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ON IRREGULAR VARIETIES WHICH CONTAIN CYCLIC INVOLUTIONS

Nota (*) *di* LEONARD ROTH (*a Londra*)

1. *Introduction.* - The present note generalises some familiar results of De Franchis and Comessatti concerning irregular multiple planes. In the first place, a classical theorem of De Franchis [4, 5] states that, on any double plane of irregularity $q > 0$, the branch curve is reducible, consisting of a number of curves belonging to a pencil; it follows from this that any surface V_2 which is a simple model of the double plane must contain a pencil, of genus q, of curves.

The theorem in question is established by computing the simple integrals of the first kind attached to V_2. Actually, it is the second of the above results which is significant, for it means that V_2 cannot possess a proper model V_2^* on its Picard-Severi variety V_q (see [9]). Thus we may conclude that the existence on V_2 of a *rational* involution I_2 of order 2 implies the non-existence of V_2^* and hence, by a theorem of Severi [10], that V_2 contains a pencil of genus q. From this the result concerning the branch curve can be deduced.

Now it appears that the De Franchis theorem is merely a special case of a proposition about superficially irregular algebraic varieties V_r of any dimension $r \geqslant 2$ which carry superficially *regular* involutions I_2. Denoting by $g_k(I_2)$ the number of linearly independent differential forms of the first kind and of degree $k (k = 1, 2, \dots, r)$ attached to the image variety of I_2, we show that:

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If a variety $V_r (r \geqslant 2)$ of superficial irregularity $q > 0$ carries an involution I_2 such that $g_1(I_2) = g_2(I_2) = 0$, then V_r must contain a congruence of subvarieties (of some dimension $\geqslant 1$), the congruence having superficial irregularity q.

Moreover, we readily see that the coincidence locus of I_2 always belongs to the congruence in question.

We show further that this result is itself a special case of the following theorem:

If a variety $V_r (r \geqslant 2)$ of superficial irregularity $q > 0$ carries an involution I_2 such that $g_1(I_2) = 0$, while for any even value of $k (\leqslant r)$, $g_k(I_2) < \binom{q}{k}$, then V_r must contain an irregular congruence (of superficial irregularity $\leqslant$ q).

The limitation $q \geqslant r$ is required for the method of proof; the restriction is, however, inessential, since whenever $q < r$, V_r must contain some congruence of superficial irregularity q ([10]).

We then indicate how the same methods can be applied to the case where V_r carries a superficially regular cyclic involution I_m of any order $m \geqslant 3$. Here, however, the results are less precise, since — in constrast with the case $m = 2$ — there are now various types of associated involution of order m on the Picardian V_q of V_r. Moreover, unlike the case $m = 2$, we have now always to deal with singular transformations of V_q, and these necessarily give rise to problems of existence. Examples of irregular cyclic planes quoted by Comessatti [2] demonstrate that general results analogous to those obtained in the case $m = 2$ cannot be established.

Finally we remark that the previous considerations may be extended to the case where the involution I_m is non cyclic, provided that the associated involution on V_q is generable by a (finite) group of automorphisms of V_q; the results then obtained are exactly similar to those mentioned above.

2. Generalities. - Consider a non-singular algebraic variety $V_r (r \geqslant 2)$ having superficial irregularity $q \geqslant r$; in all that follows the case $q < r$ can be set aside since we know that V_r will then contain a congruence of superficial irregularity q ([10]). Assuming that V_r does not contain such a congruence we may obtain for V_r a model (simple or multiple) of dimension r on the second Picardian or Picard-Severi variety V_q constructed with the period matrix associated with the linearly independent simple integrals $u_i (i = 1, 2, \ldots, q)$ of the first kind attached to V_r (see [1, 9, 13]). Denoting by x a point current on V_r, we write $u_i(x) = u_i$, and take u_i for coordinates on V_q; then the locus of the corresponding point (u) on V_q is an irreducible algebraic variety V_r^*. This will be a simple model of V_r if and only if the congruences

$$u_i(x) \equiv u_i(y) \qquad \text{(mod. periods)} \tag{1}$$

where x, y are points of V_r, in general admit a single solution. If instead for arbitrary x, the equations (1) admit $\nu (> 1)$ solutions, it follows that V_r^* is a ν-ple model of V_r: to a point of V_r^* there then corresponds a set of ν distinct points on V_r, belonging to the *fundamental involution* I_ν. In either case we shall assume that V_r^* is non-singular; such a hypothesis may possibly be restrictive.

We observe that a *necessary and sufficient condition* for the existence of V_r^* on V_q is that V_r *should not contain any congruence of superficial irregularity* q ([10]).

Suppose now that V_r carries an involution I of order 2; this generates an automorphism between points P, P' of V_r, under which all the integrals u_i must be invariant. Hence we have a transformation from $u_i(P)$ to $u_i'(P')$ of the form

$$u_i' = \sum_{j=1}^{q} \lambda_{ij} u_j + \mu_i \qquad (i = 1, 2, \ldots, q) \tag{2}$$

where λ_{ij}, μ_i are constants.

Evidently the transformation (2), applied to V_r^*, is subordinate to a transformation of the entire variety V_q which generates an involution J, likewise of order 2, on V_q. From the theory of Picard varieties it is known ([7]) that J can be represented by the canonical form

$$u_i = u_i + a_i \qquad (i = 1, 2, \ldots, p;\ p \geqslant 0) \tag{3$_1$}$$

$$u_j = -u_j \quad (j = p+1,\ p+2, \ldots, q) \tag{3$_2$}$$

where the a_i are constants (possibly zero) and where p is the superficial irregularity of J. We note that if V_q has general moduli, there are just two possibilities: either $p=0$ or $p=q$. In all other cases we have a singular transformation of V_q which can exist only for particular values of the moduli of V_q ([3]).

It is clear, by comparison of equations (1) and (3), that *if* $\nu > 1$, *I cannot belong to the fundamental involution* I_ν.

In the case $\nu=1$, the sets of I are in birational correspondence with the sets of an involution I^* of order 2 on V_r^*, which is subordinate to J. When $\nu > 1$, we have instead that *I is mapped on a* ν*-fold involution* I^* (*likewise of order* 2) *on* V_r^*, *which is subordinate to* J; this follows by comparing equations (1) and (3). In particular, when $q=r$, V_r^* coincides with V_q and I^* with J.

3. On the characters g_k, q_k. - We denote by $g_k (k=1, 2, \dots, r)$ the number of linearly independent differential forms of the first kind and of degree k which are attached to V_r. Here g_r is the geometric genus $P_g(V_r)$, while g_1 is the superficial irregularity q of V_r. The arithmetic genus $P_a(V_r)$ is then given by the Severi-Kodaira relation ([11])

$$P_a = g_r - g_{r-1} + \dots + (-1)^r g_1 . \tag{4}$$

Defining the r-dimensional irregularity q_r as the difference $P_g - P_a$, we then introduce the set of k-dimensional irregularities $q_k (k=2, 3, \dots, r-1)$ by taking appropriate linear sections of V_r and applying (4) to each in turn. We thus obtain the relations ([11, 13]):

$$\begin{cases} g_k = q_k + q_{k+1} & (k=2, 3, \dots, r-1) \\ g_1 = q_2 . \end{cases} \tag{5}$$

We say that V_r is *completely regular* if and only if $q_k = 0$ $(k=2. 3, \dots, r-1)$. Clearly a necessary and sufficient condition for the complete regularity of V_r is $g_s=0$ $(s=1, 2, \dots, r-1)$.

Suppose now that V_r is mapped on a multiple non-singular variety V_r'; in that case we have the inequalities

$$(6) \qquad g_k(V_r) \geqslant g_k(V_r') \qquad (k = 1, 2, \dots, r).$$

It follows from (5) and (6) that, *if V_r is completely regular, then so also is V_r'*. For, if for some $s (1 \leqslant s \leqslant r-1)$ we had $g_s(V_r') > 0$, then we should have $g_s(V_r) > 0$, whence V_r could not be completely regular.

One last preliminary remark: suppose that V_r contains a congruence Γ of some positive superficial irregularity $(\leqslant q_2)$; then Γ will be mapped by a congruence Γ' of subvarieties on V_r'. Now, in the case where V_r' is *superficially regular*, Γ' will perforce be superficially regular: this means that Γ' cannot correspond birationally, element for element, to Γ. Applying this result to the case we have to consider, let V_r' denote a birational image of the involution I on V_r; if I is *superficially regular*, we deduce that to a member of Γ' there will correspond *two* members of Γ, in general distinct. Moreover, the coincidence locus of I must belong to Γ, and the branch locus on V_r' must belong to Γ'.

4. On the Wirtinger involution. - Returning to the Picard variety V_q, we consider the case where the involution J is *superficially regular*; the involution, represented by equations (3_2), then has for image a generalised Wirtinger variety[1]) (in the case $r=2$, a generalised Kummer surface) which we shall denote by W_q.

Now every differential form of the first kind and of degree k attached to W_q must arise from an analogous form attached to V_q; and it is known ([12]) that every such form is given by an expression of the type $du_1 du_2 \dots du_k$. Evidently this furnishes a corresponding differential form on W_q if and only if it is invariant under the transformation (3_2). We thus

(1) The name of Wirtinger variety is usually restricted to the case where V_q has all its divisors equal to unity; we may call this the ordinary Wirtinger variety (for $r=2$, the ordinary Kummer surface).

obtain the results

$$(7)\qquad \begin{cases} g_k(W_q) = \binom{q}{k} & (k \text{ even}) \\ g_k(W_q) = 0 & (k \text{ odd}). \end{cases}$$

In these formulae, k takes the values 1, 2, ..., r. It now follows from (4) that the arithmetic genus P_a of W_q is given by

$$P_a(W_q) = (-1)^q(2^{q-1} - 1).$$

This result was obtained by Gröbner [3] for the ordinary Wirtinger variety by computing the Hilbert characteristic function for the manifold in question and then applying the Severi postulation formula.

5. First applications. - With the notation of n. 2, suppose that the involution I carried by V_r has superficial irregularity $p(0 < p \leqslant q)$; this means that precisely p linearly independent differential forms of the first kind and first degree — say $du_1, du_2, \ldots, du_p$ — take the same values at corresponding points P, P' of I. The equations (2) for I assume the form (3).

If V_q has general moduli, and thus admits only ordinary transformations, we must have $p = q$. If instead $p < q$, we have a singular transformation; evidently the involution J is now pseudo- Abelian of type p ([8]). Hence V_r^* must contain a superficially irregular congruence, and thus so also must V_r. Whence the result: *If I has superficial irregularity $p(0 < p < q)$, V_r must contain a superficially irregular congruence.*

Suppose next that I is superficially regular and — as usual — that V_r does not contain any congruence of superficial irregularity $q(= g_1)$; in this case the model V_r^* on V_q certainly exists and if $q = r$, coincides with V_q. The equations (1) now take the form (3_2), so that J is a generalised Wirtinger (or Kummer) involution, whose characters $g_k(J)$ are given by (7); in particular, then, we have $g_2(J) = \binom{q}{2}$.

Now the number $g_2(I^*)$ will equal $g_2(J)$ provided that V_r^* does not contain a superficially irregular congruence, for in that case none of the differential forms $du_i du_j$ can vanish identically on V_r^* ([12]). In any event, since we know that V_r^* certainly does not contain any congruence of superficial irregularity q (n. 2), not all the integrals u_i attached to V_r^* can be functions of one integral alone ([12]), and therefore we must have $g_2(I^*) > 0$; hence, whatever the value of ν, it follows that $g_2(I) > 0$, by equations (6). Thus

If V_r carries an involution I of the second order such that $g_1(I) = 0$, $g_2(I) = 0$, then V_r must contain a congruence of superficial irregularity q.

As remarked in n. 3, *the members of the congruence are conjugate in I, and the coincidence locus of I belongs to the congruence.*

In the case where $r \geqslant 3$, it follows from (5) that, if the characters g_1 and g_2 are both zero, then q_2 and q_3 are also zero, and vice-versa. Thus

If $V_r (r \geqslant 3)$ carries an involution I of the second order which is bidimensionally and also tridimensionally regular, then V_r must contain a congruence of superficial irregularity q.

In particular, then, if I is *unirational* or *birational,* V_r must contain a congruence of superficial irregularity q.

6. The double space S_r. - Consider first the case $r = 2$; suppose that V_2 contains a rational involution I, which means that V_2 can be mapped on a double plane S_2 of irregularity q. By the previous theorem, V_2 must contain an irrational pencil, of genus q, and the coincidence locus of I must consist of curves belonging to the pencil. Hence the branch curve consists of a number curves of curves belonging to a pencil in S_2, and the general curve of this pencil maps a pair of curves of V_2; this result is due to De Franchis [4].

Next, let $r = 3$; then the double planes in the corresponding double space S_3 are « generic » surfaces, having irregularity q. Hence, by the previous result, the branch surface in S_3 consists of a number of surfaces of a pencil, from which it follows that V_3 must contain a pencil, of genus q, of surfaces.

Proceeding by induction, we thus obtain the result: *Every double space $S_r (r \geqslant 2)$ of superficial irregularity $q > 0$ contains a pencil, of genus q, of hypersurfaces; and the branch locus in S_r consists of a number of primals belonging to a pencil.*

It is clear that the image of the pencil on V_r is a hyperelliptic curve, since to a member of the (linear) pencil in S_r which maps it there corresponds a pair of hypersurfaces, in general distinct. This type of double S_r has been studied by Gallarati [6], who has calculated the invariants $g_k(V_r)$ in the case where the base of the pencil in S_r is irreducible and non-singular.

7. Extension of previous results.

I. - Let $q = r$; in this case, if $\nu = 1$, the involution I is coincident with the generalised Wirtinger involution J, and its characters $g_k(I)$ are given by (7). If $\nu > 1$, we have $g_k(I) \geqslant g_k(J)$. It follows that, in order that the model $V_r^* (= V_q)$ should exist, the inequalities $g_k(I) \geqslant \binom{q}{k}$ must be satisfied for every even value of k. Hence,

If, when $q = r$, the variety V_r carries a superficially regular involution I of the second order such that, for any even value of k, $g_k(I) < \binom{k}{q}$, then V_r must contain a congruence of superficial irregularity q.

II. - In the case where $q > r$, we can obtain a result which is more general than that of n. 5. Previously we have allowed V_r to contain some irregular congruence (necessarily of superficial irregularity $< q$). Suppose now that V_r *contains no superficially irregular congruence whatever;* this entails that V_r^* also can contain no such congruence. On this hypothesis the differential forms of the first kind of any degree $k \leqslant r$ attached to J must give rise to precisely the same number of differential forms of the first kind and of like degree attached to I^*. We thus have, for every even value of $k \leqslant r$,

$$g_k(I^*) = \binom{q}{k}, \qquad \text{whence} \qquad g_k(I) \geqslant \binom{q}{k}.$$

Therefore, *if V_r carries a superficially regular involution I of the second order such that, for any even value of $k(\leqslant r)$, $g_k(I) < \binom{q}{k}$, then V_r must contain an irregular congruence (of superficial irregularity $\leqslant q$).*

As remarked, in n. 3, the members of this congruence must be conjugate in I, so that the coincidence locus of I belongs to the congruence in question.

8. Notes and examples. - We add a few comments upon the preceding results.

In the first place we remark that the conditions of n. 5 are not necessary in order that V_r should contain a congruence of superficial irregularity q. Thus, consider a product variety $V_r = V_t \times V_{r-t}$ $(t \geqslant 1)$, where V_t is the simple model of a double space S_t; in particular, when $t = 1$, V_t is a hyperelliptic curve. In this case V_r carries an involution I which is mapped by the product $S_t \times V_{r-t}$; hence, if we assume that $g_1(V_{r-t}) = 0$, we shall have $g_1(I) = 0$. Now in this case, $g_2(I) = g_2(V_{r-t})$, from which it follows that the character $g_2(I)$ can have any non-negative value whatever. Evidently the variety V_r contains a congruence of varieties V_{r-t}, which is mapped by the points of V_t, and which has maximum superficial irregularity $g_1(V_t) = q$.

Returning to the general case we observe that, from the correspondence between V_r and I we have (n. 3), for every $k(1 \leqslant k \leqslant r)$, $g_k(I) \leqslant g_k(V_r)$. For the particular double spaces S_r considered by Gallarati [6], we have $g_k(V_r) = 0$ $(k = 2, 3, \dots, r-1)$. This suggests an interesting problem: what are the most general conditions of validity for this last result?

In the second place, since for any birational involution I on V_r we have $g_k(I) = 0$ (all k), it follows that, in the previous example, $g_k(I) = g_k(V_r)$ $(k = 2, 3, \dots, r-1)$. This suggests another problem: under what conditions can we assert that this set of relations will hold? An analogous question can of course be raised for any involution, superficially regular or not, carried by a given variety V_r; but the answer is unknown even in the relatively simple case just considered, at any rate for a variety V_r of general character.

A certain amount is, however, known concerning involutions on a Picard variety V_q ([9]). Thus, for an involution I of any order on V_q, the sole condition $g_1(I) = q$ ensures that the image of I should also be a Picard variety. But the effect of other analogous conditions on the nature of I has not yet been investigated. The cyclic involutions — to which we now turn — on V_q have been studied by Lefschetz [7].

9. The general cyclic involution $I_m (m \geqslant 3)$. - Consider next the case where V_r carries a cyclic involution I_m of any order $m \geqslant 3$; such an involution is generated by an automorphism of V_r to which the remarks made in n. 2 apply. We have now a system of equations analogous to (3), which are of the form

$$\text{(8)} \qquad \begin{cases} u'_i = u_i + a_i & (i = 1, 2, \dots, p;\ p \geqslant 0) \\ u'_j = \varepsilon_j u_j & (j = p+1, p+2, \dots, q) \end{cases}$$

where p is the superficial irregularity of the associated involution J on the Picard - Severi variety V_q, which certainly exists provided V_r contains no congruence of superficial irregularity q; and where ε_j denotes an m th root of unity, other than unity itself ([7]).

Precisely as in n. 5 we see that: *if I_m has superficial irregularity $p (0 < p < q)$, then V_r must contain a superficially irregular congruence.* Supposing instead that $g_1(I_m) = 0$, we have $g_1(I^*_m) = 0$, in which case $p = 0$ in equations (8).

On this hypothesis, we may proceed to calculate the characters $g_k(J)$, for $k = 1, 2, \dots, r$. To begin with, we have $g_1(J) = 0$. Next, $g_2(J)$ is equal to the number of products

$\varepsilon_k \varepsilon_l$, where ε_k, ε_l are *different* numbers $(k \neq l)$ occurring in (8), which are equal to unity. And similarly for the remaining characters $g_k(J)$.

An essential difference between the present case and the preceding is that, while for $m = 2$, $p = 0$, we have an ordinary transformation of V_q, for $m > 2$, $p = 0$, we always have a singular transformation. Such transformations can exist only on varieties V_q with particular moduli ([3]); and in every case which is *a priori* possible it must be shown that the corresponding V_q can effectively be constructed. Moreover, since there is now a number of different involutions J for any given value of *m*, the results are necessarily less precise. We have the following analogue of the previous theorems:

If V_r *carries a cyclic involution* $I_m (m \geqslant 3)$ *such that* $g_k(I_m) = 0$ *(all k), then either there exists an associated involution J on* V_q *for which* $g_k(J) = 0$ *(all k), or else* V_r *contains a superficially irregular congruence.*

The proof is exactly as before. It should be noted that, in the case where the above-mentioned involution J actually exists, no general conclusion can be drawn. Thus Comessatti [2], in his classification of the irregular cyclic triple planes $(r = 2,\ m = 3)$ has shown that all such surfaces contain irrational pencils, though not necessarily of genus q; this had been previously noticed by Bagnera and De Franchis in their study of the hyperelliptic surfaces. Comessatti also quotes an example of an irregular quintuple plane $(r = 2,\ m = 5)$ which contains no irrational pencil whatever.

In conclusion, we point out that the previous methods will apply also to the case where, instead of the cyclic involution *J*, we have on V_q any superficially regular involution $J_m (m \geqslant 3)$, provided always that it is generable by a group $\mathcal{G}$ (of order *m*) of automorphisms of V_q. It is known ([9]) that a sufficient, but not a necessary, condition for J_m to be so generable is that the image variety of J_m should have some positive plurigenus. When the group $\mathcal{G}$ exists, it may be represented analytically by a number of sets of equations such as (8); in that case the characters $g_k(J_m)$ may be calculated from these equations, and we may then deduce results similar to the preceding.

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