}^b \langle \phi^*(t), \dot{u}(t) \rangle dt = \\ = -\langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle - \int_{a+\varepsilon}^b \langle \phi^*(t), \dot{u}(t) \rangle dt, \end{aligned}$$

and

$$\int_{a+\varepsilon}^b \langle (A^* \phi^*)(t), u(t) \rangle dt = \int_{a+\varepsilon}^b \langle \phi^*(t), Au(t) \rangle dt;$$

so, we get

$$\begin{aligned} -\langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \dot{\phi}^*(t), u(t) \rangle dt + \langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle + \\ + \int_{a+\varepsilon}^b \langle \phi^*(t), \dot{u}(t) \rangle dt = \int_{a+\varepsilon}^b \langle \phi^*(t), Au(t) \rangle dt + \int_a^b \langle \phi^*(t), f(t) \rangle dt + \int_a^{a+\varepsilon} \langle A^* \phi^*, u \rangle dt. \end{aligned}$$

But $\dot{u}(t) = Au(t) + f(t)$ on $a + \varepsilon \leq t \leq b$, as was proved above. Hence,

it remains

$$\begin{aligned} -\langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \phi^*(t), u(t) \rangle dt + \langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle = \\ = \int_a^{a+\varepsilon} \langle A^* \phi^*(t), u(t) \rangle dt + \int_a^{a+\varepsilon} \langle \phi^*(t), f(t) \rangle dt. \end{aligned}$$

If now let $\varepsilon \rightarrow 0$, it remains only, using continuity of u on $[a, b]$, that

$$\langle \phi^*(a), u(a) - u_a \rangle = 0, \quad \forall \phi^* \in K_{A^*}[a, b).$$

We can now take $\phi^*(t) = v_0(t)x^*$, where $v_0(t) \in C^1[a, b]$, equals 1 near a , and $= 0$ near b , and $x^* \in \mathcal{D}(A^*)$. Hence

$$\langle x^*, u(a) - u_a \rangle = 0 \quad \forall x^* \in \mathcal{D}(A^*)$$

which is a total set in $\mathfrak{X}^*$, and again, it will be $u(a) = u_a$. Q.E.D.

§ 2. — In this section we shall prove that uniqueness of Cauchy problem for strong solutions on an interval $[a, b]$ in the second dual space $\mathfrak{X}^{**}$, implies uniqueness of the weak Cauchy problem in the same interval, in the original space $\mathfrak{X}$.

If A is linear, closed operator with dense domain in the B -space $\mathfrak{X}$, we saw that the dual operator A^* is linear, defined on a total set in $\mathfrak{X}^*$; also A^* is closed on this set; in fact, let $x_n^* \in \mathcal{D}(A^*)$, $x_n^* \rightarrow x_0^* \in \mathfrak{X}^*$, $A^*x_n^* \rightarrow y_0^* \in \mathfrak{X}^*$. From relations $\langle x_n^*, Ax \rangle = \langle A^*x_n^*, x \rangle$, $\forall x \in \mathcal{D}(A)$, we get, as $n \rightarrow \infty$ $\langle x_0^*, Ax \rangle = \langle y_0^*, x \rangle$, $\forall x \in \mathcal{D}(A)$. Hence, by definition of A^* , it is $x_0^* \in \mathcal{D}(A^*)$, $A^*x_0^* = y_0^*$, so A^* is closed.

Let us assume from now on the supplementary.

HYPOTHESIS. A^* is an operator with dense domain in $\mathfrak{X}^*$.

(REMARK. This holds allways when $\mathfrak{X}$ is a reflexive B -space; the proof is similar to a classical one in Hilbert spaces).

Then, the second dual operator $A^{**} = (A^*)^*$ will be a well defined operator on a total set $\mathcal{D}(A^{**}) \subset \mathfrak{X}^{**}$ the second dual space of $\mathfrak{X}$. More precisely $\mathcal{D}(A^{**}) = \{\psi^{**} \in \mathfrak{X}^{**}, \text{ such that } \exists x^{**} \in \mathfrak{X}^{**}, \text{ satisfying relation } \langle \psi^{**}, A^*\phi^* \rangle = \langle x^{**}, \phi^* \rangle, \forall \phi^* \in \mathcal{D}(A^*)\}$ and if $\psi^{**} \in \mathcal{D}(A^{**})$,

$A^{**}\psi = \psi^{**}$. We also know the existence of a canonical map $J: \mathfrak{X} \rightarrow \mathfrak{X}^{**}$, which is linear and isometric; precisely, any element $x \in \mathfrak{X}$ defines a linear continuous functional f^{**} on $\mathfrak{X}^*$, by: $\langle f^{**}, x^* \rangle = \langle x^*, x \rangle$, $\forall x^* \in \mathfrak{X}^*$. Then put $Jx = f^{**}$, so that $\langle x^*, x \rangle = \langle Jx, x^* \rangle$, $\forall x^* \in \mathfrak{D}(A^*)$. Let now $u(t)$, $a \leq t \leq b$, be a $C^1[a, b; \mathfrak{X}]$ function such that $u(t) \in \mathfrak{D}(A)$, $\forall t \in [a, b]$ and $u'(t) = Au(t)$ on $[a, b]$. This is a strong solution on $[a, b]$, and $u(a)$ belongs necessarily to $\mathfrak{D}(A)$. Then $(Ju)(t)$ is a $C^1[a, b; \mathfrak{X}^{**}]$ function, as easily seen, and $(d/dt)(Ju) = J(du/dt)$. We prove now following

THEOREM 2.1. *Let us assume that for any function $u(t) \in C^1([a, b], \mathfrak{X})$ such that*

- i) $(Ju)(t) \in \mathfrak{D}(A^{**})$, $a \leq t < b$,
- ii) $(d/dt)(Ju) - A^{**}(Ju) = 0$ on $[a, b]$,
- iii) $(Ju)(a) = \theta$,

it is $(Ju)(t) = \theta$, $\forall t \in [a, b]$. Then, there is unicity of the forward weak Cauchy problem on $[a, b]$.

PROOF. What we must prove is the following: $v(t) \in L_{\text{loc}}^p([a, b]; \mathfrak{X})$, and

$$(2.1) \quad -\int_a^b \langle \phi^*(t), v(t) \rangle dt = \int_a^b \langle (A^*\phi^*)(t), v(t) \rangle dt, \quad \forall \phi^* \in K_{A^*}[a, b],$$

implies $v(t) = \theta$ almost everywhere on $[a, b]$.

Now, using a suggestion by professor S. Agmon (in Pisa, Italy), we start by extending $v(t)$ to $(-\infty, b)$, as follows: $\tilde{v}(t) = v(t)$ for $a \leq t < b$, $\tilde{v}(t) = \theta$ for $-\infty < t < a$. It holds now the following

LEMMA 2.1. *The extended function $\tilde{v}(t)$ verifies the integral identity*

$$(2.2) \quad \int_{-\infty}^b \langle \dot{\psi}^*(t) + (A^*\psi^*)(t), \tilde{v}(t) \rangle dt = 0$$

for any function $\psi^(t)$, $-\infty < t \leq b \rightarrow \mathfrak{X}^*$, continuously differentiable there, such that $\psi^*(t) \in \mathfrak{D}(A^*)$, $\forall t \in (-\infty, b]$, $A^*\psi^*(t)$ is $\mathfrak{X}^*$ -continuous; support ψ^* is compact in $(-\infty, b)$ (i.e. $\psi^* = \theta$ near b and near $-\infty$).*

In fact, (2.2) is the same as

$$(2.3) \quad \int_a^b \langle \dot{\psi}^*(t) + (A^* \psi^*)(t), v(t) \rangle dt = 0.$$

But the restriction to $[a, b]$ of the above considered test function $\psi^*(t)$ is obviously in the class $K_{A^*}[a, b]$, (because it was null near b , and had all regularity required properties).

Hence, by (2.1), the lemma is proved.

A second, needed result (already announced in our paper [6]) is as follows:

Take any scalar function $\alpha_\varepsilon(t) \in C^1(-\infty, \infty)$, which $= 0$ for $|t| \geq \varepsilon$; for any $w(t) \in L_{loc}^p(-\infty, b; \mathfrak{X})$ ($\mathfrak{X}$ -mesurable on $(-\infty, b)$, such that $\int_\alpha^\beta \|w\|_{\mathfrak{X}}^p \cdot dt < \infty$, $\forall \alpha > -\infty$, $\beta > \alpha$, $\beta < b$), we can consider the mollified function

$$(w * \alpha_\varepsilon)(t) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau) \alpha_\varepsilon(t-\tau) d\tau$$

which is well-defined for $-\infty < t < b - \varepsilon$, is strongly continuously differentiable, and

$$\frac{d}{dt} (w * \alpha_\varepsilon) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau) \dot{\alpha}_\varepsilon(t-\tau) d\tau, \quad -\infty < t < b - \varepsilon.$$

We have

LEMMA 2.2. *If $w(t) \in L_{loc}^p(-\infty, b; \mathfrak{X})$ verifies the integral identity:*

$$(2.4) \quad \int_{-\infty}^b \langle \dot{\psi}^*(t) + (A^* \psi^*)(t), w(t) \rangle dt = 0$$

$\forall \psi^*$ as in Lemma 2.1, then, it is $J(w * \alpha_\varepsilon) \in \mathcal{D}(A^{**})$, and $(d/dt)J(w * \alpha_\varepsilon) = A^{**}(J(w * \alpha_\varepsilon))$ holds, $\forall t \in (-\infty, b - \varepsilon)$ where J is the canonical map of $\mathfrak{X}$ in $\mathfrak{X}^{**}$.

Take in fact any fixed $t_0 \in (-\infty, b - \varepsilon)$, and consider then the functions $\psi_{t_0, \varepsilon}^*(t) = \alpha_\varepsilon(t_0 - t)x^*$, where $x^* \in \mathcal{D}(A^*)$. These are good test functions because $\alpha_\varepsilon(t_0 - t) = 0$ for

$$|t - t_0| \geq \varepsilon, \quad \text{hence in any case,} \quad \alpha_\varepsilon(t_0 - t) = 0$$

near b and near $-\infty$.

There is also $(d/dt)\psi_{t_0, \varepsilon}^* = -\dot{\alpha}_\varepsilon(t_0 - t)x^*$. Writing now (2.4), we get

$$\int_{-\infty}^b \langle \dot{\alpha}_\varepsilon(t_0 - t)x^*, w(t) \rangle dt = \int_{-\infty}^b \alpha_\varepsilon(t_0 - t) \langle A^*x^*, w(t) \rangle dt$$

or also

$$\left\langle A^*x^*, \int_{-\infty}^b \alpha_\varepsilon(t_0 - t)w(t) dt \right\rangle = \left\langle x^*, \int_{-\infty}^b \dot{\alpha}_\varepsilon(t_0 - t)w(t) dt \right\rangle, \quad \forall x^* \in \mathcal{D}(A^*),$$

or

$$\langle A^*x^*, (w^* \alpha_\varepsilon)(t_0) \rangle = \langle x^*, (w^* \alpha_\varepsilon)'(t_0) \rangle, \quad \forall x^* \in \mathcal{D}(A^*).$$

Here, if we introduce the canonical imbedding operator J , we have:

$$\langle J(w^* \alpha_\varepsilon)(t_0), A^*x^* \rangle = \langle J(w^* \alpha_\varepsilon)'(t_0), x^* \rangle, \quad \forall x^* \in \mathcal{D}(A^*).$$

Now if we use definition of $\mathcal{D}(A^{**})$ and of A^{**} , we see that $J(w^* \alpha_\varepsilon) \cdot (t_0) \in \mathcal{D}(A^{**})$, and

$$A^{**}(J(w^* \alpha_\varepsilon)(t_0)) = J(w^* \alpha_\varepsilon)'(t_0) = \frac{d}{dt} J(w^* \alpha_\varepsilon)(t_0)$$

which is the desired Lemma 2.2.

We pass now to the final steps of the proof.

Take $w(t) = \tilde{v}(t)$ the function used in Lemma 2.1; as $v(t) \in L_{\text{loc}}^p \cdot ([a, b]; \mathfrak{X})$ and $\tilde{v} = \theta$ for $t < a$, it is obvious that

$$\tilde{v}(t) = w(t) \in L_{\text{loc}}^p([-\infty, b]; \mathfrak{X}).$$

$$\cdot \left(\int_{\alpha}^{\beta} \|w(t)\|^p dt = \int_a^{\beta} \|v(t)\|^p dt < \infty \text{ for } \beta < b, \alpha < a \right).$$

Let us apply Lemma 2.2 to $\tilde{v}(t)$. We obtain that $(\tilde{v} * \alpha_\varepsilon)(t)$ is well-defined on $-\infty < t < b - \varepsilon$ where is continuously differentiable; also $J(\tilde{v} * \alpha_\varepsilon) \in \mathcal{D}(A^{**})$ and

$$(2.5) \quad \frac{d}{dt} (J(\tilde{v} * \alpha_\varepsilon)) = A^{**}(J(\tilde{v} * \alpha_\varepsilon)) \quad \text{holds on } -\infty < t < b - \varepsilon.$$

Remark also that $(\tilde{v} * \alpha_\varepsilon)(t) = \theta$ for $t \leq a - \varepsilon$, because it is

$$(\tilde{v} * \alpha_\varepsilon)(t) = \int_{t-\varepsilon}^{t+\varepsilon} \tilde{v}(\tau) \alpha_\varepsilon(t - \tau) d\tau$$

and $\tilde{v}(\tau) = \theta$ for $a - \varepsilon \leq \tau \leq a$.

Hence, (2.5) holds on $a - \varepsilon \leq t \leq b - \varepsilon$, and also $(\tilde{v} * \alpha_\varepsilon)(a - \varepsilon) = \theta$ so $J(\tilde{v} * \alpha_\varepsilon)(a - \varepsilon) = \theta$.

Now, if $(\tilde{v} * \alpha_\varepsilon)(t) = Z(t)$, we see that, in the space $\mathfrak{X}^{**}$, it is:

$$(JZ)'(t) = A^{**}JZ(t) \quad \text{on } [a - \varepsilon, b - \varepsilon], \text{ and } JZ(a - \varepsilon) = \theta.$$

Put then $t = \sigma - \varepsilon$ and $Z(t) = Z(\sigma - \varepsilon) = u(\sigma)$; when $a - \varepsilon \leq t \leq b - \varepsilon$, we get $a - \varepsilon \leq \sigma - \varepsilon \leq b - \varepsilon$, or $a \leq \sigma \leq b$; also $Ju'(\sigma) = JZ'(t)$, so that $(Ju)'(\sigma) = A^{**}(Ju)(\sigma)$ in $\mathfrak{X}^{**}$, $a < \sigma < b$, and $Ju(a) = (JZ)(a - \varepsilon) = \theta$. Applying the hypothesis of the theorem, it follows that $u(t) = \theta$ on $[a, b]$, hence, $Z(t) = \theta$ on $[a - \varepsilon, b - \varepsilon]$, that is

$$(\tilde{v} * \alpha_\varepsilon)(t) = \theta \quad \text{on} \quad [a - \varepsilon, b - \varepsilon].$$

Now, take a sequence of functions $\alpha_n(t)$ which are non-negative, $= 0$ for $|t| \geq 1/n$, continuously differentiable, such that

$$\int_{-1/n}^{1/n} \alpha_n(\sigma) d\sigma = 1.$$

We obtain then, in the usual way, as for scalar-valued functions, the relation:

$$\lim_{n \rightarrow \infty} \int_{a_1}^{b_1} \|v(t) - (\tilde{v} * \alpha_n)(t)\|^p dt = 0, \quad \forall b_1 < b, \quad a_1 > a.$$

But, for n big enough, $b - 1/n > b_1$; so $\tilde{v} * \alpha_n = \theta$ on $[a_1, b_1]$, hence

$$\int_a^{b_1} \|v(t)\|^p dt = 0 \quad \forall b_1 < b \Rightarrow v(t) = \theta \text{ a.e. on } (a, b).$$

§ 3. — We shall give now some applications of Theorem 2.1. To start with, we give a proof of the following result (see [2]). « Let $\mathfrak{X}$ be a reflexive B -space; A be the infinitesimal generator of a strongly continuous semi-group of class C_0 ; A^* be the dual operator to A . Let $u(t)$, $0 \leq t \leq T \rightarrow \mathfrak{X}$ be a strongly continuous function, verifying the integral identity

$$(3.1) \quad \int_0^T \langle \dot{\varphi}^*(t) + (A^* \varphi^*)(t), u(t) \rangle dt = 0$$

for any function $\varphi^*(t)$, $0 \leq t \leq T \rightarrow \mathfrak{X}^*$, which is continuously differentiable in $\mathfrak{X}^*$, belongs to $\mathcal{D}(A^*)$, $\forall t \in [0, T]$, $(A^* \varphi^*)$ is $\mathfrak{X}^*$ -continuous, $0 \leq t \leq T$, and $\varphi^*(t)$ is null near 0 and near T . Let also be $u(0) = \theta$; then $u(t) = \theta$, $0 \leq t \leq T$. »

Let us remark first that A is linear closed with dense domain in $\mathfrak{X}$ as any generator of a C_0 semi-group. By reflexivity of $\mathfrak{X}$ (which means, as usual, that $J(\mathfrak{X}) = \mathfrak{X}^{**}$), it follows that $\mathcal{D}(A^*)$ is dense in $\mathfrak{X}^*$, and that $A^{**}(Jx) = J(Ax)$, $\forall x \in \mathcal{D}(A)$, and $J(\mathcal{D}(A)) = \mathcal{D}(A^{**})$, (see [9]).

We shall see now that hypothesis i)-ii)-iii) of Theorem 2.1 are verified.

Take hence $u(t) \in C^1\{[0, T]; \mathfrak{X}\}$; assuming that $Ju \in \mathcal{D}(A^{**}) = J(\mathcal{D}(A))$ means: $\forall t \in [0, T]$, $\exists v(t) \in \mathcal{D}(A)$, such that $Jv(t) = Ju(t)$; as J^{-1} exists, $\Rightarrow v(t) = u(t)$; hence $u(t) \in \mathcal{D}(A)$, $0 \leq t \leq T$. Also, $A^{**} \cdot (Ju(t)) = J(Au(t))$; We assumed in ii) that $(d/dt)Ju - A^{**}(Ju) = \theta$ on $[0, T]$. But $(d/dt)Ju = J(du/dt)$, as $u \in C^1\{[0, T]; \mathfrak{X}\}$. Hence ii) becomes $J(du/dt) - J(Au) = \theta$ on $[0, T]$ which implies $u' - Au = \theta$ on $[0, T]$.

Furthermore iii) implies obviously that $u(0) = \theta$, again because J^{-1} exists ($\mathfrak{X}^{**} \rightarrow \mathfrak{X}$).

Now, the well-known unicity result for strong solutions of $(d/dt - A)w = 0$ when A is generator of a C_0 -semi-group (see for example [7], theorem 2.2.2) implies that $u(t) = \theta$ on $[0, T]$, so $Ju(t) = \theta$ on $[0, T]$ too. Hence, all conditions of theorem 2.1 are fulfilled, and by now we can conclude that:

If the relation

$$\int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle dt = 0$$

holds $\forall \phi \in K_{A^*}[0, T]$, then $u = \theta$ on $[0, T]$ (in fact, u -continuous is in L^p_{loc} , and $u = \theta$ a.e. on $[0, T] \Rightarrow u = \theta$ everywhere on $[0, T]$). Hence, it remains to check precisely that

$$(3.2) \quad \int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle dt = 0 \quad \forall \phi^* \in K_{A^*}[0, T].$$

Remember that our hypothesis here is slightly different: we assume in fact that it is

$$(3.3) \quad \int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle dt = 0$$

for test-functions regular as those in $K_{A^*}[0, T]$ but null near 0 as well as near T , which forms a subclass of $K_{A^*}[0, T]$ (denoted usually as $K_{A^*}(0, T)$). We added however the condition $u(0) = \theta$. So, it remains to prove that (3.2) holds.

Take henceforth an arbitrary $\phi^*(t) \in K_{A^*}[0, T]$. Then consider, for any $\varepsilon > 0$, a scalar-valued function $v_\varepsilon(t) \in C^1[0, T]$, which $= 0$ for $0 \leq t \leq \varepsilon$, and $= 1$ for $2\varepsilon \leq t \leq T$, satisfying also an estimate $|\dot{v}_\varepsilon(t)| \leq c/\varepsilon$, $0 \leq t \leq T$.

Then the product $v_\varepsilon(t)\phi^*(t)$ is also $= \theta$ near $t = 0$, so it is in the subclass of admissible here test-functions. We get from (3.3) the following equality

$$(3.4) \quad \int_0^T \langle \dot{v}_\varepsilon \phi^* + v_\varepsilon \dot{\phi}^* + v_\varepsilon A^* \phi^*, u \rangle dt = 0, \quad \forall \varepsilon > 0, \quad \phi^* \in K_{A^*}[0, T].$$

Obviously (3.4) reduces to the following

$$\begin{aligned} & \int_\varepsilon^{2\varepsilon} \langle \dot{v}_\varepsilon \phi^*, u \rangle dt + \int_\varepsilon^{2\varepsilon} \langle v_\varepsilon \dot{\phi}^*, u \rangle dt + \int_{2\varepsilon}^T \langle \dot{\phi}^*, u \rangle dt + \\ & + \int_\varepsilon^{2\varepsilon} \langle A^* \phi^*, u \rangle dt + \int_{2\varepsilon}^T \langle A^* \phi^*, u \rangle dt = 0, \quad \forall \varepsilon > 0, \quad \forall \phi^* \in K_{A^*}[0, T]. \end{aligned}$$

Now, for $\varepsilon \rightarrow 0$, the first integral is estimated as

$$\left| \int_{\varepsilon}^{2\varepsilon} \langle \phi^*, u \rangle dt \right| \leq \frac{c}{\varepsilon} \sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \cdot \varepsilon;$$

as $u(0) = \theta$, $u(t) \rightarrow \theta$ when $t \rightarrow 0$, hence $\sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \leq K \sup_{\varepsilon \leq t \leq 2\varepsilon} \|u(t)\| \rightarrow 0$ with ε . The other integrals containing ε are easily handled so that we obtain

$$\int_0^T \langle \phi^*, u \rangle dt + \int_0^T \langle A^* \phi^*, u \rangle dt = 0, \quad \forall \phi^* \in K_A[0, T]$$

which finishes our proof.

REMARK. The original proof of [2] was given using the adjoint semi group theory in reflexive spaces in a very natural way. We shall see later on a similar proof for the non-reflexive case (§ 5).

§ 4. — We shall deal here with the following unicity result for weakened solutions (see [3], Theorem 3.1, p. 81):

« Let be A a linear operator in the B -space $\mathfrak{X}$, such that $R(\lambda; A) = (\lambda - A)^{-1} \in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ for λ real $> \lambda_0$, and

$$\overline{\lim}_{\lambda \rightarrow +\infty} \frac{\ln \|R(\lambda)\|}{\lambda} = h_A < \infty.$$

Let $u(t)$ be a weakened solution of $u' - Au = \theta$ on the interval $0 \leq t \leq T$, such that $u(0) = \theta$, and assume also that $h_A < T$. Then $u(t) = \theta$ in $0 \leq t \leq T - h_A$. »

A slight generalization is possible, replacing $[0, T]$ by an arbitrary real interval $[a, b]$.

THEOREM 4.1. *Under the same hypothesis on A , and if $h_A < b - a$, any weakened solution $u(t)$ of $u' - Au = 0$ on $a \leq t \leq b$, such that $u(a) = \theta$, is $= \theta$ on $a \leq t \leq b - h_A$.*

We can in fact take $T = b - a$ in the above theorem; so if $u(0) = \theta$, we get $u(t) = \theta$ on $[0, b - a - h_A]$.

To prove theorem 4.1, let us put $u(t+a) = u_a(t)$; it maps the interval $0 \leq t \leq b-a$ into $\mathfrak{X}$. Also it is $\dot{u}_a(t) = u'(t+a) = Au(t+a) = Au_a(t)$ for $0 \leq t \leq b-a$.

Hence $u_a(t)$ is weakened solution on $0 \leq t \leq b-a$, and $u_a(0) = u(a) = \theta$; so, $u_a(t) = \theta$ on $0 \leq t \leq b-a-h_A$ that is $u(t+a) = \theta$ for $0 \leq t \leq b-a-h_A$, hence for $a \leq t+a \leq b-h_A$, which gives $u(t) = \theta$ for $a \leq t \leq b-h_A$.

Now we shall see a partial extension of Theorem 4.1 in general B -spaces, taking weak solutions instead of weakened. Precisely, we propose ourselves to prove the following

THEOREM 4.2. *Let A be a linear operator in the B -space $\mathfrak{X}$, such that $(\lambda - A)^{-1} \in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ for λ real $\geq \lambda_0$ and assume also that*

$$\lim_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda; A)\|}{\lambda} = h_A < \infty.$$

Let also be $\mathcal{D}(A^)$ a dense subset of $\mathfrak{X}^*$, and $\mathcal{D}(A)$ be dense in $\mathfrak{X}^*$. Assume finally that*

$$\int_a^b \langle \phi^* + A^* \phi^*, u \rangle dt = 0$$

$\forall \phi^* \in K_A[a, b)$, where $u \in L_{loc}^p([a, b); \mathfrak{X})$. Then, $u = \theta$ a.e. on $a \leq t \leq b-h_A$, provided $h_A < b-a$.

Let us start the proof by remembering Phillips's fundamental results (see [4], [8]) concerning resolvents of dual operators.

« Let T be linear closed operator with dense domain $\mathcal{D}(T) \subset \mathfrak{X}$, and T^* be its dual operator (acting on a total set in $\mathfrak{X}^*$, $\mathcal{D}(T^*)$). Then the resolvent sets $\varrho(T)$ and $\varrho(T^*)$ coincide; also, for any $\lambda \in \varrho(T)$, it is $(R(\lambda; T))^* = R(\lambda; T^*)$. »

Apply this result to our operator A which is linear closed in $\mathfrak{X}$, because we assume that $R(\lambda; A)$ exists $\in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ for $\lambda \geq \lambda_0$, λ real, and $\mathcal{D}(A)$ is dense by hypothesis. We obtain that for λ real $\geq \lambda_0$,

(*) The existence of $(\lambda - A)^{-1} \in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ does not implies in general, that $\mathcal{D}(\lambda - A) = \mathcal{D}(A)$ is dense in $\mathfrak{X}$.

It suffices to consider $\mathfrak{X} = C[0, 1]$; $A = d^2/dx^2$ defined on functions in $C^2[0, 1]$ which vanish for $x = 0$ and $x = 1$. Considering the equation $u'' = f$, $\forall f \in C[0, 1]$, we find a unique solution $u \in \mathcal{D}(A)$, depending continuously on f . However, $\mathcal{D}(A)$ is not dense in $\mathfrak{X}$.

$R(\lambda, A^*)$ also $\in \mathfrak{L}(\mathfrak{X}^*, \mathfrak{X}^*)$, and $R(\lambda; A^*) = [R(\lambda; A)]^*$. We know also that $\|[R(\lambda; A)]^*\| = \|[R(\lambda; A)]\|$ hence $\|R(\lambda; A^*)\| = \|R(\lambda; A)\|$ and consequently

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda; A^*)\|}{\lambda} = h_A \text{ too.}$$

Now, $\mathfrak{D}(A^*)$ is also dense in $\mathfrak{X}^*$, and A^* is closed. It follows that $R(\lambda; A^{**}) \in \mathfrak{L}(\mathfrak{X}^{**}, \mathfrak{X}^{**})$, $\forall \lambda$ real $\geq \lambda_0$, and for these λ , $\|R(\lambda; A^{**})\| = \|R(\lambda; A)\|$ so,

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda; A^{**})\|}{\lambda} = h_A < \infty \text{ too.}$$

Now we shall apply theorem 2.1 on the interval $a \leq t \leq b - h_A$. Let us consider consequently a function $u(t) \in C^1[a, b - h_A; \mathfrak{X}]$, such that $Ju \in \mathfrak{D}(A^{**})$, $a \leq t \leq b - h_A$, $(d/dt)(Ju) - A^{**}(Ju) = 0$ on $a \leq t \leq b - h_A$, and $(Ju)(a) = \theta$.

Let us apply now theorem 4.1 taking A^{**} instead of A which is possible by the above (remarking also that here the solutions are strong which is better than weakened). It follows that $Ju(t) = \theta$ on $a \leq t \leq b - h_A$. Hence theorem 2.1 is applicable on $[a, b - h_A]$ and we get uniqueness of weak solutions, as desired.

§ 5. — In this section we present a variant of the unicity result considered in § 3, which is valid in more general, non-reflexive B -spaces.

Let us start by remembering Phillips's theorem on dual semi-groups (see [4], [5], [8]).

Consider in the B -space $\mathfrak{X}$, a linear closed operator A with domain $\mathfrak{D}(A)$ dense in $\mathfrak{X}$, and assume that A generates a semi-group of class (C_0) of linear continuous operators $T(t)$, $0 \leq t < \infty \rightarrow \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$.

Now, as previously, the dual operator A^* of A is a closed linear transformation on $\mathfrak{D}(A^*) \subset \mathfrak{X}^*$ to $\mathfrak{X}^*$. We know that $\mathfrak{D}(A^*)$ is a total set in $\mathfrak{X}^*$, but in general $\mathfrak{D}(A^*)$ is not dense in $\mathfrak{X}^*$ so that A^* is not necessarily the infinitesimal generator of a strongly continuous semi-group in $\mathfrak{X}^*$.

Therefore it is convenient to consider the so called $\odot$ -dual space $\mathfrak{X}^\odot$ of $\mathfrak{X}$, defined by $\mathfrak{X}^\odot = \overline{\mathfrak{D}(A^*)}$ (closure in $\mathfrak{X}^*$). In the case of reflexive $\mathfrak{X}$, we have $\mathfrak{X}^\odot = \mathfrak{X}^*$, else $\mathfrak{X}^\odot$ may be a proper subset of $\mathfrak{X}^*$.

Let us define now the operator $A^\odot$ to be the restriction of the

dual operator A^* to the domain

$$(5.1) \quad \mathcal{D}(A^\circ) = [x^* \in \mathfrak{X}^*, x^* \in \mathcal{D}(A^*) \text{ such that } A^*x^* \in \mathfrak{X}^\circ].$$

Furthermore, let $T^*(t)$ be, for any $t \geq 0$, the dual operator of $T(t)$, and then $T^\circ(t)$ be the restriction of $T^*(t)$ to $\mathfrak{X}^\circ$; then $T^\circ(t) \in \mathcal{L}(\mathfrak{X}^\circ, \mathfrak{X}^\circ)$, $t \geq 0$, and it is a semi-group of class (C_0) having A° as infinitesimal generator.

Our aim is to prove the following

THEOREM 5.1. *Let $u(t)$ be a continuous function. $0 \leq t \leq T$ to $\mathfrak{X}$, such that $u(0) = \theta$, and satisfying relation*

$$(5.1) \quad \int_0^T \langle \dot{\phi}^\circ + A^\circ \phi^\circ, u(t) \rangle dt = 0$$

for any function $\phi^\circ(t)$, $0 \leq t \leq T \rightarrow \mathcal{D}(A^\circ)$, $\phi^\circ \in C^1[0, T; \mathfrak{X}^\circ]$, $A^\circ \phi^\circ \in C[0, T; \mathfrak{X}^\circ]$, $\phi^\circ = \theta$ near 0 and near T . Then $u(t) = \theta$ on $[0, T]$.

REMARK. Before giving the proof, let us consider the particular case of reflexive space $\mathfrak{X}$. Then $\mathfrak{X}^\circ = \mathfrak{X}^*$, $A^\circ = A^*$, so we find again the previously proved theorem in § 3.

PROOF OF THE THEOREM. We have firstly

LEMMA 5.1. *The relation*

$$(5.2) \quad \int_0^T \langle \dot{\phi}^\circ + A^\circ \phi^\circ, u \rangle dt = 0$$

is verified for the more general class of test-function: $\phi^\circ(t) \in C^1[0, T; \mathfrak{X}^\circ]$, $\phi^\circ(t) \in \mathcal{D}(A^\circ)$, $(A^\circ \phi^\circ)(t) \in C[0, T; \mathfrak{X}^\circ]$, $\phi^\circ(T) = \theta$.

Let us consider, $\forall \varepsilon > 0$, a scalar-valued function $v_\varepsilon(t)$, continuously differentiable on $0 \leq t \leq T$, $= 0$ for $0 \leq t \leq \varepsilon$, $T - \varepsilon \leq t \leq T$, $= 1$ for $2\varepsilon \leq t \leq T - 2\varepsilon$, such that $|v'_\varepsilon(t)| \leq c/\varepsilon$, $0 \leq t \leq T$, $|v_\varepsilon(t)| \leq 1$, $0 \leq t \leq T$; then $v_\varepsilon(t)\phi^\circ(t)$ is a test-function as required in theorem 5.1, because it vanishes near $t = 0$ and near $t = T$. We can write henceforth the relation (5.2) for $v_\varepsilon\phi^\circ$, and obtain the following:

$$\int_0^T \langle \dot{v}_\varepsilon \phi^\circ + v_\varepsilon \dot{\phi}^\circ, u \rangle dt = - \int_0^T v_\varepsilon \langle A^\circ \phi^\circ, u \rangle dt.$$

The right-hand integral splits as

$$-\int_{2s}^{T-2s} \langle A^\circ \phi^\circ, u \rangle dt - \int_s^{2s} \nu_s \langle A^\circ \phi^\circ, u \rangle dt - \int_{T-2s}^{T-s} \nu_s \langle A^\circ \phi^\circ, u \rangle dt$$

and is readily seen that

$$\lim_{s \rightarrow 0} - \int_0^T \nu_s \langle A^\circ \phi^\circ, u \rangle dt = - \int_0^T \langle A^\circ \phi^\circ, u \rangle dt.$$

The left-hand side integral equals

$$\int_0^T \dot{\nu}_s \langle \phi^\circ, u \rangle dt + \int_0^T \nu_s \langle \phi^\circ, u \rangle dt = I_1 + I_2.$$

Actually it results

$$I_1 = \int_s^{2s} \dot{\nu}_s \langle \phi^\circ, u \rangle dt + \int_{T-2s}^{T-s} \dot{\nu}_s \langle \phi^\circ, u \rangle dt = I_3 + I_4.$$

Now, $\lim_{s \rightarrow 0} I_3 = 0$, essentially because $|\dot{\nu}_s| \leq c/\varepsilon$, and $u(0) = \theta$. Also $\lim_{s \rightarrow 0} I_4 = 0$, essentially because $|\dot{\nu}_s| \leq c/\varepsilon$, and $\phi^\circ(T) = \theta$. As for I_2 , it is obviously seen to converge to $\int_0^T \langle \phi^\circ, u \rangle dt$, as $\varepsilon \rightarrow 0$. Hence, altogether, for $\varepsilon \rightarrow 0$ we get

$$\int_0^T \langle \phi^\circ, u \rangle dt + \int_0^T \langle A^\circ \phi^\circ, u \rangle dt = 0,$$

and the Lemma is proved.

We can continue now the proof of our theorem.

Let us take an arbitrarily given function $k^\circ(t) \in C^1[0, T; \mathfrak{X}^*]$. Then consider in the $\odot$ -dual space $\mathfrak{X}^\circ$, the strong inhomogeneous Cauchy

problem

$$(5.3) \quad \frac{d\psi^\circ}{dt} - A^\circ \psi^\circ = -k^\circ, \quad \psi^\circ(0) = \theta.$$

Due to the fact that A° is the generator of a (C_0) -semigroup $T^\circ(t)$ in $\mathfrak{X}^\circ$, by a well-known result of Phillips ([7], Theorem 2.2.3), the problem (5.3) has a unique solution (given by the formula $\psi^\circ(t) = -\int_0^t T^\circ(t-\sigma) \cdot k^\circ(\sigma) d\sigma$, but this is not important here).

Consider now the function $\phi^\circ(t)$, defined for $0 \leq t \leq T$ through the relation $\phi^\circ(t) = \psi^\circ(T-t)$.

It is continuously differentiable in $\mathfrak{X}^\circ$ on $0 \leq t \leq T$; it belongs to $\mathcal{D}(A^\circ)$, $\forall t \in [0, T]$, and $(A^\circ \phi^\circ)(t) = (A^\circ \psi^\circ)(T-t)$ is continuous, $0 \leq t \leq T \rightarrow \mathfrak{X}^\circ$. Finally, $\phi^\circ(T) = \psi^\circ(0) = \theta$. Hence, $\phi^\circ(T)$ is an admissible test-function, and the relation $\int_0^T \langle \phi^\circ + A^\circ \phi^\circ, u \rangle dt = 0$ is verified.

Furthermore, $d\phi^\circ/dt = -\psi^\circ(T-t)$ and consequently we get:

$$\phi^\circ(t) + A^\circ \phi^\circ(t) = -\psi^\circ(T-t) + A^\circ \psi^\circ(T-t) = k^\circ(T-t),$$

in view of (5.3). Hence, we obtained the identity

$$\int_0^T \langle k^\circ(T-t), u(t) \rangle dt = 0,$$

for any $k^\circ \in C^1[0, T; \mathfrak{X}^\circ]$, or, obviously, as $t \rightarrow T-t$ maps $C^1[0, T; \mathfrak{X}^\circ]$ onto itself,

$$\int_0^T \langle h^\circ(t), u(t) \rangle dt = 0 \quad \forall h^\circ \in C^1[0, T; \mathfrak{X}^\circ].$$

Take in particular $h^\circ(t) = \nu(t)x^*$, where $x^* \in \mathfrak{X}^\circ$. Then

$$\int_0^T \nu(t) \langle x^*, u(t) \rangle dt = 0, \quad \text{if } \nu(t) \in C^1[0, T].$$

As $\langle x^*, u \rangle$ is scalar-continuous on $[0, T]$, we obtain $\langle x^*, u(t) \rangle = 0$,

$\forall t \in [0, T]$. But we can let x^* to vary in the total set $\mathcal{D}(A^*) \subset \mathcal{X}^\circ$. It follows that $u(t) = \theta$, $\forall t \in [0, T]$.

This ends the proof of our theorem.

A simple corollary is the following

THEOREM 5.2. - *Let $u(t) \in C\{[0, T]; \mathcal{X}\}$, such that $u(0) = \theta$ and assume that*

$$(5.4) \quad \int_0^T \langle \phi^* + A^* \phi^*, u \rangle dt = 0 ,$$

for any function $\phi^(t)$, $0 \leq t \leq T \rightarrow \mathcal{D}(A^*)$, belonging to $C^1([0, T]; \mathcal{X}^*)$, such that $A^* \phi^* \in C([0, T]; \mathcal{X}^*)$ and $\phi^* = \theta$ near 0 and near T . Then $u(t) = \theta$ on $[0, T]$.*

In fact it suffices to remark that the class of test-functions considered here contains as a subset the class considered in the theorem 5.1, because A° is a certain restriction of A^* to an (eventually) smaller domain. Hence, the relation (5.2) is verified and theorem 5.2 implies $u = \theta$ on $[0, T]$.

We have also the following

THEOREM 5.3. *Let A be the generator of a (C_0) semi-group $T(t)$ in the B -space $\mathcal{X}$, and A^* ; $\mathcal{D}(A^*) \subset \mathcal{X}^* \rightarrow \mathcal{X}^*$ be its dual operator, defined on the total set $\mathcal{D}(A^*)$.*

Let $u(t)$ a continuous function. $0 \leq t \leq T \rightarrow \mathcal{X}$, such that $u(0) = u_0$ given arbitrarily in $\mathcal{X}$, and satisfying the relation

$$(5.5) \quad \int_0^T \langle \phi^* + A^* \phi^*, u \rangle dt = 0 , \quad \forall \phi^*(t) \in K_{A^*}(0, T) \text{ } ^{(1)} .$$

Then $u(t)$ has the representation $u(t) = T(t)u_0$, $0 \leq t \leq T$.

Let us consider in fact the strongly continuous function $v(t)$, $0 \leq t \leq T \rightarrow \mathcal{X}$, given by $v(t) = T(t)u_0$. Then (5.5) is valid also for this function v .

In fact, let $(u_n)_1^\infty \subset \mathcal{D}(A)$ be a sequence convergent to u_0 . Let also

(¹) This is the class of test-functions considered in Theorem 5.2.

$v_n(t) = T(t)u_n$, so that, as well-known, it is $\dot{v}_n = Av_n$, $0 \leq t \leq T$. Now

$$\int_0^T \langle \dot{\phi}^*, v_n \rangle dt = - \int_0^T \langle \phi^*, \dot{v}_n \rangle dt,$$

as obviously seen. Furthermore is $\langle A^* \phi^*, v_n \rangle = \langle \phi^*, Av_n \rangle$, $\forall t \in [0, T]$. It follows

$$\int_0^T \langle \dot{\phi}^* + A^* \phi^*, v_n \rangle dt = - \int_0^T \langle \phi^*, \dot{v}_n \rangle dt + \int_0^T \langle \phi^*, Av_n \rangle dt = 0.$$

when $n \rightarrow \infty$, $v_n(t) \rightarrow v(t)$ uniformly on $[0, T]$, as $\sup_{0 \leq t \leq T} \|T(t)\| = C_T < \infty$ so, it results:

$$\int_0^T \langle \dot{\phi}^* + A^* \phi^*, v \rangle dt = 0$$

too. If we take now $w(t) = u(t) - v(t)$, then (5.5) is verified for $w(t)$, and $w(0) = \theta$. By previous theorem, it follows $u(t) = v(t) = T(t)u_0$ on $0 \leq t \leq T$.

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