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NOTE ON A CERTAIN CREMONA TRANSFORMATION ASSOCIATED WITH A PLANE TRIANGLE

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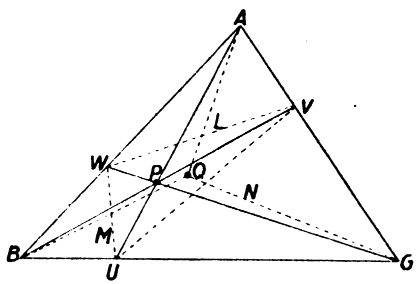
Introduction. – This short paper deals with a certain Cremona transformation, related to a given plane triangle and bearing an intrinsic geometrical significance in Affine Geometry. I am not aware whether this particular type of Cremona transformation has received much attention from previous writers.

1. – Suppose that P is an *arbitrary* point [1], lying within or without a given plane $\Delta\, ABC$ and that the three lines AP, BP, CP cut BC, CA, AB respectively at U, V, W and that L, M, N denote the middle points of VW, WU, UV. (See annexed figure).



If, then, the *areal* coordinates of the point P, referred to the $\Delta\, ABC$, be (α, β, γ), those of the points U, V, W, L, M, N can be easily shewn to be [2]:

$$U(0,\ k_1\beta,\ k_1\gamma)\ ,\qquad V(k_2\alpha,\ 0,\ k_2\gamma)\ ,\qquad W(k_3\alpha,\ k_3\beta,\ 0)$$

(*) Pervenuta in Redazione il 7 febbraio 1950.

[1] Needless to say, the usual conventions regarding the *algebraic* signs of the (areal) coordinates must be observed, no matter the point (P) is inside or outside the triangle (ABC).

[2] See Askwith's « *Analytical Geometry of the Conic Sections* » (1935). P. 277, Art. 262.

$$L\left(\frac{(k_2+k_3)\alpha}{2}, \frac{k_3\beta}{2}, \frac{k_2\gamma}{2}\right), \quad M\left(\frac{k_3\alpha}{2}, \frac{(k_3+k_1)\beta}{2}, \frac{k_1\gamma}{2}\right)$$

$$\text{and} \quad N\left(\frac{k_2\alpha}{2}, \frac{k_1\beta}{2}, \frac{(k_1+k_2)\gamma}{2}\right),$$

where $\quad k_1 \equiv \dfrac{1}{\beta+\gamma}, \quad k_2 \equiv \dfrac{1}{\gamma+\alpha} \quad$ and $\quad k_3 \equiv \dfrac{1}{\alpha+\beta}$.

Consequently the *areal* equations of the three lines AL, BM, CN are respectively:

$$\frac{y}{\dfrac{\beta}{k_2}} = \frac{z}{\dfrac{\gamma}{k_3}}, \qquad \frac{z}{\dfrac{\gamma}{k_3}} = \frac{x}{\dfrac{\alpha}{k_1}} \quad \text{and} \quad \frac{x}{\dfrac{\alpha}{k_1}} = \frac{y}{\dfrac{\beta}{k_2}},$$

shewing that AL, BM, CN are concurrent lines and that the coordinates $(\alpha', \beta', \gamma')$ of their point of concurrence (Q) are proportional to:

$$\alpha(\beta+\gamma), \qquad \beta(\gamma+\alpha), \qquad \gamma(\alpha+\beta).$$

So we may write:

$$\text{(I)} \quad \rho\alpha' = \frac{1}{\beta} + \frac{1}{\gamma}, \quad \rho\beta' = \frac{1}{\gamma} + \frac{1}{\alpha}, \text{ and } \rho\gamma' = \frac{1}{\alpha} + \frac{1}{\beta},$$

where ρ is a factor of proportionality.

Manifestly (I) is equivalent to:

$$\text{(II)} \quad \sigma\alpha = \frac{1}{\beta'+\gamma'-\alpha'}, \quad \sigma\beta = \frac{1}{\gamma'+\alpha'-\beta'}, \text{ and } \sigma\gamma = \frac{1}{\alpha'+\beta'-\gamma'},$$

where σ is a factor of proportionality.

The geometrical correspondence between the points P and Q will be characterised in a different manner in the next article.

2. - Reference to the figure of Art 1 reveals the existence of a conic (S), which touches the three sides BC, CA, AB of the triangle of reference at the points U, V, W respectively. There is no difficulty in shewing that this conic S is given by:

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}-\frac{2yz}{\beta\gamma}-\frac{2zx}{\gamma\alpha}-\frac{2xy}{\alpha\beta}=0\,.$$

If $(\alpha_1, \beta_1, \gamma_1)$ be the centre of this conic, its polar, *viz.*

$$\frac{x}{\alpha}\left(\frac{\beta_1}{\beta}+\frac{\gamma_1}{\gamma}-\frac{\alpha_1}{\alpha}\right)+\frac{y}{\beta}\left(\frac{\gamma_1}{\gamma}+\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)$$
$$+\frac{z}{\gamma}\left(\frac{\alpha_1}{\alpha}+\frac{\beta_1}{\beta}-\frac{\gamma_1}{\gamma}\right)=0$$

must be identical with the line at infinity, *viz.*

$$x+y+z=0\,.$$

For this to be possible, the relevant conditions are:

$$\frac{\beta_1}{\beta}+\frac{\gamma_1}{\gamma}-\frac{\alpha_1}{\alpha}=2k\alpha\,,\qquad \frac{\gamma_1}{\gamma}+\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}=2k\beta$$
$$\text{and}\quad \frac{\alpha_1}{\alpha}+\frac{\beta_1}{\beta}-\frac{\gamma_1}{\gamma}=2k\gamma\,,$$

where k is a factor of proportionality.

The last three equations, when solved for α_1, β_1, γ_1, give:

$$\alpha_1 : \beta_1 : \gamma_1 = \alpha(\beta+\gamma) : \beta(\gamma+\alpha) : \gamma(\alpha+\beta)$$
$$= \alpha' : \beta' : \gamma', \qquad \text{by (I) of Art 1.}$$

This proves that the two points $(\alpha', \beta', \gamma')$ and $(\alpha_1, \beta_1, \gamma_1)$ coincide. That is to say, for a given position of the point P, the other point, *viz.* Q $(\alpha', \beta', \gamma')$, defined in Art 1 as being the point of concurrence of the three lines AL, BM, CN, can with equal propriety be defined as the centre of the *unique* conic S, which touches BC, CA, AB at U, V, W respectively.

The same conclusion could also be reached from purely geometrical considerations. For, if we assume at the very start that O is the centre of the conic (S), which touches BC, CA, AB at U, V, W respectively, the figure of Art. 1 shews at once that VW is the chord of the contact of the two tangents that can be drawn to (S) from A. Consequently by a well-known lemma on conics, the line, which joins O to A, must be *conjugate* in direction to VW and must as such bisect it. In other words the line OA goes through L, or rather, the line AL goes through O. For a similar reason BM and CN must each pass through O. Thus the centre O (of S) is virtually the same as the point of concurrence Q of AL, BM, CN.

3. - We shall now make a few general observations on the geometrical kinship that subsists between the points P, Q. For this purpose we shall change the notations and call the two points $P(x, y, z)$ and $Q(x', y', z')$. Thus the birational or Cremona transformations, which convert one of them into the other, can be exhibited in either of the two equivalent forms:

$$\text{(I)} \begin{cases} \rho x' = \dfrac{1}{y} + \dfrac{1}{z}, \\[2ex] \rho y' = \dfrac{1}{z} + \dfrac{1}{x}, \\[2ex] \rho z' = \dfrac{1}{x} + \dfrac{1}{y}, \end{cases} \qquad \text{and} \qquad \text{(II)} \begin{cases} \sigma x = \dfrac{1}{y'+z'-x'}, \\[2ex] \sigma y = \dfrac{1}{z'+x'-y'}, \\[2ex] \sigma z = \dfrac{1}{x'+y'-z'}, \end{cases}$$

where ρ and σ denote factors of proportionality.

If D, E, F be respectively the middle points of the sides BC, CA, AB of the triangle of reference, the *areal* equations of the three right lines EF, FD and DE are easily seen to be:

$$y + z - x = 0 , \qquad z + x - y = 0 \quad \text{and} \quad x + y - z = 0 .$$

By a cursory glance at (I) or (II), one can now readily substantiate the following statements:

(a) that when P moves on an *arbitrary* right line, Q must move on a conic circumscribing the $\Delta\, DEF$;

and

(b) that when Q moves on an *arbitrary* right line, P must move on a conic circumscribing the $\Delta\, ABC$.

Finally, we have to take account of the *united* or *self-corresponding* point of the Cremona transformation. To do this we have simply to put:

$$x' = x \ , \qquad y' = y \ , \qquad z' = z$$

in (I) or (II). As a consequence, we get:

$$x = y = z = x' = y' = z' .$$

The obvious geometrical interpretation is that the *united* point is no else than the centroid G of the $\Delta\, ABC$. There is no difficulty in recognising that the determinate conic (S), of which the centre is the united point, *viz.* G, is designable uniquely as the ellipse of *maximum* area that can be inscribed in the triangle of reference. [Vide WILLIAMSON's: « *Differential Calculus* » (1927), Ex 1, P. 165].

4. – We shall now give a finishing touch to the present investigation by making a passing reference to Affine Geometry. Regard being had to the patent fact that an affine transformation of the unrestricted type conserves, among other things,

(i) the line at infinity

(ii) the middle point of a finite rectilinear segment

and (iii) the centre of a conic,

it appears that the geometrical character of the Cremona correspondence (I) or (II) of Art 3 remains *essentially the same*, when both the points $P(x, y, z)$ and $Q(x', y', z')$ are subjected to the most general type of *affine* transformation. It is scarcely necessary to remark that, when the affine transformation is replaced by the most general type of projective transformation (or, collineation), the essential geometrical features of the inter-relation between P and Q will *not* ordinarily remain invariant. In other words, the Cremona transformation, talked about in the present paper, is of interest in Affine Geometry but *not* in Projective Geometry.