RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

PIETRO ARDUINI

Monomorphisms and epimorphisms in abstract categories

Rendiconti del Seminario Matematico della Università di Padova, tome 42 (1969), p. 135-166

<http://www.numdam.org/item?id=RSMUP_1969__42__135_0>

MONOMORPHISMS AND EPIMORPHISMS IN ABSTRACT CATEGORIES

PIETRO ARDUINI *)

Classically [1]) a subobject of an object A, in any category $\mathcal{A}$, is a monic [2]) of codomain A. Hence: in the category of topological spaces, as in that of topological groups, the real numbers system with the discrete topology is a subobject of the real numbers system with the usual topology; in the category of ordered sets and increasing functions, a set ordered by equality is a subobject of any ordered set having the same support. And so on.

Between the concepts, which, in particular, may allow for better categorical definitions of subobject and quotient object, we recall: the concept of *bicategory* introduced by Isbell [4] and generalized by Wyler [10]; the definition of *normal monomorphism* [3]) due to Kuroš [5]; the definitions of *extremal monomorphism* and *canonical category* in Sonner [8]; Wyler's *operational categories* [9]. However none of these concepts gives rise to fully satisfactory definitions of subobject, quotient object, image and coimage [4]); in particular, Son-

---

*) Lavoro svolto nell'ambito del gruppo di ricerca n. 20 del Comitato Nazionale per la Matematica del C.N.R., a.a. 1967-8.

Istituto Matematico dell'Università, via L. B. Alberti 4, 16132 Genova (Italy).

1) From Grothendieck [1] to Mitchell [7].

2) That is a left cancellable morphism.

3) This definition is referred only to the categories with zero morphisms: in such a case, any normal monomorphism is a monomorphism with respect to the definition in our section 1, *but not vice versa.*

4) Briefly. By means of a single bicategorical structure in the sense of Isbell,

ner's definitions have the feature that the composition of consecutive extremal monomorphisms need not be an extremal monomorphism, even if the category is canonical in his sense (see our section 8).

The aim of this paper is to propose, in full generality, « good » definitions of *monomorphism* and *epimorphism*, testing them both from a practical point of view (in categories of « structured sets » subobjects must agree with subspaces; and dually) and from a theoric point of view (there are properties that a « good » definition of monomorphism or of epimorphism *must* satisfy).

Pursuing in this direction, in a succeeding paper, « good » definitions of *image* and *coimage* will be proposed and tested also from a theoretic point of view (reducing the existence of image (resp. coimage) for each morphism to the existence of a right (resp. left) adjoint of a suitable forgetful functor).

A « good » self-dual and comprehensive definition of *canonical category* (i.e. a category such that every morphism has a coimage-image factorization) will then be at hand.

With regard to the underlying formal system and to logical difficulties, which arise in categorical algebra, the naive point of view is adopted and the word « set » is indiscriminately used: one may state all the smallness hypothesises required by Von Neumann-Bernays-Gödel system, or specify, when necessary, the suitable universes in the sense of Grothendieck, or use Lawvere's category of all categories, accordingly to one's taste.

---

the image and the coimage of a continuons function, in the category of topological spaces, cannot be discriminated (see [4] page 575).

With regard to normal monomorphisms, in addition to the preceding footnote, we may remark that: a theory with the self-dual axiom " *Every morphism has both a normal image and a conormal image* » (the terminology is that of [6]) does not cover a category as that of groups; a self-dual theory, general enough to cover the category of groups, as that of Wyler [10], needs of a bicategorical machinery which seems quite complicated and redundant with respect to the examples given.

Finally the theory of operational categories deals with categories of « structured sets » only.

Recently new theories have been developed by Heller and Michalowicz (see Notices of the A.M.S. 15 (1968) page 467).

Other conventions: the dual to item *m.n.p.* is denoted or referred to by *m.n.p.* *; « *fx* » stands for « *f*(*x*) » whenever there is no danger of confusion; for undefined terms, excepting adjoint functors, reference is made to Mitchell [7].

All the proofs are easy; however they are written down in section 7, so that the paper can be read by anybody who knows such definitions as those of category, functor, natural transformation and a few others.

The content of this paper is therefore as follows:

1. Definitions and elementary properties.

1.1 DEFINITION. *Let* $\mathcal{A}$ *be any category. A morphism* $A' \to A$ *is a* **monomorphism** *iff:*

(M_I) $A' \to A$ *is monic;*

(M_{II}) *for any commutative square*

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ *epic, there is a (necessarily unique*[5]*) morphism* $X'' \to A'$

5) Because $X \to X''$ is epic.

such that the diagram

$$\begin{array}{ccc} X & \rightarrow & A' \\ \downarrow & \nearrow & \downarrow \\ X'' & \rightarrow & A \end{array}$$

commutes.

1.1.1 NOTE. The axioms (M_I) and (M_{II}) are independent.

1.1 * DEFINITION. *Let $\mathfrak{A}$ be any category. A morphism* $A \rightarrow A''$ *is an* **epimorphism** *iff:*

(E_I) $A \rightarrow A''$ *is epic;*

(E_{II}) *for any commutative square*

$$\begin{array}{ccc} A & \rightarrow & X' \\ \downarrow & & \downarrow \\ A'' & \rightarrow & X \end{array}$$

with $X' \rightarrow X$ *monic, there is a (necessarily unique* [6]*) morphism* $A'' \rightarrow X'$ *such that the diagram*

$$\begin{array}{ccc} A & \rightarrow & X' \\ \downarrow & \nearrow & \downarrow \\ A'' & \rightarrow & X \end{array}$$

commutes.

1.2 PROPOSITION. *If* $A_2 \rightarrow A_1$ *and* $A_1 \rightarrow A$ *are monomorphisms, then their composition* $A_2 \rightarrow A_1 \rightarrow A$ *is a monomorphism.*

1.3 PROPOSITION. *If the composition* $A_2 \rightarrow A_1 \rightarrow A$ *is a monomorphism, then* $A_2 \rightarrow A_1$ *is a monomorphism.*

1.3.1 COROLLARY. *A coretraction is a monomorphism.*

1.3.2 EXAMPLES. (*i*) The diagonal $\Delta : A \rightarrow A \times A$ (in a category with finite products) is a monomorphism.

(*ii*) The injections $A_i \rightarrow \mathbf{X}_i A_i$ (in a category with products and

---

[6]) Because $X' \rightarrow X$ is monic.

zero morphisms) are monomorphisms.

1.4 PROPOSITION. *An epic monomorphism is an isomorphism.*

1.5 PROPOSITION. *A morphism is an isomorphism iff it is both a monomorphism and an epimorphism.*

2. Examples.

2.1 In each of the following categories every monic is a monomorphism: sets; pointed sets; lattices (with lattice morphisms); groups; exact categories; complexes of an exact category; corrispondences in an abelian category.

2.2 Categories, whose monomorphisms are just (up to isomorphisms) inclusions of ordinary subspaces, are, e.g., the following: *(partly) ordered sets; lattices (with lattice morphisms); groups; topological spaces; pointed topological spaces; uniform spaces; topological groups; linear topological spaces; topological spaces with open* [7]) *(or closed or proper) maps; compact (Hausdorff) spaces* [8]).

2.3 Categories, whose monomorphisms are just (up to isomorphisms) inclusions of *closed* subspaces, are, e.g., the following: *Hausdorff spaces; locally compact spaces; metric spaces with continuous (or Lipschitzian) maps; locally compact abelian groups; linear Hausdorff topological spaces; linear normed spaces; Banach spaces; nuclear spaces.*

2.3.1 REMARK. One need not wonder that, from the categorical point of view, the closed subspaces of an Hausdorff space are its only subobjects. Take the following definitions:

(*a*) A is an absolute retract in the category of topological spaces iff every topological imbedding $A \rightarrow X$ is a coretraction;

(*b*) A is an absolute retract in the category of metric spaces (with continuous maps) iff every *closed* topological imbedding $A \rightarrow X$ is a coretraction.

Definitions (*a*) and (*b*) seem not to agree because of that « closed » in (*b*). With reference to 1.1, 2.2, 2.3, they are both special cases of the following:

---

[7]) In such a category, subspaces are open subsets with the relative topology, because the inclusion map must be open. Similarly in the other cases.

[8]) In such a category, subspaces are closed subsets with the relative topology, because they must be compact.

2.4 Definition. *A is an absolute retract in a category $\mathcal{A}$ iff every monomorphism in $\mathcal{A}$ of domain A is a coretraction.*

2.5 In an ordered category (that is in an ordered set regarded as a category) the identities are the only monomorphisms.

3. Other elementary properties.

In this section we relate monomorphisms with equalizers, cartesian squares [9]), limits, functor categories.

Throughout the section, unless otherwise stated, we are working in any category $\mathcal{A}$.

3.1 Proposition. *If* $K \xrightarrow{u} A$ *is an equalizer for* $A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$, *then* $K \xrightarrow{u} A$ *is a monomorphism.*

3.1.1 Corollary. *Let*

$$\begin{array}{ccc} P & \rightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \rightarrow & A \end{array}$$

be a cartesian square. If the product $A_1 \times A_2$ *exists, then the canonical morphism* $P \rightarrow A_1 \times A_2$ *is a monomorphism.*

3.2 Proposition. *Let*

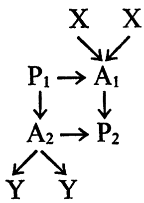
$$\begin{array}{ccc} P & \rightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \rightarrow & A \end{array}$$

be a cartesian square. If $A_1 \rightarrow A$ *is a monomorphism, then so is* $P \rightarrow A_2$.

3.2.1 Remarks. Consider the category generated by the following commutative diagram

---

[9]) Other names for the same thing are: pullback diagram, couniversal square, produit fibré, meet, co-amalgamation.



where the composition morphisms from X to P_2 are equal, and similarly the composition morphisms from P_1 to Y.

It is easily seen that the square is bicartesian (i.e. cartesian and cocartesian). However: (*i*) $A_1 \to P_2$ is an epimorphism while $P_1 \to A_2$ is not even epic; (*ii*) $P_1 \to A_2$ is a monomorphism while $A_1 \to P_2$ is not even monic.

3.3 PROPOSITION. *Let* $(A'_i \xrightarrow{\alpha_i} A_i)_{i \in I}$ *be a translation between inverse systems over an ordered set I (in a category* $\mathfrak{A}$ *with inverse limits over I). Then* $\varprojlim \alpha_i$ *is a monomorphism whenever* α_i *is a monomorphism for each* $i \in I$.

3.3.1 REMARK. Proposition 3.3 extends, in an obvious way, to the case of limits over any category *I*.

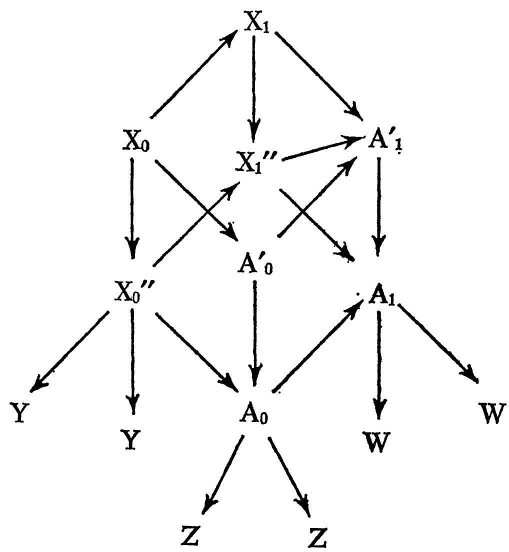
3.3.2 COROLLARY. *If* (α_i) *is a family of monomorphisms in a category with products, then* $\mathop{\mathsf{X}}_i \alpha_i$ *is a monomorphism.*

3.4 REMARKS. Given categories $\mathfrak{A}$ and *I*, let us denote by $\mathfrak{A}^I$ the category of functors from *I* to $\mathfrak{A}$ and of natural transformations between such functors. Then we have:

(*i*) Contrary to what happens for monics, in $\mathfrak{A}^I$ a pointwise monomorphism [10]) need not be a monomorphism.

Take, for example, as $\mathfrak{A}$ the category generated by the following commutative diagram

10) I.e. a natural transformation $F' \to F$ such that $F'_i \to F_i$ is a monomorphism for each object i in I.



where the composition morphisms from X_0 to Y like those from A'_0 to Z like those from A'_1 to W are equal, and as *I* the ordinal 2, i.e. the category pictured by

$$0 \rightarrow 1.$$

Then the symbols used undoubtedly suggest what we mean with functors like A', A, X, X'' and with natural transformations like $A' \rightarrow A$, $X \rightarrow X''$, $X \rightarrow A'$, $X'' \rightarrow A$ in $\mathcal{A}^2$. Now it is easy to check that $A' \rightarrow A$ is a pointwise monomorphism and $X \rightarrow X''$ is epic (but not pointwise). Since there is no morphism from X''_0 to A'_0 the square

$$\begin{array}{ccc} X & \rightarrow & A' \\ \downarrow & & \downarrow \\ X'' & \rightarrow & A \end{array}$$

cannot be filled in commutatively with a natural transformation from X'' to A', so that $A' \rightarrow A$ is not a monomorphism in $\mathcal{A}^2$.

Conversely:

(*ii*) A monomorphism in $\mathcal{A}^I$ need not be a pointwise monomorphism.

En effect let us add, in a commutative way, a morphism $X''_0 \to A'_0$ to the foregoing diagram; let $\bar{\mathcal{A}}$ be the category generated by the diagram so obtained (under the same assumptions as above) and let again I be the ordinal 2. Then it is easily seen that, in $\bar{\mathcal{A}}^I$, $X \to X''$ is an epimorphism which is not even a pointwise epic: in other words we have a counterexample dual with regard to that required.

However, with the symbols introduced above, we have trivially:

3.4.1 PROPOSITION. *In $\mathcal{A}^I$ every pointwise monomorphism is a monomorphism whenever every epic is a pointwise epic.*

Likely, in order to get that in $\mathcal{A}^I$ every epic is a pointwise epic, suitable hypothesises on the categories $\mathcal{A}$ and I must be made from time to time. For example, it is easily seen that each of the following conditions:

(*a*) *I is an ordered category and $\mathcal{A}$ has a zero object;*

(*b*) *I is a discrete category* [11])

is a sufficient one, so that we have respectively:

3.4.2 COROLLARY. *If I is an order category and $\mathcal{A}$ has a zero object, then in $\mathcal{A}^I$ every pointwise monomorphism is a monomorphism.*

3.4.3 COROLLARY. *If I is a discrete category then in $\mathcal{A}^I$ every pointwise monomorphism is a monomorphism.*

4. Preservation or reflection properties.

Contrary to monics, a faithful functor need not reflect monomorphisms (different but comparable uniform structures over the same set may induce the same topology; over a non-zero group there are at least two comparable and group-compatible topologies; and so on). The same happens also for inclusions, possibly full, of subcategories (the inclusion Hausdorff spaces $\to$ Topological spaces gives an example).

---

[11]) I.e. a category with only identies as morphisms (in other words: a set).

In this section we first give sufficient conditions in order that a functor reflects monomorphisms or epimorphisms and we relate such properties with adjointness; afterwards we relate adjointness with preservation of monomorphisms or epimorphisms; as a consequence we obtain at last that monomorphisms and epimorphisms are invariant with respect to equivalences.

4.0 With respect to a functor $T\colon \mathfrak{A} \to \mathfrak{B}$, we shall deal with the following statements:

(A) *for any morphism* $X \to A$ *and any monic* $A' \to A$, $X \to A$ *factors*[12] *through* $A' \to A$ *(in* $\mathfrak{A}$*), whenever* $TX \to TA$ *factors through* $TA' \to TA$ *(in* $\mathfrak{B}$*);*

(B) *any commutative square*

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ *epic and* $A' \to A$ *monic, can be filled in commutatively with a morphism* $X'' \to A'$ *(in* $\mathfrak{A}$*) whenever its image*

$$\begin{array}{ccc} TX & \to & TA' \\ \downarrow & & \downarrow \\ TX'' & \to & TA \end{array}$$

does (in $\mathfrak{B}$*),*

which the following lemmas hold for:

4.0.1 Lemma. (A) *implies* (B).

4.0.1* Lemma. (A*) *implies* (B)[13].

4.0.2 Lemma. *If* $T\colon \mathfrak{A} \to \mathfrak{B}$ *is a full and faithful functor, then* T *satisfies* (A), (A*) *and* (B).

The announced reflection properties are:

---

[12]) Necessarily in a unique way.

[13]) Note that (B) is self-dual.

4.1. PROPOSITION. *Let* $T\colon \mathfrak{A} \to \mathfrak{B}$ *be an epic preserving and monic reflecting functor. Then* T *reflects monomorphisms whenever satisfies any one of* (A), (A*), (B).

4.1.1 COROLLARY. *A full and faithful functor which preserves epics (resp. monics) reflects monomorphisms (resp. epimorphisms).*

4.1.2 COROLLARY. *If* $\mathfrak{A}'$ *is a full subcategory of a category* $\mathfrak{A}$, *then the inclusion functor reflects monomorphisms (resp. epimorphisms) whenever preserve epics (resp. monics).*

Reflection properties and adjointness [14]) are related as follows.

---

[14]) Hereafter, whenever we deal with a pair of adjoint functors

$$S\colon \mathfrak{B} \to \mathfrak{A}, \qquad T\colon \mathfrak{A} \to \mathfrak{B}$$

with S left adjoint to T, we understand that a quadruple of natural transformations:

$$\Phi = \Phi^{(S,T)}\colon\ hom_{\mathfrak{A}}(S-,\ -) \to hom_{\mathfrak{B}}(-,\ T-)$$
$$\Psi = \Psi^{(S,T)}\colon\ hom_{\mathfrak{B}}(-,\ T-) \to hom_{\mathfrak{A}}(S-,\ -)$$
$$\varphi = \varphi^{(S,T)}\colon\ I_{\mathfrak{B}} \to TS$$
$$\psi = \psi^{(S,T)}\colon\ ST \to I_{\mathfrak{A}}$$

so is chosen as to satisfy the following equations:

(C_0)	$\Psi = \Phi^{-1}$	
(C_1)	$\varphi_B = \Phi(id_{SB})$	*for all objects* B *in* $\mathfrak{B}$.
(C_1^*)	$\psi_A = \Psi(id_{TA})$	*for all objects* A *in* $\mathfrak{A}$.
(C_2)	$\Phi(\alpha) = T\alpha \circ \varphi_B$	*for any morfism* $SB \xrightarrow{\alpha} A$ *in* $\mathfrak{A}$ $(A \in \mathfrak{A},\ B \in \mathfrak{B})$.
(C_2^*)	$\Psi(\beta) = \psi_A \circ S\beta$	*for any morphism* $B \xrightarrow{\beta} TA$ *in* $\mathfrak{B}$ $(A \in \mathfrak{A},\ B \in \mathfrak{B})$.
(C_3)	$T\psi_A \circ \varphi_{TA} = id_{TA}$	*for all objects* A *in* $\mathfrak{A}$.
(C_3^*)	$\psi_{SB} \circ S\varphi_B = id_{SB}$	*for all objects* B *in* $\mathfrak{B}$,

4.2 PROPOSITION. *Suppose a functor T: $\mathfrak{A} \to \mathfrak{B}$ have a left adjoint S: $\mathfrak{B} \to \mathfrak{A}$. Then the following statements are equivalent.*

(*a*) *T is faithful and satisfies* (A).

(*b*) *T is faithful and satisfies* (B).

(*c*) *T reflects epics and satisfies* (B).

(*d*) *T reflects epimorphisms.*

(*e*) *If β: $\mathrm{B} \to T\mathrm{A}$ is an epimorphism, then $\alpha = \Psi(\beta)$ is an epimorphism.*

(*f*) *ψ_{A}: $ST\mathrm{A} \to \mathrm{A}$ is an epimorphism for all objects* A *of* $\mathfrak{A}$.

4.2.1 COROLLARY. *A full and faithful functor, which has a left adjoint, reflects epimorphisms.*

4.2.2 COROLLARY. *Let $\mathfrak{A}'$ a full subcategory of a category $\mathfrak{A}$. If the inclusion functor has a left adjoint, then it reflects epimorphisms.*

4.2.3 EXERCISE. The inclusion Hausdorff spaces $\to$ Topological spaces has no right adjoint.

4.2.4 COROLLARY. *Suppose a functor T: $\mathfrak{A} \to \mathfrak{B}$ have both a left adjoint S: $\mathfrak{B} \to \mathfrak{A}$ and a right adjoint U: $\mathfrak{B} \to \mathfrak{A}$. Then the following statements are equivalent.*

(*a*) *T reflects epimorphisms.*

(*b*) *T reflects monomorphisms.*

---

which together the following further equations yield:

(C4) $\Phi(S\delta) = \varphi_{\mathrm{B}} \circ \delta$ *for any morphism* $\mathrm{B}' \xrightarrow{\delta} \mathrm{B}$ *in* $\mathfrak{B}$.

(C4*) $\Psi(T\gamma) = \gamma \circ \psi_{\mathrm{A}}$ *for any morphism* $\mathrm{A} \xrightarrow{\gamma} \mathrm{A}'$ *in* $\mathfrak{A}$.

(C5) $\alpha = \psi_{\mathrm{A}} \circ ST\alpha \circ S\varphi_{\mathrm{B}}$ *for any morphism* $S\mathrm{B} \xrightarrow{\alpha} \mathrm{A}$ *in* $\mathfrak{A}$ $(\mathrm{A} \in \mathfrak{A},\ \mathrm{B} \in \mathfrak{B})$.

(C5*) $\beta = T\psi_{\mathrm{A}} \circ TS\beta \circ \varphi_{\mathrm{B}}$ *for any morphism* $\mathrm{B} \xrightarrow{\beta} T\mathrm{A}$ *in* $\mathfrak{B}$ $(\mathrm{A} \in \mathfrak{A},\ \mathrm{B} \in \mathfrak{B})$.

As a rule, the equations above will be written down or quoted without any reference to this footnote.

(*c*) $\psi_A^{(S,T)}$: $STA \to A$ *is an epimorphism for all objects* A *of* $\mathcal{A}$.

(*d*) $\varphi_A^{(T,U)}$: $A \to UTA$ *is a monomorphism for all objects* A *of* $\mathcal{A}$.

The relationship between adjointness and preservation reads as follows.

4.3 PROPOSITION. *If* T: $\mathcal{A} \to \mathcal{B}$ *has a left adjoint* S: $\mathcal{B} \to \mathcal{A}$, *then* T *preserves monomorfisms.*

4.3.1 REMARK. Let $\mathcal{A}$ be a category with products. If I is a set (i.e. a discrete category), then $\underset{i \in I}{\mathbf{X}}$ can be envisaged as a functor from $\mathcal{A}^I$ to $\mathcal{A}$, and more precisely as the right adjoint of a suitable « constant » functor. Hence combining 4.3 and 3.4.3 we get another proof of 3.3.2.

Likewise, if $\mathcal{A}$ is a category with zero objects, 3.3 can be proved combining 4.3 and 3.4.2. However, if $\mathcal{A}$ has no zero object the same argument does not apply: en effectt, since any category has inverse limits over the ordinal 2, the obstruction to such a proof is given by the remark (*i*) of 3.4. Hence, after all, proposition 3.3 (and *a fortiori* its generalization 3.3.1) stands apart.

Combining 4.2.1 and 4.3 we obtain the following:

4.4 PROPOSITION. *If a functor* T *is full, faithful and has a left adjoint, then* T *preserves monomorphisms and reflects epimorphisms.*

In particular

4.4.1 COROLLARY. *Let* $\mathcal{A}'$ *a full subcategory of a category* $\mathcal{A}$. *If the inclusion functor has a left adjoint, then it preserves monomorphisms and reflects epimorphisms.*

4.5 Recall that an equivalence between a category $\mathcal{A}$ and a category $\mathcal{B}$ (see Grothendieck [1]) is a quadruple (S, T, σ, τ) consisting of covariant functors

$$S: \mathcal{B} \to \mathcal{A}, \qquad T: \mathcal{A} \to \mathcal{B}$$

and natural equivalences

$$\sigma\colon I_{\mathfrak{B}} \to TS, \qquad \tau\colon I_{\mathfrak{A}} \to ST$$

such that

$$\sigma_{TA}^{-1} \circ T\tau_A = id_{TA} \textit{ for all objects } A \textit{ of } \mathfrak{A},$$

$$\tau_{SB}^{-1} \circ S\sigma_B = id_{SB} \textit{ for all objects } B \textit{ of } \mathfrak{B}.$$

Now it is easily seen that, if (S, T, σ, τ) is such a quadruple, then S is both a left and a right adjoint of T with all the φ, ψ natural equivalences. Hence combining 4.2.4, 4.3 and 4.3* we get:

4.5.1 PROPOSITION. *Monomorphisms and epimorphisms are invariant with respect to equivalences.*

5. Injectives, projectives.

5.1 DEFINITIONS. *An object* Q *in a category* $\mathfrak{A}$ *is* **injective** *iff for every diagram*

$$\begin{array}{l} A' \to A \\ \downarrow \\ Q \end{array}$$

with $A' \to A$ *a monomorphism, there is a (not necessarily unique) morphism* $A \to Q$ *making the diagram commutative.*

A category $\mathfrak{A}$ *has* **enough injectives** *iff very object* A *in* $\mathfrak{A}$ *admits a monomorphism* $A \to Q$, *with* Q *an injective.*

The dual to injective is **projective.**

5.1.1 EXAMPLE. The injectives in the category of ordered sets are just the complete lattices. Furthermore the category has enough injectives.

5.1.2 NOTE. The definition of injective in 5.1 is obtained from Mitchell's one simply by taking « monomorphism » instead of « monic » (see [7]).

(With respect to Mitchell's definition, the only injective in the category of ordered set is, up to isomorphisms, the ordinal 1).

Conversely our definition agrees with that of Heller (see [2]).

As usual we get:

5.2 Proposition. *A retract of an injective is an injective.*

5.3 Proposition. *Every injective is an absolute retract (in any category). Conversely in a category with enough injectives, every absolute retract is an injective.*

5.4 Proposition. *If* $Q = \mathsf{X}_i Q_i$ *and if each* Q_i *is injective, then* Q *is injective. Conversely, in a category with zero morphisms, if* Q *is injective then each* Q_i *is injective.*

Other properties can be reached in a standard way. For example we have:

5.5 Proposition. *Suppose a functor* T: $\mathcal{A} \rightarrow \mathcal{B}$ *have a left adjoint* S: $\mathcal{B} \rightarrow \mathcal{A}$*, which preserves monomorphisms. Then* T *preserves injectives.*

5.5.1 Corollary. *Let* $\mathcal{A}'$ *be a full subcategory of a category* $\mathcal{A}$*. If the inclusion functor has a left adjoint which preserves monomorphisms, then an object* Q' *in* $\mathcal{A}'$ *is injective in* $\mathcal{A}'$ *iff it is injective in* $\mathcal{A}$.

6. Subobjects, quotient objects.

After a survey of the properties of monomorphisms listed in the preceding sections, the following definition seems correct.

6.1 Definition. *Let* A *be an object in any category* $\mathcal{A}$*. A morphism* α *in* $\mathcal{A}$ *is a* **subobject** *of* A *iff* α *is a monomorphism of codomain* A.

The domain of α will be denoted A_α and will be called, by a standard « abus de langage », a subobject of A.

Such locutions as « inclusion », « contained » and such symbols as «$\subset$» are at hand.

6.1.1 Other definitions and Remarks. Given subobjects α_1 and α_2 of an object A, we shall write $\alpha_1 \leq \alpha_2$ if there is a morphism γ such that $\alpha_1 = \alpha_2\gamma$.

As usual: $\alpha \leq \alpha$; $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_3$ implies $\alpha_1 \leq \alpha_3$, so that the set of subobjects of an object is a pre-ordered set.

Further: « $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ » is equivalent to « there is an isomorphism γ such that $\alpha_1 = \alpha_2\gamma$ ».

Again: unions and intersections of families of subobjects of an object A are to be understood in their usual meaning in pre-ordered sets. So $A_\alpha = \bigcup_i A_{\alpha_i}$ is equivalent to $A_{\alpha_i} \subset A_\alpha$ for each i and $A_\alpha \subset A_{\alpha'}$, for any $A_{\alpha'}$, which contains each A_{α_i}. (Obviously A_α is determined up to isomorphisms).

6.1.2 Note. All properties of monomorphisms seen in the preceding sections may be rewritten in terms of subobjects. In order to obtain stronger properties of subobjects, a more rich structure than that of category is needed.

For example by 3.2, 1.2 and by definitions it is easily seen that *given subobjects* α_1 *and* α_2 *of an object* A, *if*

$$\begin{array}{ccc} P & \rightarrow & A_{\alpha_1} \\ \downarrow & & \downarrow \\ A_{\alpha_2} & \rightarrow & A \end{array}$$

is a cartesian square, then the composition from P *to* A *is the intersection of* α_1 *and* α_2. Nevertheless the vice versa may fail unless the category has images (see our introduction).

Therefore we must put off a more detailed study of subobjects.

By duality we have:

6.1* Definition. *Let* A *be an object in any category* $\mathcal{A}$. *A morphism* α *in* $\mathcal{A}$ *is a* **quotient object** *of* A *iff* α *is an epimorphism of domain* A.

The codomain of α will usually be denoted by A^α and will be called a quotient object of A; α itself will be referred to as the projection of A onto A^α.

Definitions and remarks, dual to those for subobjects, obviously stand.

7. Proofs.

Hereafter, whenever we write « It is known that ... » we understand « and the proof may be found in [7] ».

7.0 Some useful trivialities.

In order to prove that a morphism satisfies (M_{II}) or dually (E_{II}), the following obvious lemma or its dual or their corollary are frequently used.

LEMMA. *Suppose that in the diagram*

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & \nearrow & \downarrow \\ X'' & \to & A \end{array}$$

the morphism $X \to X''$ *be epic. If the square and the upper triangle commute, then also the lower triangle commutes.*

COROLLARY. *Suppose that in the diagram*

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & \nearrow & \downarrow \\ X'' & \to & A \end{array}$$

$X \to X''$ *be epic and* $A' \to A$ *be monic. Then the diagram commutes iff the square and any one of the triangles commute.*

7.1 Proofs relative to section 1.

PROOF OF 1.1.1 The most of the categories listed in section 2 have monics which are not monomorphisms.

Conversely the morphism $A' \to A$ satisfies (M_{II}) but not (M_I) in the category pictured by:

$$\begin{array}{ccc} & & Z \\ & \nearrow & \upuparrows \\ X & \rightrightarrows & A' \\ \downarrow & \swarrow & \downarrow \\ A & \rightrightarrows & Y \end{array}$$

where:

(*i*) distinct letters (resp. arrows) denote different objects (resp. morphisms);

(*ii*) all triangles commute.

PROOF. OF 1.2 It is known that if $A_2 \rightarrow A_1$ and $A_1 \rightarrow A$ are monic, then so is their composition $A_2 \rightarrow A_1 \rightarrow A$.

Take now any commutative diagram

$$\begin{array}{ccc} X & \rightarrow & A_2 \\ \big| & & \downarrow \\ \big| & & A_1 \\ \downarrow & & \downarrow \\ X'' & \rightarrow & A \end{array}$$

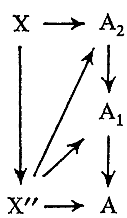
with $X \rightarrow X''$ epic. Since $A_1 \rightarrow A$ satisfies (M_{II}), there is a morphism $X'' \rightarrow A_1$ such that the diagram

$$\begin{array}{ccc} X & \rightarrow & A_2 \\ \big| & & \downarrow \\ \big| & & A_1 \\ \downarrow & \nearrow & \downarrow \\ X'' & \rightarrow & A \end{array}$$

commutes. Again, since $A_2 \rightarrow A_1$ satisfies (M_{II}), there is a morphism $X'' \rightarrow A_2$ such that the diagram

$$\begin{array}{ccc} X & \rightarrow & A_2 \\ \downarrow & \nearrow & \downarrow \\ X'' & \rightarrow & A_1 \end{array}$$

and therefore also the diagram



commutes. Hence the composition $A_2 \rightarrow A_1 \rightarrow A$ satisfies (M_{II}).

PROOF OF 1.3 It is known that if the composition $A_2 \rightarrow A_1 \rightarrow A$ is monic, then so is $A_2 \rightarrow A_1$.

Take now any commutative square

$$\begin{array}{ccc} X & \rightarrow & A_2 \\ \downarrow & & \downarrow \\ X'' & \rightarrow & A_1 \end{array}$$

with $X \to X''$ epic. Since the composition $A_2 \to A_1 \to A$ satisfies M_{II}, the diagram

$$\begin{array}{ccc} X & \longrightarrow & A_2 \\ \Big| & & \downarrow \\ \Big| & & A_1 \\ \downarrow & & \downarrow \\ X'' \to A_1 & \to & A \end{array}$$

which obviously commutes, can be filled in commutatively with a morphism $X'' \to A_2$. Hence in the diagram

$$\begin{array}{ccc} X & \to & A_2 \\ \downarrow & \nearrow & \downarrow \\ X'' & \to & A_1 \end{array}$$

both the square and the upper triangle commute; then it follows by 7.0 that the whole diagram commutes, so that 1.3 is proved.

Proof of 1.3.1 By definition, $A' \to A$ is a coretraction iff there is a morphism $A \to A'$ such that the composition $A' \to A \to A'$ be equal to $id_{A'}$, which is obviously a monomorphism. Hence 1.3 applies.

Proof of 1.3.2 The diagonal $A \to A \times A$ and the iniections $A_i \to \mathop{\mathsf{X}}\limits_i A_i$ are coretractions, so that 1.3.1 applies.

Proof of 1.4 If $A' \to A$ is an epic monomorphism, then by (M_{II}) there is a morphism $A \to A'$ such that the diagram

$$\begin{array}{ccc} A' & \to & A' \\ \downarrow & \nearrow & \downarrow \\ A & \to & A \end{array}$$

where the horizontal arrows are identity morphisms, commutes. In other words $A' \to A$ admits $A \to A'$ as inverse, i.e. it is an isomorphism.

Proof of 1.5 An isomorphism is both a coretraction and a retraction; hence by 1.3.1 and 1.3.1* respectively it is both a monomorphism and an epimorphism. The vice versa is a special case of 1.4.

7.2 Proofs relative to section 2.

Proof. of 2.1 Let us take the category of sets. It is known that, in this category, the monics are the injections (the functions one-one into), and the epics are the surjections (functions onto). That being stated, let $A' \to A$ be a monic;

in order to prove that it satisfies (M_{II}), let

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow \\ X'' & \rightarrow & A \end{array}$$

be any commutative square with $X \xrightarrow{f} X''$ an epic. Now, for each $x'' \in X''$, the set $\varphi(f^{-1}(x''))$ has at least one point because f is surjective; it has at most one point because $A' \rightarrow A$ is injective and the square commutes. The required function $X'' \rightarrow A'$ is therefore at hand.

The same argument applies to the category of pointed sets. Similarly for lattices.

With regard to the category of the correspondences in an abelian category, remember that, in such a category, every monic is a coretraction (see, for example, [3]) and hence a monomorphism (see our 1.3.1).

Finally, to the other categories listed in section 2.1, apply the following:

LEMMA. *Let $\mathcal{A}$ be a category with zero morphisms, such that every epic be the cokernel of some morphism. Then, in $\mathcal{A}$, every monic is a monomorphism (and every epic is an epimorphism).*

En effect, let

$$\begin{array}{ccc} X & \rightarrow & A' \\ \downarrow & & \downarrow \\ X'' & \rightarrow & A \end{array}$$

be any commutative square with $X \rightarrow X''$ an epic and $A' \rightarrow A$ a monic; and let $Y \rightarrow X$ any morphism which admits $X \rightarrow X''$ as cokernel. Since the composition $Y \rightarrow X \rightarrow X''$ is zero, then so is $Y \rightarrow X \rightarrow X'' \rightarrow A$; since the square commutes the same holds for $Y \rightarrow X \rightarrow A' \rightarrow A$; finally also $Y \rightarrow X \rightarrow A'$ is zero (because $A' \rightarrow A$ is monic). Hence, by the universal property of cokernels, $X' \rightarrow A$ factors through $X \rightarrow X''$, so that the conclusion follows from 7.0.

PROOF OF 2.2 Let us first consider the category of topological spaces. As in the category of sets, monics are injections and epics, surjections.

Let $A' \rightarrow A$ be a monomorphism and let $A' \rightarrow I \rightarrow A$ its factorization through the image, so that $I \rightarrow A$ is the inclusion of I as a subspace of A and $A' \rightarrow I$ is epic. Hence, by (M_{II}), the square

$$\begin{array}{ccc} A' & \rightarrow & A' \\ \downarrow & & \downarrow \\ I & \rightarrow & A \end{array}$$

where the top row is the identity on A′, can be filled in commutatively with a continuous function $I \to A'$; hence $A' \to I$ is an epic coretraction and therefore, by 1.3.1 and 1.4, an isomorphism. In other words $A' \to A$ is the composition of an isomorphism, $A' \to I$, and the inclusion of a subspace of A, $I \to A$.

Conversely, let A′ be a subspace of a topological space A and let $A' \to A$ be the inclusion. Then $A' \to A$ is monic; in order to prove that it satisfies (M_{II}), take any commutative square

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ an epic and denote by F the forgetful functor from Topological spaces to Sets. We have seen, in the proof of 2.1, that he square

$$\begin{array}{ccc} FX & \to & FA' \\ \downarrow & & \downarrow \\ FX'' & \to & FA \end{array}$$

can be filled in commutatively with a function $FX'' \to FA'$. Now any U′, open in A′, can be expressed in the form $U' = A' \cap U$, U an open in A; further the inverse image U″ of U by means of $FX'' \to FA$ is open in X″ (because $X'' \to A$ is continuous). Finally, since $FX \to FX''$ is surjective and the square commutes, it is easy to check that U″ is also the inverse image of U′ by means of $FX'' \to FA'$; this one is therefore the image under F of a continuous function $X'' \to A'$ which fills in commutatively the starting square.

Let us now consider, for example, the category of topological groups. Once again monics are injections and epics, surjections.

In order to prove that every monomorphism is, up to an isomorphism, the inclusion of a subspace, the model used for topological spaces fits in again.

Conversely, to prove that every inclusion satisfies (M_{II}), let us denote by Σ the usual commutative square

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ an epic and $A' \to A$ our inclusion of A′ as a topological subgroup of A; moreover let us consider the following commutative square of forgetful functors:

$$\begin{array}{ccc} \boxed{\text{Top. groups}} & \to & \boxed{\text{Groups}} \\ \downarrow & & \downarrow \\ \boxed{\text{Top. spaces}} & \to & \boxed{\text{Sets}} \end{array}$$

Now $A' \to A$ in Groups [15]) is again an inclusion so that (by 2.1) Σ in Groups can be filled in commutatively in the usual way; on the other hand $A' \to A$ is an inclusion also in Top. spaces and therefore (by the homologous proof given above for such category) Σ can be filled in commutatively in the usual way also in Top. spaces. Because of the commutativity of the above square of functors, the same thing happens also for Σ and the proof is complete.

Similar arguments apply to the other categories listed in 2.2.

PROOF OF 2.3 Let us consider the category of Hausdorff spaces and let F be the forgetful functor from Hausdorff spaces to Sets. The monics are again the injections, whereas the epics are the continuous functions $X \to X''$ such that $FX \to FX''$ has an image dense in X'' [16]).

Given a monomorphism $A' \to A$, let I be the closure in A, with its relative topology, of the image of $FA' \to FA$; then $A' \to A$ admits a unique factorization $A' \to I \to A$ with $A' \to I$ an epic and $I \to A$ the inclusion of I as a subspace of A. Hence, by (M_{II}), the square

$$\begin{array}{ccc} A' & \to & A' \\ \downarrow & & \downarrow \\ I & \to & A \end{array}$$

where the top row is the identity on A', can be filled in commutatively with a continuous function $I \to A'$. It follows that $A' \to I$ is an epic coretraction and therefore, by 1.3.1 and 1.4, an isomorphism. In other words A' is isomorphic to a *closed* subspace of A.

Conversely, let A' a *closed* subspace of A and let $A' \to A$ the inclusion. In order to prove that $A' \to A$ satisfies (M_{II}), given any commutative square

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

---

[15]) Here « $A' \to A$ in Groups » stands for « the image of $A' \to A$ in Groups under the forgetful functor from Top. groups to Groups ». And so on.

[16]) Other admissible wordings for « the continuous ... in X'' » are, e.g., the following: « the continuous functions whose image in Sets in dense X'' » (see footnote [15]); « the continuous functions whose set theoretical image is dense in their codomain ». On the contrary, to avoid misunderstandings, such wordings as « the continuous functions $X \to X''$ whose image is dense in X'' » are not permissible, why the categorical Image (in Hausdorff spaces) of the epic $X \to X''$ will be X''.

with $X \to X''$ an epic, let us first construct a function $FX'' \to FA'$ filling in commutatively the square

$$\begin{array}{ccc} FX & \to & FA' \\ \downarrow & & \downarrow \\ FX'' & \to & FA \end{array}$$

It is sufficient to prove that the image $a \in FA$ of any $x'' \in FX''$, under $FX'' \to FA$, belongs to FA'. En effect if U is a neighbourhood of $a \in FA$, its inverse image, V'', under $FX'' \to FA$ is a neighbourhood of x'', whose inverse image V, under $FX \to FX''$, is not empty (because $X \to X''$ is epic). Hence the image U' of V under $FX \to FA'$ is not empty; further, since the above square commutes, we have $U' \subset U$, so that a belongs to the adherence of A', i.e. to A', because it is closed. The required function $FX'' \to FA$ is so at hand. Now if H is closed in A', it is closed also in A (since A' is so) and therefore also its inverse image K'' under $FX'' \to FA$ is closed; but K'' is also the inverse image of H under $FX'' \to FA'$ which is therefore the image under F of a continuous function filling in commutatively the starting square. This complete the proof.

Similar arguments or similar techniques to that used for Top. groups, in the proof of 2.2, apply to the other categories listed in 2.3.

Proof of 2.5 An ordered category is a category such that:

(*i*) for every couple of objects (A_1, A_2), there is at most one morphism $A_1 \to A_2$;

(*ii*) if $A_1 \to A_2$ and $A_2 \to A_1$ are morphisms, then $A_1 = A_2$.

Hence in an ordered category every morphism is both monic and epic. If now $A' \to A$ is a monomorphism, applying (M_{II}) to the square

$$\begin{array}{ccc} A' & \to & A' \\ \downarrow & & \downarrow \\ A & \to & A \end{array}$$

we get a morphism $A \to A'$, so that (by (*ii*)) $A = A'$ and hence (by (*i*)) $A' \to A$ is the identity.

7.3 Proofs relative to section 3.

Proof of 3.1 It is known that equalizers are monic. Take now any commutative square

$$\begin{array}{ccc} X & \to & K \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ an epic. Since $K \to A$ is an equalizer for $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$, $K \to A \overset{\alpha}{\to} B$ is equal to $K \to A \overset{\beta}{\to} B$ so that $X \to K \to A \overset{\alpha}{\to} B$ is equal to $X \to K \to A \overset{\beta}{\to} B$; hence $X \to X'' \to A \overset{\alpha}{\to} B$ is equal to $X \to X'' \to A \overset{\beta}{\to} B$ (because the above square commutes) and then $X'' \to A \overset{\alpha}{\to} B$ is equal to $X'' \to A \overset{\beta}{\to} B$ (because $X \to X''$ is epic). Now the last equality yields that $X'' \to A$ factors through $K \to A$ (because of the universal property of $K \to A$ as equalizer for $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$). In other words there exists a morphism $X'' \to K$ such that the lower triangle in the diagram

$$\begin{array}{ccc} X & \to & K \\ \downarrow & \nearrow & \downarrow \\ X'' & \to & A \end{array}$$

commutes. Since also the square commutes the result follows from 7.0.

PROOF OF 3.1.1 It is known that $P \to A_1 \times A_2$ is the equalizer of $A_1 \times A_2 \to A_1 \to A$ and $A_1 \times A_2 \to A_2 \to A$ (where $A_1 \times A_2 \to A_1$ and $A_1 \times A_2 \to A_2$ are the projections). Hence 3.1 applies.

PROOF. OF 3.2. It is known that, in our hypothesises, $P \to A_2$ is monic. Take now any commutative square

$$\begin{array}{ccc} X & \to & P \\ \downarrow & & \downarrow \\ X'' & \to & A_2 \end{array}$$

with $X \to X''$ an epic. Then also the diagram

$$\begin{array}{ccccc} X & \to & P & \to & A_1 \\ \downarrow & & \downarrow & & \downarrow \\ X'' & \to & A_2 & \to & A \end{array}$$

commutes; further, since $A_1 \to A$ satisfies (M_{II}), it can be filled in commutatively with a morphism $X'' \to A_1$; therefore the hypothesis that the right square is cartesian implies, in particular, that $X'' \to A_2$ factors through $P \to A_2$; in other words there exists a morphism $X'' \to P$ such that the lower triangle in the diagram

$$\begin{array}{ccc} X & \to & P \\ \downarrow & \nearrow & \downarrow \\ X'' & \to & A_2 \end{array}$$

commutes. Since also the square commutes, the result follows by 7.0.

PROOF OF 3.3 Let us state, briefly, $A' = \varprojlim A'_i$; $A = \varprojlim A_i$; $\alpha = \varprojlim \alpha_i$. It is known that α is monic. In order to prove that α satisfies (M_{II}) take any commutative square

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ an epic. Since each $A'_i \to A_i$ satisfies (M_{II}), for each i there is a morphism $X'' \to A'_i$ making commutative the diagram

$$\begin{array}{ccccc} X & \longrightarrow & A' & \longrightarrow & A'_i \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ X'' & \longrightarrow & A & \longrightarrow & A_i \end{array}$$

Now, using the hypothesis that each $A'_i \to A_i$ is monic, it is easy to check that the family $(X'' \to A'_i)_{i \in I}$ is I-compatible; hence there is a unique morphism $X'' \to A'$ such that $X'' \to A \to A'_i$ be equal to $X'' \to A'_i$ for each i; therefore $X'' \to A' \to A'_i \to A_i$ is equal to $X'' \to A \to A_i$ for each i and such a family of morphisms from X'' to A_i is I-compatible; this yields that $X'' \to A' \to A$ is equal to $X'' \to A$. The conclusion now follows by 7.0.

PROOF OF 3.3.1 The proof, in the case of limits over any category I, differs from the previous one only in the meaning of « limit » and of « family I-compatible ».

PROOF OF 3.3.2 Apply 3.3 for I ordered by equality.

PROOF OF 3.4.1 Let $A' \to A$ be any natural transformation in $\mathcal{A}^I$ such that $A'_i \to A_i$ be a monomorphism for each object i in I. Then it is known that $A' \to A$ is monic. If now

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

is any commutative square in $\mathcal{A}^I$ with $X \to X''$ an epic, since $X \to X''$ turns out to be a pointwise epic, then for each object i in I there is a morphism $X''_i \to A'_i$ making commutative the diagram

$$\begin{array}{ccc} X_i & \to & A'_i \\ \downarrow & \nearrow & \downarrow \\ X''_i & \to & A_i \end{array}$$

Now it is easy to check, by means of appropriate cubical diagrams, that the family $(X''_i \to A'_i)_i$ defines the required natural transformation $X'' \to A'$, so that also (M_{II}) is proved.

7.4 Proofs relative to section 4.

Proof of 4.0.1 Apply 7.0.

Proof of 4.0.2 En effect let $TX'' \to TA'$ be a morphism which fills in commutatively the square in $\mathfrak{B}$; then it is the image under T of a morphism $X'' \to A'$ (by definition of full functor) which fills in commutatively the square in $\mathfrak{A}$ (since, as it is known, a faithful functor reflects commutative diagrams).

Proof of 4.1 Consider a morphism $A' \to A$ in $\mathfrak{A}$ and suppose that $TA' \to TA$ be a monomorphism in $\mathfrak{B}$; we have to prove that $A' \to A$ is a monomorphism.

Now $A' \xrightarrow{\beta} A$ is monic by the hypothesis that T reflects monics. In order to prove that it satisfies (M_{II}) consider any commuative square

$$\begin{array}{ccc} X & \to & A' \\ \downarrow & & \downarrow \\ X'' & \to & A \end{array}$$

with $X \to X''$ an epic. Since T preserves epics we get a commutative square

$$\begin{array}{ccc} TX & \to & TA' \\ \downarrow & & \downarrow \\ TX'' & \to & TA \end{array}$$

with $TX \to TX''$ an epic, which can be filled in commutatively with a morphism $TX'' \to TA'$ since, by hypothesis, $TA' \to TA$ satisfies (M_{II}). The conclusion now follows applying (B), which is in any case satisfied because of 4.0.1 and 4.0.1*.

Proof of 4.1.1 It is known that a faithful functor reflects monics and epics. By 4.0.2 a full and faithful functor satisfies (A), (A*) and (B). Hence 4.1 (resp. 4.1*) applies.

Proof of 4.1.2 It is a particular case of 4.1.1.

Proof of 4.2 (*a*) implies (*b*): by 4.0.1.

(*b*) implies (*c*): it is known that a faithful functor reflects epics.

(*c*) implies (*d*): it is known that a functor, whith has a left adjoint, preserves monics; hence 4.1* applies.

(*d*) implies (*e*): by (C_0) and (C_2) the triangle

$$\begin{array}{ccc} B & \longrightarrow & TSB \\ {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle T\alpha} & \\ TA & & \end{array}$$

commutes. Hence, if β is an epimorphism, it follows from 1.3* that $T\alpha$ is an epimorphism. Since T reflects epimorphisms, this means that α is an epimorphism.

(*e*) implies (*f*): this follows by taking $\beta = id_{TA}$.

(*f*) implies (*a*): it is known that, in our hypothesises, T is faithful. Let us now prove that T satisfies (A). Given a morphism $X \to A$ and a monic $A' \to A$ in $\mathcal{A}$ such that

$$\begin{array}{ccc} & & TA' \\ & \nearrow & \downarrow \\ TX & \longrightarrow & TA \end{array}$$

commutes in $\mathcal{B}$ ($TX \to TA'$ being a suitable morphism of $\mathcal{B}$) mapping by means of S and making use of the naturality ψ, we get in $\mathcal{A}$ the following commutative diagram

$$\begin{array}{ccccc} & & STA' & & \\ & \nearrow & \downarrow & \searrow{\scriptstyle\psi_{A'}} & \\ STX & \longrightarrow & STA & & A' \\ & \searrow{\scriptstyle\psi_X} & & \searrow{\scriptstyle\psi_A} & \downarrow \\ & & X & \longrightarrow & A \end{array}$$

Now ψ_X satisfies (E_{II}) and $A' \to A$ is monic, hence the diagram

$$\begin{array}{ccccc} STX & \longrightarrow & STA' & \xrightarrow{\psi_{A'}} & A' \\ {\scriptstyle\psi_X}\downarrow & & & & \downarrow \\ X & & \longrightarrow & & A \end{array}$$

can be filled in commutatively with a morphism $X \to A'$, so that $X \to A$ factors through $A' \to A$ [17]).

[17]) In order to prove that (*f*) implies (*a*) the hypothesis that S is a left adjoint for T needs not. In fact it is sufficient to know that there are a functor S: $\mathcal{B} \to \mathcal{A}$ and a natural transformation ψ: $ST \to I_{\mathcal{A}}$ satisfying statement (*f*).

Proofs of 4.2.1, 4.2.2, 4.2.3 The corollary 4.2.1 follows by 4.0.2 and 4.2; 4.2.2 is a special case of 4.2.1; 4.2.3 follows by 4.2.2*, since the functor Hausdorff spaces → Topological spaces does not reflect monomorphisms (see 2.2 and 2.3).

Proof of 4.2.4 Remark that statement (*b*) of proposition 4.2 is self-dual; then combine 4.2 and 4.2*.

Proof. of 4.3 Let $A' \to A$ be a monomorphism in $\mathcal{A}$. It is known that $TA' \to TA$ is monic in $\mathcal{B}$; in order to prove that it satisfies (M_{II}), take in $\mathcal{B}$ any commutative square

$$\begin{array}{ccc} Y & \to & TA' \\ \downarrow & & \downarrow \\ Y'' & \to & TA \end{array}$$

with $Y \to Y''$ an epic. Mapping by means of S and using the naturality of ψ we get in $\mathcal{A}$ the following commutative diagram

$$\begin{array}{ccccc} SY & \longrightarrow & STA' & \xrightarrow{\psi_{A'}} & A' \\ \downarrow & & \downarrow & \psi_A & \downarrow \\ SY'' & \longrightarrow & STA & \longrightarrow & A \end{array}$$

where $SY \to SY''$ is epic (because S, having a right adjoint, preserves epics). Now, since by hypothesis $A' \to A$ satisfies (M_{II}), there is a morphism $SY'' \to A'$ such that the diagram

$$\begin{array}{ccccc} SY & \longrightarrow & STA' & \longrightarrow & A' \\ \downarrow & & & \nearrow & \downarrow \\ SY'' & \longrightarrow & STA & \longrightarrow & A \end{array}$$

commutes (in $\mathcal{A}$). Mapping by means of T and using the naturality of φ we obtain in $\mathcal{B}$ the following commutative diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{\varphi_Y} & TSY & \longrightarrow & TSTA' & \longrightarrow & TA' \\ \downarrow & & \downarrow & & & \nearrow & \downarrow \\ Y'' & \xrightarrow[\varphi_{Y''}]{} & TSY'' & \longrightarrow & TSTA & \longrightarrow & TA \end{array}$$

Now the conclusion follows by $(C_5{}^*)$.

7.5 Proofs relative to section 5.

Proof of 5.1.1 For any ordered set E, let $\widehat{E}$ be the set of all lower segments [18]) of E ordered by inclusion and let $\varphi_E : E \to \widehat{E}$ be the function defined by $\varphi_E(x) = \{y \mid y \in E,\ y \leq x\}$ $(x \in E)$.

Taking for granted that:

(*a*) *For any ordered set* E, $\widehat{E}$ *is a complete lattice;*

(*b*) *For any ordered set* E, φ_E *is a monomorphism* (in the category of ordered sets), we have:

(*i*) *Every complete lattice is an injective object* (in the category of ordered sets).

En effect suppose that in the diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & \\ Q & & \end{array}$$

α be a monomorphism (so that, for any $a'_1, a'_2 \in A'$, $a'_1 \leq a'_2$ iff $\alpha(a'_1) \leq \alpha(a'_2)$ holds) and Q be a complete lattice. Then it is easily seen that the equation

$$\bar{f}(a) = \sup_{\substack{a' \in A' \\ \alpha(a') \leq a}} f(a') \qquad (a \in A)$$

defines an increasing function $\bar{f} : A \to Q$ making commutative the diagram above.

(*ii*) *Every injective object is a complete lattice* (in the category of ordered sets).

En effect, if Q is an injective object, since $\varphi_Q : Q \to \widehat{Q}$ is a monomorphism (see statement (*b*) above), the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi_Q} & \widehat{Q} \\ {\scriptstyle id_Q}\downarrow & & \\ Q & & \end{array}$$

can be filled in commutatively with a morphism $g : \widehat{Q} \to Q$. Now it is easily seen that, for any subset X of Q, $g(\bigcup_{x \in X} \varphi_Q(x))$ and $g(\bigcap_{x \in X} \varphi_Q(x))$ are respectively the least

[18]) A lower segment of an ordered set E is a subset Σ of E, which the following property

$$(\forall x)(\forall y)(x \in \Sigma \ \&\ y \in E \ \&\ y \leq x \Rightarrow y \in \Sigma)$$

holds for.

upper bound and the greatest lower bound of X in Q. Hence Q is a complete lattice.

(*iii*) *The category of ordered sets has enough injectives.*

This is an obvious consequence of (*a*), (*b*) and (*i*).

Proof of 5.1.2 Let Q be any injective object (in the sense of Mitchell) in the category of ordered sets. It is easily seen that Q cannot be empty. Take therefore $x_1, x_2 \in Q$ and let A′, A_1, A_2 be ordered sets having $\{x_1, x_2\}$ as support and ordered respectively by: equality; $x_1 \leq x_2$; $x_2 \leq x_1$. Let at last $A' \to A_1$, $A' \to A_2$, $A' \to Q$ be the canonical injections. Then with reference to the diagrams

$$\begin{array}{ccc} A' & \to & A_1 \\ \downarrow & & \\ Q & & \end{array} \qquad\qquad \begin{array}{ccc} A' & \to & A_2 \\ \downarrow & & \\ Q & & \end{array}$$

it is easily seen that both $x_1 \leq x_2$ and $x_2 \leq x_1$ hold in Q. Hence $x_1 = x_2$ and Q has just one element.

Proof of 5.2 Let Q be a retract of an injective Q′ and let $Q \to Q' \to Q$ a retraction diagram (so that the composition is id_Q). If $A' \to A$ is a monomorphism and $A' \to Q$ is any morphism, then $A' \to Q$ (which is equal to the composition $A' \to Q \to Q' \to Q$), by the injectivity of Q′ is equal to the composition $A' \to A \to Q' \to Q$ for some morphism $A \to Q'$. This establishes the injectivity of Q.

Proof of 5.3 If Q is injective, then given any monomorphism $Q \to A$ there is a morphism $A \to Q$ such that $Q \to A \to Q$ is id_Q. In other words Q is an absolute retract.

Conversely suppose that Q be an absolute retract, i.e. that every monomorphism $Q \to A$ be a coretraction. If the category has enough injectives then we may take A injective, so that Q is a retract of an injective and therefore, by 5.2, an injective.

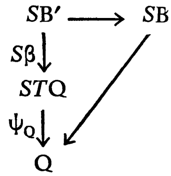
Proof of 5.4 By definition of product, a morphism $X \to Q$ is uniquely determined by a family of morphisms $X \to Q_i$, for each of which $X \to Q_i$ is equal to $X \to Q \to Q_i$. Suppose that Q_i be injective for each *i*, and let $A' \to A$ be a monomorphism. Then any family $A \to Q_i$, such that all the diagrams

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \\ Q & \swarrow & \\ \downarrow & & \\ Q_i & & \end{array}$$

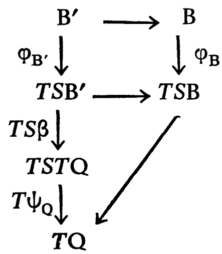
commute, determines a morphism $A \to Q$ with the required property.

The converse follows from 5.2, since (in a category with zero morphisms) injections into products are coretractions.

Proof of 5.5 Given any injective Q in $\mathfrak{A}$, take any monomorphism $B' \to B$ and any morphism $B' \xrightarrow{\beta} TQ$ in $\mathfrak{B}$. Since S preserves monomorphisms there is a morphism $SB \to Q$ in $\mathfrak{A}$ making commutative the diagram



Hence also the diagram



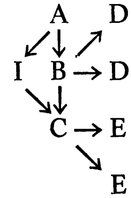
commutes. Applying (C_5*), we see that TQ is injective.

Proof of 5.5.1 It is obvious that an injective in a category $\mathfrak{A}$ is an injective in any full subcategory.

The converse, in our hypothesises, is a special case of 5.5.

8. Appendix.

Consider the category $\mathfrak{A}$ generated by the following commutative diagram



where the composite morphisms from A to D, like those from B to E, like thos from I to E are equal. Then the following statements are only a matter of computation:

(*i*) $\mathcal{A}$ is canonical in the sense of Sonner [8];

(*ii*) A $\rightarrow$ B and B $\rightarrow$ C are extremal monomorphisms (*ibid.*);

(*iii*) the composition A $\rightarrow$ B $\rightarrow$ C is *not* an extremal monomorphism.

REFERENCES

[1] GROTHENDIECK, A.: *Sur quelque points d'algèbre homologique.* Tôhoku Math. J., **9** (1957), 119-221.

[2] HELLER, A.: *Homological Algebra in Abelian categories.* Ann. of Math. II, **68** (1958), 484-525.

[3] HILTON, P.: *Correspondences and exact squares.* Proc. of the Conf. on Cat. Alg. La Jolla, 1965, 254-271.

[4] ISBELL, J. R.: *Some remarks concerning categories and subspaces.* Can. J. of Math. **9** (1957), 563-577.

[5] KUROŠ, A. G.: *Direct decomposition in algebraic categories.* Trudy Moskov Mat. Obšč. **8** (1959), 391-412; Am. Math. Soc. Transl. (2) **27** (1963), 231-255.

[6] KUROŠ, A. G., LIVŠIC, A. H., ŠULGEĬFER E. G.: *Foundations of the theory of categories.* Uspehi Mat. Nauk **15** (1960), No. 6, 3-52; Russian Math. Surveys **15** (1960), 1-45.

[7] MITCHELL, B.: *Theory of Categories.* Academic Press, New York and London, 1965.

[8] SONNER, J.: *Canonical categories.* Proc. of the Conf. on Cat. Alg. La Jolla, 1965, 272-294.

[9] WYLER, O.: *Operational categories.* Proc. of the Conf. on Cat. Alg. La Jolla, 1965, 295-316.

[10] WYLER, O.: *Weakly Exact Categories.* Arch. der Math. **17** (1966), 9-19. (1966), 9-19.

Manoscritto pervenuto in redazione il 2 settembre 1968.