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AUTOMORPHISMS OF ABELIAN p-GROUPS AND HYPO RESIDUAL FINITENESS

JUTTA HAUSEN

1. Introduction.

A group X is called residually finite if the intersection of all subgroups of X of finite index is trivial.

If $\mathcal{C}$ is a nonempty isomorphism inherited class of groups then, following the standard terminology for group theoretical properties, we call X a hypo $\mathcal{C}$-group, if it possesses a well ordered descending chain of subgroups X_μ :

$$X = X_0, \qquad X_{\mu+1} \trianglelefteq X_\mu, \qquad X_\lambda = \bigcap_{\mu<\lambda} X_\mu \text{ for } \lambda \text{ a limit ordinal,}$$

such that $X_\mu / X_{\mu+1}$ is a $\mathcal{C}$-group for every ordinal μ and $X_\sigma = 1$ for sufficiently large σ.

In [2] we have studied those abelian p-groups G whose automorphism group $A(G)$ is hypo residually finite. It was shown that $A(G)$ possesses this property if and only if every divisible and every bounded pure subgroup of G has finite rank.

The present paper is concerned with an estimation of the shortest length of all descending chains of $A(G)$ which have residually finite factors. Each of those chains will be shown to have length at least λ,

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where λ is the length of the chain of higher residua $\mathfrak{R}_\mu A(G)$ of $A(G)$ introduced in [2].

Our main result is the following connection between λ and the Ulm type τ of G, or rather the normal expansion of τ.

THEOREM. *Let the Ulm type τ of the abelian p-group G have the normal expansion*

$$(*) \qquad \tau=\omega^{\alpha}\cdot n+\omega^{\alpha_1}\cdot n_1+ \dots +\omega^{\alpha_k}\cdot n_k\,,$$

where $\alpha>\alpha_1> \dots >\alpha_k$ are ordinals; $n, n_1, \dots, n_k\geq 0$ are integers; $n<2^m$ for some integer m. Let λ be the least ordinal such that $\mathfrak{R}_\lambda A(G)=1$. Then

$$\lambda\leq\omega\alpha+m+2.$$

Note, that every ordinal τ posses a normal expansion of the form (*) and furthemore, that this expansion is unique for $\tau\neq 0$ (cf. [3], p. 323, Theorem 2).

Notation and terminology will be standard with possibly the following exceptions: We write $Y\trianglelefteq X$ if Y is a normal subgroup of the group X. The automorphism group of X is denoted by $A(X)$. If $\Delta\leq A(X)$ and $W\trianglelefteq Y\leq X$ are Δ-admissible subgroups then $\Delta\,|_{Y/W}$ denotes the group of automorphisms of Y/W induced by Δ. We write $Y(\Delta-1)$ for the set of all $y\delta-y$ where $y\in Y$ and $\delta\in\Delta$. The set of all $\alpha\in A(X)$ fixing $Y\trianglelefteq X$ elementwise and inducing the identity automorphism in X/Y is called the stabilizer of Y in X. Stabilizers are wellknown to be abelian (cf. [4], p. 88, Satz 19). As always, ω denotes the first infinite ordinal.

Throughout this paper G will be an abelian p-group for some fixed prime p.

2. Descending chains with residually finite factors.

For X a group, let $\mathfrak{R}X$ denote the intersection of all (normal) subgroups of X of finite index. $\mathfrak{R}X$ is called the residuum of X. By definition, X is residually finite if and only if its residuum is trivial. If $Y\trianglelefteq X$ and X/Y is residually finite, then $\mathfrak{R}X\leq Y$ (cf. [2], Lemma 3.5). It follows immediately that X is hypo residually finite if and only if the chain of iterated residua, the so-called higher residua, leads from X to 1 in a finite or possibly transfinite number of steps (see Lemma 2.2).

DEFINITION. Let $\mathfrak{R}_0X=X$. If $\mathfrak{R}_\mu X$ is defined for $0\leq\mu<\sigma$, then $\mathfrak{R}_\sigma X=\bigcap_{\mu<\sigma}\mathfrak{R}_\mu X$ if σ is a limit ordinal, and $\mathfrak{R}_\sigma X=\mathfrak{R}\mathfrak{R}_{\sigma-1}X$ otherwise. The $\mathfrak{R}_\mu X$ are called higher residua of X.

Some easy consequences of these definitions we state as

LEMMA 2.1. i) *If* $Y\leq X$ *then* $\mathfrak{R}_\mu Y\leq\mathfrak{R}_\mu X$ *for all* μ.

ii) *If* $Y\trianglelefteq X$ *and* $\mathfrak{R}_\mu(X/Y)=1$ *then* $\mathfrak{R}_\mu X\leq Y$.

PROOF. [2], Lemmas 3.1 and 3.5.

The relation between the chain of higher residua and arbitrary descending chains with residually finite factors is contained in the following result.

LEMMA 2.2. *Let*

$$X=X_0\trianglerighteq X_1\trianglerighteq\dots\trianglerighteq X_\mu\trianglerighteq X_{\mu+1}\trianglerighteq\dots\trianglerighteq X_\sigma=Y \tag{**}$$

be a well-ordered descending chain of subgroups of X *such that* $X_\mu/X_{\mu+1}$ *is residually finite for all* $\mu<\sigma$ *and* $X_\lambda=\bigcap_{\mu<\lambda}X_\mu$ *if* λ *is a limit ordinal. Then* $\mathfrak{R}_\mu X\leq X_\mu$ *for all* $\mu\leq\sigma$.

PROOF. The proof is by simple and straightforward induction on μ. Clearly, for $\mu=0$ the proposition holds. Suppose, that $\mathfrak{R}_\nu X\leq X_\nu$ for all $0\leq\nu<\mu\leq\sigma$. Then, if μ is a limit ordinal,

$$\mathfrak{R}_\mu X=\bigcap_{\nu<\mu}\mathfrak{R}_\nu X\leq\bigcap_{\nu<\mu}X_\nu=X_\mu$$

by our induction hypothesis. If $\mu=\nu+1$ we obtain

$$\mathfrak{R}_\mu X=\mathfrak{R}\mathfrak{R}_\nu X\leq\mathfrak{R}X_\nu\leq X_{\nu+1}=X_\mu$$

using the induction hypothesis, the residual finiteness of $X_\nu/X_{\nu+1}$, and Lemma 2.1.

COROLLARY 2.3. *Let* λ *be the least ordinal such that* $\mathfrak{R}_\lambda X=1$. *Then every well-ordered descending chain* (**) *of subgroups* X_μ *of* X *with residually finite factors has length* $\geq\lambda$.

We recall the definition of a hypo $\mathfrak{C}$-group, for $\mathfrak{C}$ a group theoretical property, given in the introduction. The following result is added for the sake of completeness.

PROPOSITION 2.4. *A group is hypo residually finite if and only if it is hypo finite.*

PROOF. Clearly, since every finite group is residually finite, every hypo finite group is hypo residually finite.

Conversely, if X is hypo residually finite, then by Lemma 3.2 of [2], every nontrivial subgroup of X possesses a nontrivial finite epimorphic image. Using set theoretical arguments it is easy to see, that such a group is hypo finite. This proves the proposition.

Since we are concerned with the lengths of descending chains with residually finite factors we will prefer and continue to use the term « hipo residually finite » instead of « hypo finite ».

3. Subgroups of $A(G)$ fixing $G/p^{\omega}G$ elementwise.

Let as usual $p^{\omega}G$ denote the set of all elements of G of infinite height. Since $p^{\omega}G$ is a characteristic subgroup, the set of all automorphisms γ of G inducing the identity mapping $G/p^{\omega}G$ is a normal subgroup of $A(G)$, denoted by $\Gamma(G)$.

In [2] we proved the somewhat surprising fact that for reduced G the residual behaviour of $A(G)$ (in the sense we are concerned with) depends entirely on the first step in its chain of higher residua. It was shown that $A(G)$ is hypo residually finite if and only if $A(G)/\Gamma(G)$ is residually finite or, equivalently $\mathfrak{R}_1 A(G) = \mathfrak{R}A(G) \leq \Gamma(G)$. Contrary to this, there is always on ordinal ν such that $\mathfrak{R}_\nu \Gamma(G) = 1$ without any further condition imposed on G except that it is reduced (or that the maximal divisible subgroup of G has finite rank).

Therefore, an estimation of the least ordinal ν satisfying $\mathfrak{R}_\nu \Gamma(G) = 1$ is our first aim.

Crucial for the course of all further proofs will be the following generalization of Lemma 4.3 in [2].

We remark that the subgroups G^{μ} of G are defined as in [1], p. 118: $G^0 = G$, $G^{\lambda} = \bigcap_{\mu<\lambda} G^{\mu}$ if λ is a limit ordinal and $G^{\lambda} = p^{\omega}(G^{\lambda-1})$ otherwise.

Since $\mathfrak{R}P = p^{\omega}P$ if P is an abelian p-group (see [2], Lemma 3.7), $G^{\mu} = \mathfrak{R}_{\mu}G$ in our notation for every ordinal μ, and

$$G = G^0 \geq G^1 \geq \dots \geq G^{\mu} \geq G^{\mu+1} \geq \dots \geq G^{\tau} = G^{\tau+1}$$

is the chain of higher residua of G. The ulm type of G is the least ordinal τ such that $G^{\tau} = G^{\tau+1}$.

LEMMA 3.1. *Let η be an endomorphism of G such that $G\eta \leq G^{\mu}$ for some ordinal μ. Then*

$$G^{\nu}\eta \leq G^{\mu+\nu}$$

for every ordinal ν.

PROOF. The proof will be by induction on ν. For $\nu = 0$ the proposition is obvious. So, suppose, that $G^{\nu}\eta \leq G^{\mu+\nu}$ for every $0 \leq \nu < \lambda$. We distinguish two cases.

CASE 1. λ is a limit ordinal. In this case, $G^{\lambda} = \bigcap_{\nu<\lambda} (G^{\nu})$ and likewise, $G^{\mu+\lambda} = (G^{\mu})^{\lambda} = \bigcap_{\nu<\lambda} (G^{\mu})^{\nu}$. Consequently,

$$G^{\lambda}\eta = \Big(\bigcap_{\nu<\lambda} G^{\nu}\Big)\eta \leq \bigcap_{\nu<\lambda} (G^{\nu}\eta), \tag{1}$$

and, using the induction hypothesis,

$$\bigcap_{\nu<\lambda} (G^{\nu}\eta) \leq \bigcap_{\nu<\lambda} G^{\mu+\nu} = \bigcap_{\nu<\lambda} (G^{\mu})^{\nu} = (G^{\mu})^{\lambda}. \tag{2}$$

Comparing (1) and (2) we obtain

$$G^{\lambda}\eta \leq (G^{\mu})^{\lambda} = G^{\mu+\lambda}$$

as claimed.

CASE 2. $\lambda = \nu + 1$. Then, by definition, $G^{\lambda} = G^{\nu+1} = \bigcap_{n\geq 0} p^n G^{\nu}$, and G^{λ} is the subgroup of elements of infinite height in G^{ν}. Let $x \in G^{\lambda}$. Then there are $y_n \in G^{\nu}$ such that

$$x = p^n y_n \qquad \text{for } n = 1, 2, \dots,$$

and hence

$$x\eta = p^n y_n \eta \in p^n(G^\nu \eta) \qquad \text{for } n=1, 2, \ldots . \tag{3}$$

Using thet induction hypothesis $G^\nu\eta \le G^{\mu+\nu}$ it follows that

$$x\eta \in p^n(G^{\mu+\nu}), \qquad n=1, 2, \ldots$$

for every $x\in G^\lambda$ and consequently

$$G^\lambda\eta \le \bigcap_{n\ge 0} p^n G^{\mu+\nu} = p^\omega G^{\mu+\nu} = G^{\mu+\nu+1} = G^{\mu+\lambda}.$$

So, in both cases we have carried the induction one step further, proving the lemma.

We are now ready to prove the vital

LEMMA 3.2. *Let Δ be a group of automorphisms of G fixing G/G^μ elementwise for some ordinal μ. Then $\mathfrak{R}_m\Delta$ induces the identity automorphism in $G/G^{\mu\cdot 2^m}$ for every integer $m\ge 0$.*

PROOF. The proof will be by induction on m. For $m=0$ the statement obviously is true. So, suppose, that

$$G(\mathfrak{R}_m\Delta - 1) \le G^{\mu\cdot 2^m} \qquad \text{for some } m\ge 0.$$

Then, for every $\alpha\in\mathfrak{R}_m\Delta$, we have $G(\alpha-1)\le G^{\mu\cdot 2^m}$, and consequently, by Lemma 3.1,

$$G^{\mu\cdot 2^m}(\alpha-1) \le G^{\mu\cdot 2^m+\mu\cdot 2^m} = G^{\mu\cdot 2^{m+1}}$$

(cf. [3], p. 293). Therefore, α induces in $G/G^{\mu\cdot 2^{m+1}}$ an automorphism which fixes both $G^{\mu\cdot 2^m}/G^{\mu\cdot 2^{m+1}}$ and the factor group

$$(G/G^{\mu\cdot 2^{m+1}})/(G^{\mu\cdot 2^m}/G^{\mu\cdot 2^{m+1}}) \cong G/G^{\mu\cdot 2^m}$$

elementwise. This being true for all $\alpha\in\mathfrak{R}_m\Delta$ it follows that the group of automorphisms of $G/G^{\mu.2^{m+1}}$, which is induced by $\mathfrak{R}_m\Delta$, is contained in the stabilizer Σ of $G^{\mu 2^m}/G^{\mu 2^{m+1}}$ in $G/G^{\mu 2^{m+1}}$. Since Σ is residually finite (see [2], Lemma 4.1) Lemma 2.1 implies that

$$G(\mathfrak{R}\mathfrak{R}_m\Delta - 1) \leq G^{\mu \cdot 2^{m+1}}$$

and hence that

$$\mathfrak{R}_{m+1}\Delta = \mathfrak{R}\mathfrak{R}_m\Delta$$

induces the identity automorphism in $G/G^{\mu \cdot 2^{m+1}}$. This proves the lemma.

COROLLARY 3.3. *Let $\Delta \leq A(G)$ such that $G(\Delta - 1) \leq G^{\mu}$ for some ordinal μ. Then $\mathfrak{R}_\omega\Delta$ induces the identity automorphism in $G/G^{\mu\omega}$.*

PROOF. By Lemma 3.2, since $\mathfrak{R}_\omega\Delta = \bigcap_{m \geq 0} \mathfrak{R}_m\Delta$,

$$G(\mathfrak{R}_\omega\Delta - 1) \leq G(\mathfrak{R}_m\Delta - 1) \leq G^{\mu \cdot 2^m} \qquad \text{for all } m \geq 0.$$

Consequently,

$$G(\mathfrak{R}_\omega\Delta - 1) \leq \bigcap_{m \geq 0} G^{\mu \cdot 2^m} = G^{\mu\omega}$$

(cf. [3], p. 300, Theorem 3).

Another result in the same direction is the following lemma. We recall that the set of all automorphisms of G inducing the identity mapping in $G/p^\omega G$ was denoted by $\Gamma(G)$.

LEMMA 3.4. *Let $\Delta \leq \Gamma(G)$. Then $\mathfrak{R}_{\omega\alpha}\Delta$ induces the identity automorphism in G/G^{ω^α} for every ordinal α.*

PROOF. Again, the proof is by transfinite induction the proposition being true for $\alpha = 0$. Suppose, that

(H) $$G(\mathfrak{R}_{\omega\nu}\Delta - 1) \leq G^{\omega^\nu} \qquad \text{for all } 0 \leq \nu < \lambda,$$

and consider

CASE 1. λ is a limit ordinal. Then $\omega\lambda$ and ω^λ also are limit ordinals, being the least upper bounds of the ordinals $\omega\nu$, $\nu < \lambda$ (cf. [3], p. 300, Theorem 3) and ω^ν, $\nu < \lambda$ (cf. [3], p. 312) respectively. Consequently,

(1) $$\mathfrak{R}_{\omega \cdot \lambda}\Delta = \bigcap_{\mu < \omega\lambda} \mathfrak{R}_\mu\Delta = \bigcap_{\nu < \lambda} \mathfrak{R}_{\omega\nu}\Delta$$

and

$$G^{\omega^\lambda} = \bigcap_{\mu<\omega^\lambda} G^\mu = \bigcap_{\nu<\lambda} G^{\omega^\nu}. \tag{2}$$

Using the induction hypothesis (H) we obtain from (1) and (2) that

$$G(\mathfrak{K}_{\omega\lambda}\Delta-1)=G(\bigcap_{\nu<\lambda}\mathfrak{K}_{\omega\nu}\Delta-1)\leq\bigcap_{\nu<\lambda}G^{\omega^\nu}=G^{\omega^\lambda}$$

and hence that $\mathfrak{K}_{\omega\lambda}\Delta$ induces the identity automorphism in G/G^{ω^λ} as claimed.

CASE 2. $\lambda=\nu+1$. Then $G^{\omega^\lambda}=G^{\omega^{\nu+1}}=G^{\omega^\nu\cdot\omega}$. Put $\mathfrak{K}_{\omega\nu}\Delta=\theta$. By induction hypothesis we have $G(\theta-1)\leq G^{\omega^\nu}$. Applying Corollary 3.3 we obtain

$$G(\mathfrak{K}_\omega\theta-1)\leq G^{\omega^\nu\cdot\omega}=G^{\omega^{\nu+1}}=G^{\omega^\lambda},$$

and since

$$\mathfrak{K}_\omega\theta=\mathfrak{K}_\omega\mathfrak{K}_{\omega\nu}\Delta=\mathfrak{K}_{\omega\nu+\omega}\Delta=\mathfrak{K}_{\omega(\nu+1)}\Delta=\mathfrak{K}_{\omega\cdot\lambda}\Delta$$

(cf. [3], p. 293), we finally conclude that

$$G(\mathfrak{K}_{\omega\lambda}\Delta-1)\leq G^{\omega^\lambda}$$

as stated in the proposition.

THEOREM 1. *If Δ is a group of automorphisms of G fixing $G/p^\omega G$ elementwise, then $\mathfrak{K}_{\omega\alpha+m}\Delta$ induces the identity automorphism in $G/G^{\omega^\alpha\cdot 2^m}$ for every ordinal α and every integer $m\geq 0$.*

PROOF. Lemma 3.4 implies that $G(\mathfrak{K}_{\omega\alpha}\Delta-1)\leq G^{\omega^\alpha}$ for every ordinal α. Put $\mathfrak{K}_{\omega\alpha}\Delta=\theta$. By Lemma 3.2, $\mathfrak{K}_m\theta$ induces the identity automorphism in $G/G^{\omega^\alpha\cdot 2^m}$ for every integer $m\geq 0$. Hence $\mathfrak{K}_m\theta=\mathfrak{K}_m\mathfrak{K}_{\omega\alpha}\Delta=$ $=\mathfrak{K}_{\omega\alpha+m}\Delta$ fixes $G/G^{\omega^\alpha\cdot 2^m}$ elementwise as claimed.

We are now ready for an estimation of the least ordinal σ satisfying $\mathfrak{K}_\sigma\Gamma(G)=1$. In view of Theorem 1 we consider the unique normal expansion of the Ulm type τ of G into decreasing powers of ω:

$$(*) \qquad \tau=\omega^{\alpha}\cdot n+\omega^{\alpha_1}\cdot n_1+\ \dots\ +\omega^{\alpha_k}\cdot n_k\,,$$

where $\alpha>\alpha>\ \dots\ >\alpha_k$ are ordinals $n, n_1, \dots, n_k$ are non-negative integers, $n\neq 0$ (cf. [3], p. 323, Theorem 2).

The following is a consequence of Theorem 1.

COROLLARY 3.5. *Let G be reduced and let*

$$(*) \qquad \tau=\omega^{\alpha}\cdot n+\omega^{\alpha_1}\cdot n_1+\ \dots\ +\omega^{\alpha_k}\cdot n_k$$

be the normal expansion of its Ulm type $\tau\neq 0$. If $n\leq 2^m$, then $\mathfrak{R}_{\omega\alpha+m+1}\Gamma(G)=1$. Moreover, if either $n<2^m$ or $n_1=\ \dots\ =n_k=0$, then $\mathfrak{R}_{\omega\alpha+m}\Gamma(G)=1$.

PROOF. Put $\beta=\omega^{\alpha_1}n_1+\ \dots\ +\omega^{\alpha_k}n_k$. Then

$$\tau=\omega^{\alpha}n+\beta \text{ where } \beta<\omega^{\alpha}$$

(cf. [3], p. 324, 6.). It follows that

$$\tau=\omega^{\alpha}n+\beta<\omega^{\alpha}2^m+\omega^{\alpha}=\omega^{\alpha}(2^m+1)<\omega^{\alpha}2^{m+1}$$

([3], p. 278, 1.; p. 295, Theorem 2; p. 293), and hence, by Theorem 1, $\mathfrak{R}_{\omega\alpha+m+1}\Gamma(G)=1$ as claimed. Since

$$\tau=\omega^{\alpha}n+\beta<\omega^{\alpha}n+\omega^{\alpha}=\omega^{\alpha}(n+1)$$

([3], p. 278), and

$$\tau\leq\omega^{\alpha}2^m+\beta$$

([3], p. 295, Theorem 1), Theorem 1 implies in particular that $\mathfrak{R}_{\omega\alpha+m}\Gamma(G)=1$ if $n+1\leq 2^m$ or $\beta=0$. This proves the corollary.

4. The length of the chain of higher residua of $A(G)$.

A connection between the least ordinal σ satisfying $\mathfrak{R}_{\sigma}\Gamma(G)=1$ and the length of the chain of higher residua of $A(G)$ is established by the following

THEOREM 2. *For an abelian p-group G the following properties are equivalent.*

i) *$A(G)$ is hypo residually finite.*

ii) *$\mathfrak{R}A(G)\leq\Gamma(G)$ and $\Gamma(G)$ is hypo residually finite.*

iii) *Every divisible and every bounded pure subgroup of G has finite rank.*

PROOF. The equivalence of i) and iii) is contained in Theorem A in [2]. Since hypo residual finiteness is inherited by subgroups and extensions (Lemma 2.1), i) is a consequence of ii). So, let us assume the validity of i) (and hence of iii)). Then, by Theorem 3 in [2], the automorphism group of $G/p^{\omega}G$ is residually finite, and, since $A(G)/\Gamma(G)$ is essentially the group of automorphisms of $G/p^{\omega}G$ induced by $A(G)$, so is $A(G)/\Gamma(G)$. Lemma 2.1 implies $\mathfrak{R}A(G)\leq\Gamma(G)$. Therefore, the equivalent statements i) and iii) together imply ii). This proves the theorem.

PROPOSITION 4.1. *Let the Ulm type τ of the reduced abelian p-group G have the form $\tau=\omega^{\alpha}n+\beta$ where $\beta<\omega^{\alpha}$ and $n\leq 2^{m}$. If $A(G)$ is hypo residually finite, then $\mathfrak{R}_{\omega\alpha+m+1}A(G)=1$. Moreover, if τ is infinite and either $n<2^{m}$ or $\beta=0$, then $\mathfrak{R}_{\omega\alpha+m}A(G)=1$.*

PROOF. By Theorem 2, the hypo residual finiteness of $A(G)$ implies

$$\mathfrak{R}A(G)\leq\Gamma(G). \tag{1}$$

Applying Lemma 2.1 we obtain

$$\mathfrak{R}_{1+\mu}A(G)=\mathfrak{R}_{\mu}\mathfrak{R}A(G)\leq\mathfrak{R}_{\mu}\Gamma(G) \tag{2}$$

for every ordinal μ, and, since $1+\mu=\mu$ for transfinite μ,

$$\mathfrak{R}_{\mu}A(G)\leq\mathfrak{R}_{\mu}\Gamma(G) \qquad \text{for } \mu\geq\omega. \tag{3}$$

Hence, if τ is infinite, the entire proposition follows from corollary 3.5.

If $\tau=n\leq 2^{m}$ is finite, then by (2),

$$\mathfrak{R}_{1+m}A(G)\leq\mathfrak{R}_{m}\Gamma(G)$$

and, since $G(\mathfrak{R}_{m}\Gamma(G)-1)\leq G^{2^{m}}$ (Lemma 3.2), $\mathfrak{R}_{m+1}A(G)=1$ as stated above.

We are now ready to prove our main result stated as Theorem in the introduction.

THEOREM 3. *Let the automorphism group of an abelian p-group G be hypo residually finite. Let the Ulm type τ of G have the normal expansion*

$$\tau=\omega^{\alpha}\cdot n+\omega^{\alpha_1}\cdot n_1+ \dots +\omega^{\alpha_k}\cdot n_k ,$$

where $\alpha>\alpha> \dots >\alpha_k$ are ordinals, $n, n_1, \dots, n_k\geq 0$ are integers, and $n\leq 2^m$ for some integer $m\geq 0$. Then $\mathfrak{R}_{\omega\alpha+m+2}A(G)=1$. Moreover, if τ is infinite and either $n_1=n_2= \dots =n_k=0$ or $n<2^m$, then $\mathfrak{R}_{\omega\alpha+m+1}A(G)=1$.

PROOF. Let

$$G=D+R,\ D \text{ divisible},\ R \text{ reduced}.$$

The hypo residual finiteness of $A(G)$ implies that D has finite rank (Theorem 2), and hence, that $A(D)$ is residually finite ([2], Lemma 4.2). Since D is a characteristic subgroup of G we obtain, using Lemma 2.1.,

$$\mathfrak{R}A(G)\,|_D =1. \tag{1}$$

If $\tau=0$ then $G=D$ and $A(G)$ is residually finite in accordance with our proposition. So, suppose that $\tau>0$. Since the Ulm type of G/D is equal to τ, Proposition 4.1 implies

$$\mathfrak{R}_\sigma A(G/D)=1,$$

where $\sigma=\omega\alpha+m+1$ or $\sigma=\omega\alpha+m$ if τ is infinite and either $n_1= \dots =n_k=0$ or $n<2^m$. Hence, for this σ,

$$\mathfrak{R}_\sigma A(G)\,|_{G/D} =1, \tag{2}$$

applying Lemma 2.1 again. Since $\sigma>0$ it follows from (1) and (2) that $\mathfrak{R}_\sigma A(G)$ is contained in the stabilizer $\Sigma(G:D)$ of D in G which is residually finite ([2], Lemma 4.1):

$$\mathfrak{R}_\sigma A(G)\leq\Sigma(G:D), \qquad \mathfrak{R}\Sigma(G:D)=1. \tag{3}$$

Applying Lemma 2.1 we obtain

$$\mathfrak{R}_{\sigma+1}A(G)=1$$

and consequently $\mathfrak{R}_{\omega\alpha+m+2}A(G)=1$ always and, for infinite τ and either $n_1=n_2=\ldots=n_k=0$ or $n<2^m$, even $\mathfrak{R}_{\omega\alpha+m+1}A(G)=1$ as stated above.

This concludes the proof of our theorem.

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