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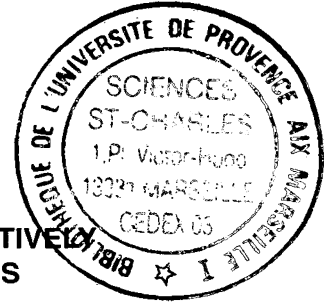
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## DUALITY FOR INCREASING POSITIVELY HOMOGENEOUS FUNCTIONS AND NORMAL SETS (\*) <sup>(1)</sup>

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*Abstract.* – A nonlinear duality operation is defined for the class of increasing positively homogeneous functions defined on the positive orthant (including zero). This class of function and the associated class of normal sets are used extensively in Mathematical Economics. Various examples are provided along with a discussion of duality for a class of optimization problems involving increasing functions and normal sets. © Elsevier, Paris

**Keywords:** Increasing functions, positively homogeneous functions, duality, conjugacy operators, normal sets, mathematical economics.

*Résumé.* – Nous définissons une opération de dualité non linéaire pour la classe des fonctions croissantes homogènes positives définies sur l'orthant positif (zéro inclus). Cette classe de fonctions, tout comme la classe associée d'ensembles normaux, est très souvent utilisée en Économie Mathématique. Nous donnons plusieurs exemples avec une discussion de la dualité pour une classe de problèmes d'optimisation impliquant des fonctions croissantes et des ensembles normaux. © Elsevier, Paris

**Mots clés :** Fonctions croissantes, fonctions homogènes positives, dualité, opérateurs conjugués, ensembles normaux, économie mathématique.

### 1. INTRODUCTION

In this paper we investigate normal subsets of the cone

$$\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n : (\forall i) x_i > 0\} \cup \{0\}$$

and increasing functions defined on this cone. In particular we shall discuss the class of increasing and positively homogeneous functions. These classes of sets and functions often arise in the study of various problems in

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Mathematical Economics [6, 8]. A subset  $\Omega$  of the cone  $\mathbb{R}_{++}^n$  is said to be *normal* if

$$(x \in \Omega, x' \in \mathbb{R}_{++}^n, x' \leq x) \Rightarrow x' \in \Omega. \quad (1)$$

Note that the ordering here is the usual coordinatewise order relation in  $\mathbb{R}^n$ . The property (1) is usually termed *free disposal* in economic theory [6, 8]. This property is often applied to subsets of the cone  $\mathbb{R}_+^n$  of all  $n$ -vectors with nonnegative coordinates. It will be more convenient in this paper to consider  $\mathbb{R}_{++}^n$ , however it is clear that all the results presented here can be easily transferred to the more general setting with natural modifications.

In the study of various models of economic equilibrium and dynamics it is usual to consider convex normal sets (*see* [6, 8, 11]). However there are significant problems which arise in economics for which nonconvex normal sets are required. For example in the study of the asymptotic behaviour of paths for Von Neumann type models of economic dynamics it is necessary to consider specially structured nonconvex normal sets [11].

Increasing positively homogeneous of degree one (*IPH*) functions also play an important role in the study of models in Mathematical Economics. For example they are used as production functions under the assumption of *constant returns to scale*.

There are two approaches to the relationship between normal sets and the class of *IPH* functions. Firstly it is often convenient to study normal sets and *IPH* functions within a very general framework of generalized (or abstract) convexity and Minkowski duality (*see* [5, 12, 1, 2]). The second approach involves the study of *IPH* functions as the Minkowski gauge of appropriate normal sets. In this paper we will unite both approaches in a very simple and elegant way in order to study a special kind of nonlinear duality for *IPH* functions and normal sets. The results obtained will be applied to the study of lower semicontinuous increasing functions defined on the cone  $\mathbb{R}_{++}^n$ . It should be noted the level sets  $\{x \in \mathbb{R}_{++}^n : f(x) \leq c\}$  of such functions are closed and normal (where  $f$  is l.s.c. and increasing on  $\mathbb{R}_{++}^n$  and  $c \in \mathbb{R}$ ).

There is a clear analogy between the class of *IPH* functions and the class of sublinear functions and between normal sets and convex sets. One of the main tools for the study of sublinearity of functions and the convexity of sets are linear functions. For example a function  $p$  defined on a locally convex Hausdorff topological vector space is l.s.c and sublinear if and only if there is

a set of continuous linear functions  $U$  such that  $p(x) = \sup \{v(x) : v \in U\}$ . In this paper we will be interested in the class of functions of the form

$$h(x) = \min \{h_i x_i : i = 1, 2, \dots, n\} \tag{2}$$

to replace the linear functions  $x \mapsto \sum_i h_i x_i$ . Here  $h \in \mathbb{R}_{++}^n$ . The class of linear functions leads to the development of sublinear functions and convex sets whereas (as we shall show) functions of the form (2) lead to the development of *IPH* functions and normal sets. We should note that in this structure increasing functions will play the role of quasiconvex functions in the sense that a function is quasiconvex if and only if it has convex level sets and a function is increasing if and only if it has normal level sets.

Functions of the form (2) were first used in the study of normal subsets of the cone  $\mathbb{R}_+^n$  and *IPH* functions on this cone in [1, 2]. For the case of functions defined on  $\mathbb{R}_{++}^n$  see the paper [12]. We shall require some of the results from this paper here and these results are discussed in the next section.

We consider  $\mathbb{R}_{++}^n$  as a topological space. Thus when we discuss closed subsets of this cone or the closure of subsets we mean closure in the topological space  $\mathbb{R}_{++}^n$ .

**2. PRELIMINARIES**

Let us consider the cone

$$\mathbb{R}_{++}^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : (\forall i) x_i > 0\} \cup \{0\}$$

and the set  $\mathcal{H}$  of all functions  $h$  defined on  $\mathbb{R}_{++}^n$  by the formula:

$$h(x) = \langle h, x \rangle \tag{3}$$

where the coupling functional  $\langle \cdot, \cdot \rangle$  is defined as follows:

$$\langle h, x \rangle = \min \{h_i x_i : i = 1, \dots, n\},$$

with  $h = (h_1, \dots, h_n) \in \mathbb{R}_{++}^n$ . We will identify the vector  $h \in \mathbb{R}_{++}^n$  and the function  $h \in \mathcal{H}$  which is generated by this vector using (3). We can introduce two natural order relations on the set  $\mathcal{H}$ . First:  $h_1 \geq h_2$  if  $h_1(x) \geq h_2(x)$  for all  $x \in \mathbb{R}_{++}^n$  (the functional order relation) and secondly:  $h_1 \geq h_2$  where  $h_1 = (h_1^1, h_2^1, \dots, h_n^1)$ ,  $h_2 = (h_1^2, h_2^2, \dots, h_n^2)$  if  $h_i^1 \geq h_i^2$  for all  $i = 1, \dots, n$  (vectorial order relation). It is straightforward to check that these two relations coincide so we can identify  $\mathcal{H}$  and  $\mathbb{R}_{++}^n$  as ordered sets.

We are now interested in functions  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$  of the form:

$$f(x) = \sup_{h \in U} \langle h, x \rangle \quad (4)$$

where  $U$  is a subset of  $\mathcal{H} = \mathbb{R}_{++}^n$ . Since  $h(x) = \langle h, x \rangle$  is an *IPH* function it follows that a function  $f$  of the form (4) is also *IPH* and  $f \not\equiv +\infty$  (for example  $f(0) = 0$ ). Let us denote the set of all *IPH* functions with the same notation, *IPH*. Thus we have the following definition:

DEFINITION 2.1: A function  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  is an *IPH* function if the following are satisfied:

1.  $x \geq y$  implies  $f(x) \geq f(y)$ .
2.  $f(\lambda x) = \lambda f(x)$  for  $\lambda > 0$ .
3.  $f$  is not identically  $+\infty$ .

From the properties of *IPH* functions it immediately follows that  $f(0) = 0$  for all *IPH* functions.

Essentially the following proposition can be found in [12].

PROPOSITION 2.2 [12]:  $f \in \text{IPH}$  if and only if there is a nonempty subset  $U$  of the cone  $\mathbb{R}_{++}^n$  such that  $f$  has the form (4).

*Remark:* In [12, Proposition 3.1] it was noted that a function  $p$  defined on the cone  $\mathbb{R}_+^n$  can be expressed in the form (4) if and only if  $p$  is *IPH* and vanishes on the boundary of  $\mathbb{R}_+^n$ . The same proof is suitable for the more specific case used in this paper.

DEFINITION 2.3: The subset  $U$  of the cone  $\mathbb{R}_{++}^n = \mathcal{H}$  is called a *support set* if there is a function  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  such that

$$U = \{h : (\forall x \in \mathbb{R}_{++}^n) h(x) \leq f(x)\}.$$

PROPOSITION 2.4 [12]: The subset  $U$  of the cone  $\mathbb{R}_{++}^n$  is a support set if and only if this set is closed (in  $\mathbb{R}_{++}^n$ ) and normal.

### 3. NORMAL SETS AND LEVEL SETS OF IPH FUNCTIONS

In this section we shall show that the level sets of an *IPH* function are normal and that the function can be expressed as the Minkowski gauge of its level set.

First we describe some properties of *IPH* functions. Let  $f$  be an *IPH* function, then:

1.  $f(x) \geq 0$  for all  $x \in \mathbb{R}_{++}^n$ . This is clear since  $f$  is increasing and  $f(0) = 0$ .

2. If  $f \in \text{IPH}$  and there exists a point  $\bar{x} \in \mathbb{R}_{++}^n$  such that  $f(\bar{x}) = +\infty$  then  $f(x) = +\infty$  for all  $x \in \mathbb{R}_{++}^n \setminus \{0\}$ . Indeed if  $x \in \mathbb{R}_{++}^n$  then there is a  $\lambda > 0$  such that  $x \geq \lambda \bar{x}$  therefore  $f(x) \geq f(\lambda \bar{x}) = \lambda f(\bar{x}) = +\infty$ .

3. If there exists a point  $\bar{x} \in \mathbb{R}_{++}^n \setminus \{0\}$  such that  $f(\bar{x}) = 0$  then  $f(x) = 0$  for all  $x \in \mathbb{R}_{++}^n$ . In fact for each  $x \in \mathbb{R}_{++}^n$  there is a  $\lambda > 0$  such that  $x \leq \lambda \bar{x}$ . Hence  $0 \leq f(x) \leq \lambda f(\bar{x}) = 0$ .

Thus for an *IPH* function  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  there are three possibilities:

- (i)  $f$  maps  $\mathbb{R}_{++}^n \setminus \{0\}$  into  $(0, +\infty)$ ,
- (ii)  $f(x) = +\infty$  for all  $x \in \mathbb{R}_{++}^n \setminus \{0\}$ ,
- (iii)  $f(x) = 0$  for all  $x \in \mathbb{R}_{++}^n$ .

4. If  $f \in \text{IPH}$  then  $f$  is continuous on the set  $\mathbb{R}_{++}^n \setminus \{0\}$ .

To see this assume that  $f$  maps  $\mathbb{R}_{++}^n \setminus \{0\}$  into  $(0, +\infty)$ . Let  $x \in \mathbb{R}_{++}^n$ ,  $x \neq 0$  and  $x_n \rightarrow x$ . Take  $\varepsilon > 0$ . For sufficiently large  $n$  we have

$$(1 - \varepsilon)x \leq x_n \leq (1 + \varepsilon)x.$$

Hence

$$(1 - \varepsilon)f(x) \leq f(x_n) \leq (1 + \varepsilon)f(x).$$

Thus  $f(x_n) \rightarrow f(x)$ .

5.  $f$  is l.s.c. at the point  $x = 0$ .

We now recall the definition of a normal set.

DEFINITION 3.1: The subset  $\Omega$  of the cone  $\mathbb{R}_{++}^n$  is called normal if

$$(x \in \Omega, x' \in \mathbb{R}_{++}^n, x' \leq x) \Rightarrow x' \in \Omega.$$

The following property of *IPH* functions will be essential in the sequel.

PROPOSITION 3.2: For an *IPH* function  $f$  the level set  $S_1(f) = \{x \in \mathbb{R}_{++}^n : f(x) \leq 1\}$  is normal and closed (in the topological space  $\mathbb{R}_{++}^n$ ).

*Proof:* Since  $f$  is increasing it follows that  $S_1(f)$  is normal. The closure follows since  $f$  is l.s.c.

Note that for a nonnegative positively homogenous of degree one function  $f$  we have, for all  $c \geq 0$ :

$$S_c(f) = \{x : f(x) \leq c\} = c \cdot S_1(f).$$

Here, by definition, we have  $0 \cdot \Omega = \bigcap_{\lambda > 0} \lambda \Omega$ . So if  $f_1, f_2 \in IPH$  then  $f_1 = f_2$  if and only if  $S_1(f_1) = S_1(f_2)$ .

Let  $U$  be a closed normal subset of the space  $\mathbb{R}_{++}^n$ . Clearly  $U$  is star-shaped with respect to zero, i.e. if  $x \in U$  then  $\lambda x \in U$  for all  $\lambda \in [0, 1]$ . Let  $\mu_U$  be the Minkowski gauge of the set  $U$ . Thus

$$\mu_U(x) = \inf \{\lambda > 0 : x \in \lambda U\} \quad (x \in \mathbb{R}_{++}^n)$$

PROPOSITION 3.3: *Let  $U$  be a closed normal subset of the space  $\mathbb{R}_{++}^n$ . Then  $\mu_U \in IPH$  and  $U = S_1(\mu_U)$ .*

*Proof:* Clearly  $\mu_U$  is positively homogeneous. Since  $U$  is star-shaped and closed it follows easily that  $U = S_1(\mu_U)$ . So we only have to verify that  $\mu_U$  is increasing. Let  $x \leq y$  and  $\mu_U(y) = c$ . Then  $y \in \lambda \Omega$  for all  $\lambda \geq c$ . Hence

$$\mu_U(x) = \inf \{\lambda > 0 : x \in \lambda \Omega\} \leq c = \mu_U(y).$$

Thus the result is established.  $\square$

COROLLARY 3.4: *The mapping  $U \mapsto \mu_U$  is a one-to-one correspondence between the collection of all closed normal sets and the set of all IPH functions.*

DEFINITION 3.5: Let  $\Omega$  be a subset of  $\mathbb{R}_{++}^n$ . The set  $N(\Omega) = \{x \in \mathbb{R}_{++}^n : (\exists x' \in \Omega) x \leq x'\}$  is called the *normal hull* of the set  $\Omega$ .

It is easy to check that the closure (in  $\mathbb{R}_{++}^n$ ) of the normal hull  $N(\Omega)$  is also a normal set.

PROPOSITION 3.6: *If  $\Omega'$  is a closed normal subset of  $\mathbb{R}_{++}^n$  and  $\Omega' \supseteq \Omega$  then  $\Omega' \supseteq \text{cl} N(\Omega)$ .*

*Proof:* It follows immediately from the definition.  $\square$

#### 4. CONJUGATE SETS AND FUNCTIONS

We now introduce the following conjugacy notion both for normal sets and IPH functions.

DEFINITION 4.1: Let  $\Omega$  be a subset of  $\mathbb{R}^n_{++}$ . The set

$$\Omega^* = \{h \in \mathbb{R}^n_{++} : (\forall x \in \Omega) \langle h, x \rangle \leq 1\}$$

is called the *conjugate set with respect to  $\Omega$* .

Clearly we can consider the set  $\Omega^*$  as analogous to the polar set in convex analysis.

We note the following properties of conjugate sets which follow directly from the definition.

1.  $\Omega^*$  is closed and normal.
2.  $\Omega^* = (\text{cl } N(\Omega))^*$ .
3.  $\{0\}^* = \mathbb{R}^n_{++}$ ;  $(\mathbb{R}^n_{++})^* = \{0\}$ .
4. Let  $I$  be an arbitrary index set and  $\Omega_i \subseteq \mathbb{R}^n_{++}$  for each  $i \in I$ . Then  $(\cup_{i \in I} \Omega_i)^* = \cap_{i \in I} \Omega_i^*$ .

We now define the conjugate function to an *IPH* function.

DEFINITION 4.2: Let  $p$  be an *IPH* function defined on  $\mathbb{R}^n_{++}$ . Then the function  $p^*$  defined on  $\mathbb{R}^n_{++}$  as follows

$$p^*(h) = \sup_{x \neq 0} \frac{\langle h, x \rangle}{p(x)} \tag{5}$$

is called the *conjugate function of  $p$* .

Clearly  $p^*$  is analogous to the polar function of a nonnegative sublinear function [10] (in particular it is analogous to the conjugate norm).

Let  $p$  be an *IPH* function and let  $\Omega = S_1(p) = \{x : p(x) \leq 1\}$ . Then

$$p^*(h) = \sup_{x \in \Omega} \langle h, x \rangle.$$

Proposition 2.2 shows that  $p^*$  is an *IPH* function. Since  $p$  is *IPH* it follows that  $\Omega = S_1(p)$  is a closed normal set. Also Proposition 3.2 shows that  $\Omega$  is a support set. Now let us describe the level set  $S_1(p^*)$  of the conjugate function  $p^*$ . We have:

$$\begin{aligned} S_1(p^*) &= \{h \in \mathbb{R}^n_{++} : p^*(h) \leq 1\} \\ &= \{h \in \mathbb{R}^n_{++} : (\forall n \in \Omega) \langle h, x \rangle \leq 1\} = \Omega^*. \end{aligned}$$

Thus the following assertion is valid.



PROPOSITION 4.3: For an IPH function  $p$  the level set  $S_1(p^*)$  of the conjugate function  $p^*$  is the conjugate set of the level set  $S_1(p)$  of the given function  $p$ .

It is possible to give an explicit description of the conjugate function in this case. We will use the following notation, for  $h = (h_1, \dots, h_n) \in \mathbb{R}_{++}^n \setminus \{0\}$  denote by  $\frac{1}{h}$  the vector  $(\frac{1}{h_1}, \dots, \frac{1}{h_n})$ .

THEOREM 4.4: For an IPH function  $p$  we have, for  $h \neq 0$ :

$$p^*(h) = \frac{1}{p(\frac{1}{h})}.$$

*Proof:* Let  $h \in \mathbb{R}_{++}^n$ ,  $h \neq 0$ . We introduce the vectors:

$$\bar{x} = \frac{1}{h}, \quad \bar{h} = p(\bar{x})h.$$

Let  $x \in \mathbb{R}_{++}^n$ . Put  $q(x) = \min_i \frac{x_i}{\bar{x}_i}$ . Further denote by  $i_0$  the index such that  $q(x) = x_{i_0}/\bar{x}_{i_0}$ . Let us consider the vector  $u = (u_1, \dots, u_n)$  with  $u_i = x_i - q(x)\bar{x}_i$ ,  $i = 1, \dots, n$ . We have

$$x = q(x)\bar{x} + u, \quad u \geq 0, \quad u_{i_0} = 0.$$

Since  $p$  is increasing it follows that

$$p(x) \geq p(q(x)\bar{x}) = q(x)p(\bar{x}).$$

Also, by applying the relations  $u_i \geq u_{i_0} = 0$  and  $\bar{x}_i = 1/h_i$  (for all  $i$ ), we have

$$\begin{aligned} \langle \bar{h}, x \rangle &= \min [p(\bar{x})h_i] \cdot [q(x)\bar{x}_i + u_i] \\ &= p(\bar{x}) \min (q(x) + u_i \cdot h_i) \\ &= p(\bar{x})q(x). \end{aligned}$$

Therefore

$$\langle \bar{h}, x \rangle \leq p(x) \quad (\forall x \in \mathbb{R}_{++}^n)$$

It follows from the definition of  $\bar{h}$  and  $\bar{x}$  that  $\langle \bar{h}, \bar{x} \rangle = p(\bar{x})$ .

Recall that  $\bar{h} = p(1/h)h$ . So we have, for all  $x \in \mathbb{R}_{++}^n$ ,

$$\langle h, x \rangle \leq \frac{1}{p(\frac{1}{h})} p(x),$$

moreover

$$\langle h, \bar{x} \rangle = \frac{1}{p\left(\frac{1}{h}\right)} p(\bar{x}).$$

Therefore

$$p^*(h) = \sup_{x \neq 0} \frac{\langle h, x \rangle}{p(x)} = \frac{1}{p\left(\frac{1}{h}\right)} \tag{6}$$

Thus the result follows.  $\square$

*Remark:* Define for  $h = 0$ ,  $p(1/h) = p(+\infty) = +\infty$ . Since  $p^*(h) = 0$  it follows that the formula (6) holds for  $h = 0$  also.

The following corollaries follow immediately from Theorem 4.4.

**COROLLARY 4.5:** *Let  $\Omega$  be a closed normal subset of  $\mathbb{R}_{++}^n$  and let  $p$  be an IPH function such that  $\Omega = S_1(p)$ . Then*

$$\Omega^* = \{h : p^*(h) \leq 1\} = \{h : p(1/h) \geq 1\}.$$

**COROLLARY 4.6:** *For an IPH function  $p$  we have  $p^{**} = p$  and for a closed normal set  $\Omega$  we have  $\Omega^{**} = \Omega$ .*

**COROLLARY 4.7:** *If  $\Omega \subseteq \mathbb{R}_{++}^n$  then  $\Omega^{**} = \text{cl } N(\Omega)$ .*

Let us consider some examples.

*Exemples 4.8:* Consider the following examples of IPH functions and their conjugates.

1. Let  $x \in \mathbb{R}_{++}^n$  and define the following family of IPH functions for  $0 < k < +\infty$ :

$$p_k(x) = \left(\sum_i x_i^k\right)^{1/k}.$$

Clearly for  $k \geq 1$  the function  $p_k$  is the  $\ell_k$  norm on  $\mathbb{R}^n$  and is consequently a continuous sublinear function. For  $0 < k < 1$  the function  $p_k$  is known as a CES function in economic theory (see, for example, [4]). However in this case it is a superlinear function (and hence nonconvex) IPH function. It is straight forward to show that for  $h \in \mathbb{R}_{++}^n \setminus \{0\}$  we have

$$p_k^*(h) = \frac{1}{p_k(1/h)} = \frac{1}{\left(\sum_i \left(\frac{1}{h_i}\right)^k\right)^{1/k}}$$

Consider, in particular, the  $\ell_1$  norm  $p_1$  in which case the conjugate function is

$$p_1^*(h) = \frac{1}{\frac{1}{h_1} + \dots + \frac{1}{h_n}}.$$

Note that if we consider, from simple DC circuit theory, the problem of calculating the total resistance from a set of resistors placed in series or parallel then the function  $p_1$  denotes the total resistance when they are placed in series and  $p_1^*$  denotes the total resistance when they are placed in parallel.

2. Consider the *IPH* function defined as follows:

$$p(x) = C x^{\alpha_1} x^{\alpha_2} \dots x^{\alpha_n}, \quad \sum_i \alpha_i = 1, \quad \alpha_i \geq 0.$$

In this case the conjugate function is defined as follows:

$$p^*(h) = \frac{1}{C \left(\frac{1}{h_1}\right)^{\alpha_1} \dots \left(\frac{1}{h_n}\right)^{\alpha_n}} = \frac{1}{C} h_1^{\alpha_1} \dots h_n^{\alpha_n}.$$

Thus if  $C = 1$  we find  $p = p^*$ . Note that functions such as  $p$  are concave *IPH* functions and they arise as the so-called Cobb-Douglas production functions in economics (*see*, for example, [4]).

3. Consider, following the first example above, the  $\ell_{+\infty}$  norm on  $\mathbb{R}^n$  restricted to the cone  $\mathbb{R}_{++}^n$ :

$$p_{+\infty}(x) = \max_i x_i.$$

Then the conjugate function is given as follows:

$$p^*(h) = \frac{1}{\max_i \frac{1}{h_i}} = \frac{1}{\min_i h_i} = \min h_i.$$

We will now apply Theorem 4.4 to establish the following assertion.

**PROPOSITION 4.9:** *If  $p$  is an *IPH* convex function then  $p^*$  is concave.*

*Proof:* 1) Let  $l = (l_1, \dots, l_n) \in \mathbb{R}_+^n \setminus \{0\}$ . Clearly the function

$$\tilde{l}(x) = \sum_{i=1}^n l_i x_i$$

is an *IPH* function. We have, for  $h \in \mathbb{R}_{++}^n$ ,

$$\tilde{l}^*(h) = \frac{1}{\tilde{l}(\frac{1}{h})} = \begin{cases} \frac{1}{\frac{1}{h_1} + \dots + \frac{1}{h_n}} & h = (h_1, h_2, \dots, h_n) \neq 0 \\ 0 & h = 0 \end{cases}$$

Let us check that  $\tilde{l}^*(h)$  is a concave function. For

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_{++}^n$$

set

$$H_n(x) = \begin{cases} \frac{1}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} & x = (x_1, x_2, \dots, x_n) \neq 0 \\ 0 & x = 0 \end{cases}$$

We first verify that  $H_i$  is concave on  $\mathbb{R}_{++}^n \setminus \{0\}$ . We shall use induction on  $n$ . Calculating the Hessian of  $H_2$  it is easy to check that this function is concave on  $\mathbb{R}_{++}^2 \setminus \{0\}$ . Clearly  $H_2$  is also increasing. So the equality

$$H_{n+1}(x_1, x_2, \dots, x_n, x_{n+1}) = H_2(H_n(x_1, x_2, \dots, x_n), x_{n+1})$$

shows that concavity of  $H_n$  implies concavity of  $H_{n+1}$ . Thus  $H_n$  is concave on  $\mathbb{R}_{++}^n \setminus \{0\}$  for all  $n$ . It is easy to check that this function is concave on  $\mathbb{R}_{++}^n$ . We have for  $h \in \mathbb{R}_{++}^n$

$$\tilde{l}^*(h) = H_\kappa(Ah)$$

where  $Ah = (h_i/l_i)_{i \in I}$ ,  $I = \{i : l_i > 0\}$ ,  $\kappa = |I|$ . Since  $A$  is a linear operator it follows that  $\tilde{l}^*$  is a concave function as required.

2) Now let  $p \in IPH$ ,  $p \neq 0$  and assume that  $p$  is convex. Thus  $p$  is sublinear and increasing. It is well known that for an increasing sublinear function  $f$  the following representation is valid:

$$p(x) = \sup_{l \in \Omega} \tilde{l}(x),$$

where  $\Omega = \partial p(0) \cap \mathbb{R}_+^n$  and  $\partial p(0) = \{l \in \mathbb{R}^n : (\forall x \in \mathbb{R}_+^n) \tilde{l}(x) \leq p(x)\}$  is the usual subdifferential of the function  $p$  at the point  $x = 0$ . We have

$$\begin{aligned} p^*(h) &= \frac{1}{p\left(\frac{1}{h}\right)} \\ &= \frac{1}{\sup_{l \in \Omega} l\left(\frac{1}{h}\right)} \\ &= \inf_{l \in \Omega} \frac{1}{l\left(\frac{1}{h}\right)} \\ &= \inf_{l \in \Omega} l^*(h) \end{aligned}$$

Since  $l^*$  is concave it follows that  $p^*$  is also concave.  $\square$

*Remark:* Example 4.8 (2) above shows that the converse to Proposition 4.9 is not valid. It is possible that both  $p$  and  $p^*$  are concave functions.

We will now give an economic interpretation to the duality under consideration in a simple case.

Assume there is a price vector  $l = (l_1, \dots, l_n)$  in an economical system. We will call this vector the vector of basic prices. Assume that  $l_i > 0$  for all  $i$ . For an arbitrary price vector  $h = (h_1, \dots, h_n)$  we can define lower and upper estimates of the deviation of the vector  $h$  from the basic vector  $l$  in the following way. Begin by defining:

$$p_1(h) = \max \{\lambda : \lambda l \leq h\}, \quad p_2(h) = \min \{\lambda : \lambda l \geq h\}.$$

Clearly

$$p_1(h) = \min_i \frac{h_i}{l_i}, \quad p_2(h) = \max_i \frac{h_i}{l_i}.$$

We have, for  $x \in \mathbb{R}_{++}^n$

$$p_1^*(x) = \frac{1}{p_1\left(\frac{1}{x}\right)} = \max_i l_i x_i, \quad p_2^*(x) = \frac{1}{p_2\left(\frac{1}{x}\right)} = \min_i l_i x_i.$$

We can consider the quantity  $l_i x_i$  as the cost of  $x_i$  units of product  $i$  under the basic price  $l_i$ . So  $p_1^*(x)$  is an upper estimate and  $p_2^*(x)$  is a lower estimate of the cost of purchasing the production vector  $x$  using the basic

price vector  $l$ . We can apply the usual sublinear duality (called *polarity* in [10]) to the function  $p_2$ . Let  $p_2^0$  be the polar function to  $p_2$ :

$$p_2^0(x) = \max_{p_2(h) \leq 1} \sum_i h_i x_i.$$

It is easy to check that, by applying the usual  $(\ell_1, \ell_\infty)$  duality, that  $p_2^0(x) = \sum_i l_i x_i$ . Since  $p_1$  is a superlinear function we must apply a different duality (polarity) operation in order to obtain a polar  $p_1^\square$  for  $p_1$  (in the sense of convex analysis), namely:

$$p_1^\square(x) = \min_{p_1(h) \geq 1} \sum_i h_i x_i.$$

Clearly  $p_1^\square(x) = \sum_i l_i x_i$ . Thus, for all  $x \in \mathbb{R}_{++}^n$ ,

$$p_1^*(x) \leq p_1^\square(x) = p_2^0(x) \leq p_2^*(x)$$

where  $p_1^*$  and  $p_2^*$  are the best estimates of  $p_1^\square = p_2^0$ .

The dual operation  $*$  permits us to apply the same conjugation scheme to both convex and concave *IPH* functions. For example it is not possible to apply duality in the sense of convex analysis to functions of the following form:

$$q_l(h) = \alpha \min_i \frac{h_i}{l_i} + \beta \max_i \frac{h_i}{l_i}.$$

with  $\alpha > 0, \beta > 0$ . However we can easily compute the conjugate function  $q_l^*$ :

$$\begin{aligned} q_l^*(x) &= \frac{1}{\alpha \min_i \frac{1}{l_i x_i} + \beta \max_i \frac{1}{l_i x_i}} \\ &= \frac{1}{\frac{\alpha}{\max_i l_i x_i} + \frac{\beta}{\min_i l_i x_i}} \\ &= \frac{(\max_i l_i x_i) (\min_i l_i x_i)}{\alpha \min_i l_i x_i + \beta \max_i l_i x_i} \\ &= \frac{1}{q_{1/l}(x)} (\max_i l_i x_i) (\min_i l_i x_i). \end{aligned}$$

**5. INCREASING FUNCTIONS AND NORMAL LEVEL SETS**

Recall that a function  $f$  defined on a linear space  $X$  is called *quasiconvex* if its level sets  $\{x : f(x) \leq c\}$  are convex for all  $c$ . We will now consider the

analogue of this construction for functions defined on  $\mathbb{R}_{++}^n$  which possess level sets which are normal (for all  $c$ ). It is straightforward to check that a function possesses this property if and only if the function is increasing.

DEFINITION 5.1: Let  $I_0$  denote the set of lower semicontinuous increasing functions defined on  $\mathbb{R}_{++}^n$  which vanish at the origin.

If  $f \in I_0$  then the level sets  $S_c(f) = \{x : f(x) \leq c\}$  and  $T_c(f) = \{x : f(x) < c\}$  are normal for all  $c > 0$ . In addition the zero level set  $S_0(f) = \{x : f(x) \leq 0\} = \{0\}$  is trivially normal. Note that by the lower semicontinuity assumption  $S_c(f)$  are closed for all  $c \geq 0$ .

Recently there has been considerable attention in the literature on various definitions of a *conjugate* to a quasiconvex function defined on a l.c.H.t.v.s  $X$  (see, for example [3, 9, 13, 7, 14]). One such approach is particularly suitable for nonnegative l.s.c. quasiconvex functions  $q$  with  $q(0) = 0$  [13]. By this approach the conjugate is defined as follows:

$$q^*(v) = \frac{1}{\inf_{v(x) > 1} q(x)}$$

where  $v$  is a continuous linear functional on  $X$ . It can be shown [13] that the level sets of the conjugate function are the polar sets of the level sets of the original function. More precisely, for  $c > 0$ ,

$$\begin{aligned} S_{1/c}(q^*) &= (T_c(q))^0 \\ T_{1/c}(q^*) &= \bigcup_{c' > c} (S_{c'}(f))^0 \end{aligned}$$

Now we give a similar definition of conjugate applicable to the situation under study in this paper.

DEFINITION 5.2: Let  $f$  be a nonnegative function mapping  $\mathbb{R}_{++}^n$  into  $\mathbb{R}_{+\infty}$  with  $f(0) = 0$ . The function  $f^*$  defined on  $\mathbb{R}_{++}^n$  as follows:

$$f^*(h) = \begin{cases} \frac{1}{\inf_{(h,x) > 1} f(x)} & h \neq 0 \\ 0 & h = 0 \end{cases}$$

is called the *conjugate* of  $f$ . We also define  $f^* \equiv 0$  if  $f(x) \equiv +\infty$  for all  $x \in \mathbb{R}_{++}^n$ .

We note the following properties of the conjugate function.

1. Let  $f_1, f_2$  be nonnegative real-valued functions defined on  $\mathbb{R}_{++}^n$  with  $f_1 \geq f_2$  then  $f_1^* \leq f_2^*$ .

2. Let  $I$  be an arbitrary index set and, for all  $i \in I$ ,  $f_i : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  be a family of functions with  $f_i(0) = 0$ ; let  $f(x) = \inf_{i \in I} f_i(x)$  ( $x \in \mathbb{R}_{++}^n$ ). Then, for all  $h \in \mathcal{H}$ ,

$$f^*(h) = \sup_{i \in I} f_i^*(h).$$

This easily follows since:

$$\begin{aligned} f^*(h) &= \frac{1}{\inf_{\langle h, x \rangle > 1} f(x)} \\ &= \frac{1}{\inf_x \inf_{i \in I} f_i(x)} \\ &= \frac{1}{\inf_{i \in I} \inf_x f_i(x)} \\ &= \sup_{i \in I} \frac{1}{\inf_x f_i(x)} \\ &= \sup_{i \in I} f_i^*(h). \end{aligned}$$

3. Let  $f \in I_0$  and continuous at the point  $1/h$ , then

$$f^*(h) = \frac{1}{f\left(\frac{1}{h}\right)}.$$

Since  $\langle h, x \rangle = \min_{i=1, \dots, n} h_i x_i$  we have  $\langle h, x \rangle > 1$  if and only if  $x_i > 1/h_i$  for all  $i$ . Applying the continuity and monotonicity of the function  $f$  we have

$$\inf_{\langle h, x \rangle > 1} f(x) = \inf \left\{ f(x) : x \geq \frac{1}{h} \right\} = f\left(\frac{1}{h}\right).$$

4. If  $p$  is an *IPH* function then both of the definitions of conjugate functions coincide. Indeed both definitions yield:

$$p^*(h) = \frac{1}{p\left(\frac{1}{h}\right)}.$$

Here we apply the continuity of  $p$  at the point  $h \neq 0$  (see property 4 for *IPH* functions in section 3).

The following proposition expresses the main property of the conjugate function.

**PROPOSITION 5.3:** *Let  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  be a function with the property  $f(0) = 0$ . Then for all  $c > 0$  we have*

$$\begin{aligned} S_{1/c}(f^*) &= (T_c(f))^* \\ T_{1/c}(f^*) &= \bigcup_{c' > c} (S_{c'}(f))^*. \end{aligned}$$



*Proof:* Take  $c > 0$ . If  $h \in S_{1/c}(f^*)$  then  $f(x) \geq c$  for all  $x$  such that  $\langle h, x \rangle > 1$ . Hence the inequality  $f(x) < c$  implies  $\langle h, x \rangle \leq 1$ . Thus

$$\sup_{x \in T_c(f)} \langle h, x \rangle \leq 1.$$

Hence  $h \in (T_c(f))^*$ . Also, if  $h \in (T_c(f))^*$  then the inequality  $\langle h, x \rangle > 1$  implies  $f(x) \geq c$ . Therefore

$$\inf_{\langle h, x \rangle > 1} f(x) \geq c \quad \text{and} \quad h \in S_{1/c}(f^*).$$

Now let  $h \in T_{1/c}(f^*)$  then  $\inf_{\langle h, x \rangle > 1} f(x) > c$  and there is  $c' > c$  such that  $f(x) > c'$  for all  $x$  with  $\langle h, x \rangle > 1$ . If  $x \in S_{c'}(f)$ , that is  $f(x) \leq c'$ , then  $\langle h, x \rangle \leq 1$ . Thus

$$h \in (S_{c'}(f))^* \subseteq \bigcup_{c' > c} (S_{c'}(f))^*.$$

Hence

$$T_{1/c}(f^*) \subseteq \bigcup_{c' > c} (S_{c'}(f))^*.$$

Similar arguments show that the reverse inclusion also holds.  $\square$

**COROLLARY 5.4:** Let  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$  with  $f(0) = 0$  then  $f^* \in I_0$ . Indeed the level sets  $S_c(f^*)$  are closed and normal for all  $c \geq 0$ .

Recall that  $S_0(f) = \{0\}$ .

**COROLLARY 5.5:** If  $f \in I_0$  then  $f^{**} = f$ .

*Proof:* Let  $c > 0$ . Since  $f \in I_0$  it follows that the set  $S_c(f)$  is closed and normal so  $S_c(f) = (S_c(f))^{**}$ . Applying Proposition 5.3 and property 4 (section 4) for conjugate sets we have

$$\begin{aligned} S_c(f^{**}) &= (T_{1/c}(f^*))^* \\ &= \left( \bigcup_{c' > c} (S_{c'}(f))^* \right)^* \\ &= \bigcap_{c' > c} (S_{c'}(f))^{**} \\ &= \bigcap_{c' > c} S_{c'}(f). \end{aligned}$$

It is easy to check that  $\cap_{c' > c} S_{c'}(f) = S_c(f)$ . Therefore  $S_c(f^{**}) = S_c(f)$  for all  $c > 0$ . So  $f = f^{**}$ .  $\square$

DEFINITION 5.6: Let  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  and let

$$\bar{f}(x) = \sup \{ \tilde{f}(x) : \tilde{f} \in I_0, \tilde{f} \leq f \}.$$

The function  $\bar{f}$  is called the l.s.c. *increasing hull* of  $f$ . Clearly  $\bar{f} \in I_0$  and  $\bar{f}$  is the greatest member of  $I_0$  majorized by  $f$ .

PROPOSITION 5.7: For  $f : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{+\infty}$  with  $f(0) = 0$  we have

$$\bar{f} = f^{**}.$$

*Proof:* Let  $\tilde{f} \in I_0, \tilde{f} \leq f$ . We have  $\tilde{f}^* \geq f^*$  and therefore  $\tilde{f} = \tilde{f}^{**} \leq f^{**}$ . Since  $f^{**} \in I_0$  it follows that  $f^{**} = \bar{f}$ .  $\square$

## 6. DUAL OPTIMIZATION PROBLEMS

Often in optimization theory and practice we encounter problems with convex constraints (*i.e.* constraints of the form  $x \in \Omega$  where  $\Omega$  is a convex set). In the study of some Difference Convex (DC) optimization problems we require reverse convex constraints, that is constraints of the form  $x \notin \Omega$  for  $\Omega$  a convex set (*see* [14, 15]).

Recently Thach [14] considered a problem involving the minimization of a quasiconvex function subject to a reverse convex constraint which was a dual to a quasiconvex maximization problem under convex constraints. Such a primal problem is an example of a global optimization problem. We shall now consider analogous results for maximization of an  $I_0$  function on a normal set.

Consider the maximization problem:

$$(P) \quad f(x) \rightarrow \max \quad \text{subject to } x \in \Omega,$$

where  $f \in I_0$  is a continuous function and  $\Omega$  is a closed normal subset of  $\mathbb{R}_{++}^n$ . The following problem of minimization under reverse normal constraints:

$$(D) \quad f^*(h) \rightarrow \min \quad \text{subject to } h \notin \Omega^*$$

is called a dual problem to (P).

THEOREM 6.1: Let  $f$  and  $\Omega$  be as above, then

$$\sup_{x \in \Omega} f(x) = \frac{1}{\inf_{h \notin \Omega^*} f^*(h)}.$$

*Proof:* Let  $p$  be the Minkowski gauge of  $\Omega$ . Then  $p$  is an IPH function and  $\Omega = \{x : p(x) \leq 1\}$ . Since  $p$  is continuous we have

$$\text{int } \Omega = \{x : p(x) < 1\}.$$

Since  $f$  is continuous we have

$$\begin{aligned} \sup_{x \in \Omega} f(x) &= \sup_{p(x) \leq 1} f(x) = \sup_{p(x) < 1} f(x) \\ &= \sup_{p(1/h) < 1} f\left(\frac{1}{h}\right) = \frac{1}{\inf_{p(1/h) < 1} f\left(\frac{1}{h}\right)}. \end{aligned}$$

It follows from Corollary 4.5 that  $\Omega^* = \{h : p\left(\frac{1}{h}\right) \geq 1\}$ . Therefore  $\sup_{x \in \Omega} f(x) = (\inf_{h \notin \Omega^*} f^*(h))^{-1}$  as required.

*Remark:* This result may be useful in studying some nonconvex extremal problems. Assume for example that  $\Omega$  is a convex set and that we have a function  $f \in I_0$  such that both  $f$  and  $f^*$  are concave functions (in particular functions from Example 4.8 (1) (with  $k < 1$ ) and 4.8(2) possess this property). It is easy to see by applying Proposition 4.9 that the set  $\{h : h \notin \Omega^*\}$  is convex. Therefore the problem (P) is a convex programming problem. At the same time (D) is not convex. Thus Theorem 6.1 allows the study of the nonconvex problem (D) using the convex problem (P).

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