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Multi-Valued Contraction Mappings in Complete Metric Spaces.

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Let (X, d) be a metric space. For any nonempty subsets A, B of X, we define

$$D(A, B) = \inf\{d(a, b)|a \in A, b \in B\} ,$$

$$\delta(A, B) = \sup\{d(a, b)|a \in A, b \in B\} ,$$

$$H(A, B) = \max\Big\{\sup\{D(a, B)|a \in A\}, \sup\{D(A, b)|b \in B\}\Big\} .$$

Let $CB(X)$ be the set of all nonempty closed and bounded subsets of X. The space $CB(X)$ is a metric space with respect to the above defined distance H (see K. Kuratowski [1] p. 214). Then we have the following theorem which is a generalization of S. Reich result [2] (or see I. Rus [3]).

THEOREM 1. *Let (X, d) be a complete metric space, and let $f: X \to CB(X)$ be a multi-valued mapping with the following condition: for every $x, y \in X$,*

$$H(f(x), f(y)) \leqslant \alpha\big(D(x, f(x)) + D(y, f(y))\big) + \beta\big(D(x, f(y)) + D(y, f(x))\big) + \gamma D(x, y) ,$$

where α, β, γ are non-negative and $2\alpha + 2\beta + \gamma < 1$. Then f has a fixep point, i.e. there is a point x such that $x \in f(x)$.

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PROOF. Let x_0 be a point in X, and $x_1 \in f(x_0)$. If

$$H(f(x_0), f(x_1)) = 0 ,$$

then we have $f(x_0) = f(x_1)$, since H is a metric on $CB(X)$. Therefore we have $x_1 \in f(x_1)$. This contains the proof of the case $\alpha = \beta = \gamma = 0$.

Next we suppose $0 < 2\alpha + 2\beta + \gamma$ and

$$H(f(x_0), f(x_1)) > 0 .$$

Put $p = (2\alpha + 2\beta + \gamma)^{\frac{1}{2}}$, then $0 < p < 1$.

Let $h = H(f(x_0), f(x_1))/p$, then we have

$$h > H(f(x_0), f(x_1)) .$$

By the definition of H, we have

$$h > H(f(x_0), f(x_1)) \geqslant D(x_1, f(x_1)) .$$

Therefore there is a point x_2 of $f(x_1)$ such that

$$h > d(x_1, x_2) .$$

Hence

$$\begin{aligned} d(x_1, x_2) < p^{-1} H(f(x_0), f(x_1)) & \\ & \leqslant p^{-1}\{\alpha[D(x_0, f(x_0)) + D(x_1, f(x_1))] \\ & + \beta[D(x_0, f(x_1)) + D(x_1, f(x_0))] + \gamma D(x_0, x_1)\} \\ & \leqslant p^{-1}\{\alpha[d(x_0, x_1) + d(x_1, x_2)] \\ & + \beta[d(x_0, x_2) + d(x_1, x_1)] + \gamma d(x_0, x_1)\} \\ & \leqslant p^{-1}\{\alpha[d(x_0, x_1) + d(x_1, x_2)] \\ & + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma d(x_0, x_1)\} . \end{aligned}$$

Hence we have

$$(p - (\alpha + \beta))\, d(x_1, x_2) < (\alpha + \beta + \gamma)\, d(x_0, x_1) .$$

Therefore

$$d(x_1, x_2) < qd(x_0, x_1) ,$$

where $q = (\alpha + \beta + \gamma)/(p - (\alpha + \beta))$ and $0 < q < 1$.

For x_1, x_2, we have two eases:

1) $H(f(x_1, f(x_2)) = 0$,

2) $H(f(x_1), f(x_0)) > 0$.

If we have the first case, then $x_2 \in f(x_2)$, which completes the proof.

If $H(f(x_1), f(x_2)) > 0$, by a similar method, there is a point x_3 of $f(x_2)$ such that

$$d(x_2, x_3) < qd(x_1, x_2) .$$

In general, if $H(f(x_i), f(x_{i+1})) = 0$ for some i, then $x_i \in f(x_i)$. If, for all i $(i = 0, 1, \dots)$, $H(f(x_i), f(x_{i+1})) > 0$, there is a point $x_{i+2} \in f(x_{i+1})$ satisfying

$$d(x_{i+1}, x_{i+2}) < qd(x_i, x_{i+1}) .$$

Hence, for $n > m$,

$$d(x_n, x_m) \leqslant \frac{q^m}{1-q} d(x_0, x_1) .$$

This shows that $\{x_n\}$ is a Cauchy sequence. The completeness of X implies the existence of the limit of $\{x_n\}$. Let x' be the limit of $\{x_n\}$, then

$$\begin{aligned} D(x', f(x')) &\leqslant d(x', x_{n+i}) + d(x_{n+1}, f(x')) \\ &\leqslant d(x', x_{n+1}) + H(f(x_n), f(x')) . \end{aligned}$$

Hence, by the assumption, we have

$$\begin{aligned} (1) \quad D(x', f(x')) \leqslant d(x', x_{n+1}) & \\ &+ \alpha[D(x_n, f(x_n)) + D(x', f(x'))] \\ &+ \beta[D(x_n, f(x')) + D(x', f(x_n))] + \gamma D(x_n, x') \\ &\leqslant d(x', x_{n+1}) + \alpha[d(x_n, x_{n+1}) + D(x', f(x'))] \\ &+ \beta[D(x_n, f(x')) + d(x', x_{n+1})] + \gamma d(x_n, x') . \end{aligned}$$

Let $n \to \infty$, then (1) implies the following relation.

$$D(x', f(x')) \leqslant \alpha D(x', f(x')) + \beta D(x', f(x')) .$$

From $1 - \alpha - \beta > 0$, we have

$$D(x', f(x')) = 0 ,$$

which means $x' \in f(x')$. This completes the proof.

Let $BN(X)$ be the set of all nonempty bounded subset of X. Then we have a fixed point theorem.

THEOREM 2. *Let (X, d) be a complete metric space. If $f\colon X \to BN(X)$ is a multi-valued function which satisfies*

$$\delta(f(x), f(y)) \leqslant \alpha[H(x, f(x)) + H(y, f(y))] + \beta[H(x, f(y)) + H(y, f(x))] + \gamma d(x, y) ,$$

for every x, y in X, where α, β, γ are non-negative and $2\alpha + 4\beta + \gamma < 1$, then f has a unique fixed point, i.e. for some x', $f(x') = \{x'\}$.

Theorem 2 is a generalization of S. Reich result [2]. The proof is due to an idea by S. Reich [2].

PROOF. If $\alpha = \beta = \gamma = 0$, then the result is trivial. We suppose $0 < 2\alpha + 4\beta + \gamma$. Now put $p = (2\alpha + 4\beta + 1)^{\frac{1}{2}}$. Then we have $p < 1$. Hence there is a single-valued function $g\colon X \to X$ such that $g(x)$ is a point y in $f(x)$ which satisfies

$$d(x, y) = d(x, g(x)) \geqslant pH(x, f(x)) .$$

For such a function g,

$$\begin{aligned} d(g(x), g(y)) &\leqslant \delta(f(x), f(y)) \\ &\leqslant \alpha[H(x, f(x)) + H(y, f(y))] \\ &\quad + \beta[H(x, f(y)) + H(y, f(x))] + \gamma d(x, y) \\ &\leqslant \alpha p^{-1}[d(x, g(x)) + d(y, g(y))] \\ &\quad + \beta p^{-1}[2d(x, y) + d(x, g(x)) + d(y, g(y))] + \gamma d(x, y) \\ &\leqslant (\alpha + \beta) p^{-1}[d(x, g(x)) + d(y, g(y))] + (2\beta p^{-1} + \gamma) d(x, y) . \end{aligned}$$

Hence we have

$$d(g(x), g(y)) \leqslant (\alpha + \beta) p^{-1}[d(x, g(x)) + d(y, g(y))] + (2\beta p^{-1} + \gamma) d(x, y) . \tag{2}$$

The assumption $2\alpha + 4\beta + \gamma < 1$ implies $2(\alpha + \beta) p^{-1} + 2\beta p^{-1} + \gamma < 1$. By a well known theorem, g has a fixed point x', i.e. $g(x') = x'$. For the point x',

$$0 = (x', g(x')) \geqslant pH(x', f(x')) .$$

Hence $x' \in f(x')$.

If $z \in f(z)$, and $H(z, f(z)) > 0$, then

$$\delta(f(y), f(y)) \leqslant 2(\alpha + \beta) H(y, f(y)) < H(y, f(y)) ,$$

which is impossible. Hence we have $f(z) = \{z\}$.
To show $z = x'$, consider

$$\delta(f(z), f(x')) \leqslant \beta [H(z, f(x')) + H(x', f(z))] + \gamma d(z, x') \leqslant (2\beta + \gamma) d(z, x') .$$

Hence we have $z = x'$, which shows that f has a unique fixed point. The proof is complete.

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