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ON THE OPERATIONAL REPRESENTATION OF SOME HYPERGEOMETRIC POLYNOMIALS

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1. The object of the present paper is to obtain operational representation of some hypergeometric polynomials. We incidentally find connections between different types of classical orthogonal polynomials. We have used here the method of p-multiplied Laplace transform. The notations and formulae used are given below.

A function $f(p)$ is said to be the operational representation of a given function $h(x)$ if

(1.1) $$f(p) = p \int_0^\infty e^{-px} h(x)\, dx, p > 0,$$

provided the integral converges. The relation between the *original* $h(x)$ and its *image* $f(p)$ is denoted by

(1.2) $$h(x) \doteq\!\!\!> f(p) \text{ or } f(p) <\!\!\!\doteq h(x)$$

(1.3) $$x^n \doteq\!\!\!> \frac{\Gamma(n+1)}{p^n}.$$

2. Let $M_n(x)$ denote the polynomial

(2.1) $${}_{p+1}F_q \left[\begin{matrix} -n, a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} \; x \right]$$

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and $G_n(x)$ denote the Polynomial

$$(2.2) \qquad {}_{p+2}F_q \left[\begin{matrix} -n, a+n, a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix} \; \frac{1-x}{2} \right]$$

Now,

$$x^{n+a-1} M_n(x) = \sum_{r=0}^{n} \frac{(-n)_r (a_1)_r \dots (a_p)_r}{r! \, (b_1)_r \dots (b_q)_r} x^{n+a+r-1}$$

Finding the operational representation of the right side term by term,

Left side

$$\doteqdot \sum_{r=0}^{n} \frac{(-n)_r (a_1)_r \dots (a_p)_r}{r! \, (b_1)_r \dots (b_q)_r} \frac{\Gamma(a+n+r)}{p^{n+a+r-1}}$$

$$= \frac{\Gamma(n+a)}{p^{n+a-1}} \sum_{r=0}^{n} \frac{(-n)_r (a_1)_r \dots (a_p)_r}{r! \, (b_1)_r \dots (b_q)_r} \frac{(a+n)_r}{p^r}$$

$$= \frac{\Gamma(n+a)}{p^{n+a-1}} G_n\left(1 - \frac{2}{p}\right).$$

Therefore,

$$(2.3) \qquad x^{n+a-1} M_n(x) \doteqdot \frac{\Gamma(n+a)}{p^{n+a-1}} G_n\left(1 - \frac{2}{p}\right)$$

Particular cases:

(i) Putting $p = 0, q = 1, b_1 = 1 + \alpha, a = 1 + \alpha + \beta$ in (2.3) we get

$$(2.4) \qquad x^{n+\alpha+\beta} L_n^{(\alpha)}(x) \doteqdot \frac{\Gamma(n+\alpha+\beta+1)}{p^{n+\alpha+\beta}} P_n^{(\alpha,\beta)}\left(1 - \frac{2}{p}\right)$$

where $L_n^{(\alpha)}(x)$ is the generalised Laguerre polynomial and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [[5] p. 62, p. 102.]

(ii) Putting $\alpha = \beta = \lambda - \frac{1}{2}$ in (2.4) and using the definition of $P_n^{\lambda}(x)$ [[5], p. 81].

$$(2.5) \qquad x^{n+2\lambda-1} L_n^{\left(\lambda-\frac{1}{2}\right)}(x) \doteqdot \frac{\Gamma(2\lambda)\,\Gamma\left(n+\lambda+\frac{1}{2}\right)}{\Gamma\left(\lambda+\frac{1}{2}\right) p^{n+2\lambda-1}} P_n^{\lambda}\left(1 - \frac{2}{p}\right)$$

(iii) Putting $\alpha = \beta = 0$ in (2.4) we get

$$x^n L_n(x) \doteqdot \frac{n!}{p^n} P_n\left(1 - \frac{2}{p}\right) \tag{2.6}$$

where $P_n(x)$ is the Legendre polynomial. This is a known operational representation [[2], p. 56].

(iv) Putting $\alpha = \beta = \frac{1}{2}$ in (2.4) we get

$$x^{n+\frac{1}{2}} H_{2n+1}(\sqrt{x}) \doteqdot (-1)^n \frac{2\,(2n+1)\,!}{p^{n+1}} U_n\left(1 - \frac{2}{p}\right) \tag{2.7}$$

and putting $\alpha = \beta = -\frac{1}{2}$, we get

$$x^{n-1} H_{2n}(\sqrt{x}) \doteqdot (-1)^n \frac{(2n)\,!}{np^{n-1}} T_n\left(1 - \frac{2}{p}\right) \tag{2.8}$$

where $H_n(x)$ is the Hermite polynomial [[5], p. 105] and $T_n(x)$ and $U_n(x)$ are the Tchebichef polynomials of the first and second kind [[5], p. 60].

3. We have

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right) \tag{3.1}$$

Putting $\frac{x+1}{x-1} = t$ and multiplying both sides by t^β

$$t^\beta (t-1)^n P_n^{(\alpha,\beta)}\left(\frac{t+1}{t-1}\right) = \binom{n+\alpha}{n} \sum_{r=0}^{n} \frac{(-n)_r (-n-\beta)_r}{r\,!\,(\alpha+1)_r} t^{n+\beta-r}$$

Since

$$(-n-\beta)_r = \frac{(-1)^r \Gamma(n+\beta+1)}{\Gamma(n+\beta-r+1)},$$

the right side

$$\doteqdot \frac{\Gamma(n+\beta+1)}{p^{n+\beta}} \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; -p),$$

$$t^\beta (t-1)^n P_n^{(\alpha,\beta)}\left(\frac{t+1}{t-1}\right) \doteqdot \frac{\Gamma(n+\beta+1)}{p^{n+\beta}} L_n^{(\alpha)}(-p) \tag{3.2}$$

By giving different values to α, β we get Operational representations of different kinds of polynomials.

When $\alpha = \beta = 0$, we get the known operational representation [[3], p. 52].

$$(3.3) \qquad (t-1)^n P_n\left(\frac{t+1}{t-1}\right) \doteq \frac{n!}{p^n} L_n(-p)$$

4. We have

$$(4.1) \quad P_n^\lambda(x) = \sum_{r=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^r \Gamma(n-r+\lambda)}{r!\,\Gamma(\lambda)\,\Gamma(n-2r+1)} (2x)^{n-2r} \ldots \left[\lambda > -\frac{1}{2}\right].$$

Putting $x = \frac{1}{\sqrt{p}}$ and multiplying by $\frac{1}{p^{\lambda+\frac{1}{2}n-1}}$, we get

$$\frac{\Gamma(\lambda) P_n^\lambda\left(\frac{1}{\sqrt{p}}\right)}{p^{\lambda+\frac{1}{2}n-1}} = \sum_{r=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^r}{r!} \frac{2^{n-2r}}{(n-2r)!} \frac{\Gamma(n-r+\lambda)}{p^{n-r+\lambda-1}}$$

the right side

$$\doteq \sum_{r=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^r}{r!} \frac{2^{n-2r}}{(n-2r)!} x^{n-r+\lambda-1}$$

$$= x^{\frac{1}{2}n+\lambda-1} \sum_{r=0}^{\left[\frac{1}{2}n\right]} \frac{(-1)^r}{r!} \frac{(2\sqrt{x})^{n-2r}}{(n-2r)!}$$

$$= x^{\frac{1}{2}n+\lambda-1} \frac{H_n(\sqrt{x})}{n!}$$

Therefore,

$$(4.2) \qquad \frac{x^{\frac{1}{2}n+\lambda-1}}{n!} H_n(\sqrt{x}) \doteq \frac{\Gamma(\lambda) P_n^\lambda\left(\frac{1}{\sqrt{p}}\right)}{p^{\lambda+\frac{1}{2}n-1}}$$

By giving different values to λ we get different operational representations.

5.

(5.1) $$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

Therefore,

$$P_n\left(1 - \frac{2\lambda^2}{p^2}\right) = \sum_{r=0}^{n} \frac{(-n)_r (n+1)_r}{r!\,(1)_r} \frac{\lambda^{2r}}{p^{2r}}$$

Since

$$(2r)! = 2^{2r}(1)_r\left(\frac{1}{2}\right)_r,$$

the right side

$$\doteq {}_2F_3\left(-n, n+1; 1, 1, \frac{1}{2}; \frac{1}{4}\lambda^2 x^2\right).$$

We know [[1], p. 186]

(5.2) $${}_1F_1(a; \varrho; z)\,{}_1F_1(a; \varrho; -z) =$$

$$= {}_2F_3\left[a, \varrho - a; \varrho, \frac{1}{2}\varrho, \frac{1}{2}(\varrho+1); \frac{1}{4}z^2\right]$$

Putting $a = -n$, $\varrho = 1$, $z = \lambda x$, we get

(5.3) $$L_n(\lambda x)\, L_n(-\lambda x) =$$

$$= {}_2F_3\left[-n, n+1; 1, 1, \frac{1}{2}; \frac{1}{4}\lambda^2 x^2\right] \doteq P_n\left(1 - \frac{2\lambda^2}{p^2}\right)$$

This was obtained by Shabde [4] by a different method.

6. From (2.4) and (4.2) we get

(6.1) $$\int_0^\infty e^{-xy}\, x^{n+\alpha+\beta}\, L_n^{(\alpha)}(x)\, dx =$$

$$= \frac{\Gamma(n+\alpha+\beta+1)}{y^{n+\alpha+\beta+1}} P_n^{(\alpha,\beta)}\left(1 - \frac{2}{y}\right), \; y > 0$$

$$(6.2) \qquad \int_0^\infty e^{-ax}\, x^{\frac{1}{2}n+\lambda-1}\, H_n(\sqrt{x})\, dx =$$

$$\frac{\Gamma(\lambda)\,\Gamma(n+1)}{a^{\lambda+\frac{1}{2}n+1}}\, P_n^{\lambda}\left(\frac{1}{\sqrt{a}}\right),\; a > 0$$

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