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HAMILTONIAN PROBLEMS IN EDGE-COLORED COMPLETE GRAPHS AND EULERIAN CYCLES IN EDGE-COLORED GRAPHS: SOME COMPLEXITY RESULTS (*)

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Abstract. – *In an edge-colored, we say that a path (cycle) is alternating if it has length at least 2 (3) and if any 2 adjacent edges of this path (cycle) have different colors. We give efficient algorithms for finding alternating factors with a minimum number of cycles and then, by using this result, we obtain polynomial algorithms for finding alternating Hamiltonian cycles and paths in 2-edge-colored complete graphs. We then show that some extensions of these results to k-edge-colored complete graphs, $k \geq 3$, are NP-complete. related problems are proposed. Finally, we give a polynomial characterization of the existence of alternating Eulerian cycles in edge-colored graphs. Our proof is algorithmic and uses a procedure that finds a perfect matching in a complete k-partite graph.*

Keywords: Complexity, NP-completeness, graph, Hamiltonicity.

Résumé. – *Dans un graphe arêtes-coloré, on dit qu'un chemin (cycle) est alternant s'il est au moins de longueur 2 (3) et si toute paire d'arêtes adjacentes de ce chemin (cycle) sont de couleurs différentes.*

Nous donnons des algorithmes efficaces trouvant des facteurs alternants comportant un nombre minimum de cycles et, en utilisant ce résultat, nous obtenons des algorithmes polynomiaux pour la recherche de cycles et chemins hamiltoniens alternants dans des graphes complets 2-arêtes-colorés. Nous montrons ensuite que certaines extensions de ces résultats aux graphes complets k-arêtes-colorés, $k \geq 3$ sont NP-complets. D'autres problèmes similaires sont proposés. Enfin, nous donnons une caractérisation polynomiale de l'existence de cycles eulériens alternants dans des graphes arêtes-colorés. Nous donnons une preuve algorithmique (constructive) qui utilise une procédure trouvant un couplage parfait dans un graphe k-parties complet.

Mots clés : Complexité, NP-complétude, graphe, hamiltonicité.

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1. INTRODUCTION

We study in this paper the existence of alternating hamiltonian and Eulerian cycles and paths in edge-colored complete graphs.

Formally, in what follows, K_n^c denotes an edge-colored complete graph of order n, with vertex set $V(K_n^c)$ and edge set $E(K_n^c)$. The set of used colors is denoted by $\Psi = \{\chi_1, \chi_2, \ldots\}$. If A and B are subsets of $V(K_n^c)$, then AB denotes the set of edges between A and B. An AB-edge is an edge between A and B, *i.e.*, it has one extremity in A and the other one in B. Whenever the edges between A and B are monochromatic, then their color is denoted by $\chi(AB)$. If $A = \{x\}$ and $B = \{y\}$, then for simplicity we write xy (resp. $\chi(xy)$) instead of AB (resp. $\chi(AB)$). If x denotes a vertex of K_n^c and χ_i is a color of Ψ, then we define the χ_i-degree of x to be the number of vertices y such that $\chi(xy) = \chi_i$. The χ_i-degree of x is denoted by $\chi_i(x)$. Whenever, the edges of K_n^c are colored by precisely two colors, then, for simplicity, these colors are called *red* and *blue* and are denoted by r and b, respectively.

A path P is said to be *alternating* if it has length at least *two* and any two adjacent edges of P have different colors. Similarly, we define alternating cycles and alternating Hamiltonian (Eulerian) cycles and paths. An *alternating factor* F is a collection of pair-wise vertex-disjoint alternating cycles $C_1, C_2, \ldots, C_m$, $m \geq 1$, covering the vertices of the graph. All cycles and paths considered in this paper are elementary, *i.e.*, they go through a vertex exactly once, unless otherwise specified.

The notion of alternating paths was originally raised by Bollobas and Erdös in [4], where they proved that if no set of k edges of K_n^c incident to a same vertex are monochromatic, then K_n^c contains an alternating Hamiltonian cycle provided that n is greater than a constant c_k. Results in almost the same vein are proved in [6]. Also, in [1], necessary and sufficient conditions are presented (*see* theorem 1 below). However, the problem of characterizing alternating Hamiltonian instances K_n^c, or at least establishing nontrivial sufficient conditions for the existence of such cycles, is still open, whenever the edges of K_n^c are colored by more than two colors. Some further results on alternating cycles and paths are proved in [2, 3, 11, 13, 14, 15].

This type of problems, except their proper theoretical interest, have many applications, for example in social sciences (a color represents a relation between two individuals) and in cryptography where a color represents a specified type of transmission. Also, it turns out that the notion of alternance is implicitely used in some classical problems of graph theory. Let us think, for example, to a given instance of Edmond's well-known algorithm for finding a *maximum matching*. The edges of the current matching can be colored *red*, any other edge can be colored *blue*, and then the searched *augmenting path* is just an alternating path.

In section 2, we deal with Hamiltonian problems on 2–edge-colored complete graphs K_n^c. Namely, by using known results on matchings, we obtain $O(n^3)$ algorithms for finding an alternating factor, if any, with a minimum number of alternating cycles in K_n^c. As an immediate consequence, we obtain an $O(n^3)$ algorithm for finding alternating Hamiltonian cycles and paths, or else for proving that such cycles or paths do not exist. As a byproduct of this result, we obtain an $O(n^3)$ algorithm for finding Hamiltonian cycles in *bipartite tournaments* (another algorithm for finding Hamiltonian cycles in bipartite tournaments is proved in [16]). To see why alternating cycles can be used in order to obtain cycles in bipartite tournaments, let us consider a bipartite tournament $B(X, Y, E)$ with bipartition classes X, Y and arc set $E(B)$. Let now K_n^c denote a complete 2-edge-colored graph obtained from B as follows: we define

$$V(K_n^c) = X \cup Y \qquad \text{and} \qquad E(K_n^c) = E_b(K_n^c) \cup E_r(K_n^c),$$

where:

$$E_b(K_n^c) = \{xy | xy \in E(B), x \in X \text{ and } y \in Y\} \cup \{xy | x, \ y \in X\}$$

and

$$E_r(K_n^c) = \{xy | xy \in E(B), x \in Y \text{ and } y \in X\} \cup \{xy | x, \ y \in Y\}.$$

Now, it is easy to see that B has a Hamiltonian cycle if and only if K_n^c has an alternating Hamiltonian cycle.

The following results are used in section 2.

THEOREM 1. M. Bánkfalvi and Z. Bánkfalvi [1]: *Let K_{2p}^c be a 2-edge-colored complete graph with vertex-set $V(K_{2p}^c) = \{x_1, x_2, \ldots, x_{2p}\}$. Assume that $r(x_1) \leq r(x_2) \leq \cdots \leq r(x_{2p})$. The graph K_{2p}^c contains an alternating factor with a minimum number m of alternating cycles if and only if there are m numbers k_i, $2 \leq k_i \leq p-2$, such that, for each i, $i = 1, 2, \ldots, m$, we have:*

$$r(x_1) + r(x_2) + \cdots + r(x_{k_i}) + b(x_{2p}) + b(x_{2p-1}) \\ + b(x_{2p-2}) + \cdots + b(x_{2p-k_i+1}) = k_i^2.$$

LEMMA 1. Manoussakis and Tuza [16]: *Let B be a bipartite tournament. Assume that B contains two pairwise vertex-disjoint cycles W_1 and W_2. If there is at least one arc oriented from W_1 to W_2 and another one oriented from W_2 to W_1, then B contains a cycle W such that $V(W) = V(W_1) \cup V(W_2)$. Furthermore, findind W can be done in $O(|V(W_1)||V(W_2)|)$ time.*

In section 3, we give some NP-complete results for Hamiltonian problems on k-edge-colored complete graphs, $k \geq 3$, and we propose related problems.

Finally, in section 4, we present algorithmic results regarding the existence of alternating Eulerian cycles and paths in directed edge-colored graphs.

2. ALTERNATING HAMILTONIAN CYCLES AND PATHS IN 2-EDGE-COLORED COMPLETE GRAPHS

In this section, we suppose that K_n^c admits F, an alternating factor consisting of m alternating cycles $C_1, C_2, \ldots, C_m$, $m \geq 2$. It turns out to be convenient, for technical reasons, to divide the vertices of each alternating cycle C_i into two classes X_i and Y_i, where $X_i = \{x_{i1}, x_{i2}, \ldots, x_{is}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{is}\}$ such that the edge $x_{ij}\, y_{ij}$ is *red* and the edge $y_{ij}\, x_{i(j+1)}$ is *blue*, for each $j = 1, 2, \ldots, s$ (where $2s$ is the length of the cycle and j is considered modulo s). Furthermore, if C_1 and C_2 are two cycles of F (if any) with classes X_1, Y_1 and X_2, Y_2 respectively, then we say that C_1 *dominates* C_2 if either all $X_1\, C_2$ edges are *red* and all $Y_1\, C_2$ edges are *blue* or all $X_1\, C_2$ edges are *blue* and all $Y_1\, C_2$ edges are *red.*

In the first part of this section, we prove lemma 2 and we establish procedure 1 for complete graphs whose vertices are covered by two pairwise vertex-disjoint alternating cycles. These preliminary results are useful for algorithm 1 given later.

LEMMA 2: *Let K_n^c be a 2-edge-colored complete graph. Assume that there exist two pair-wise vertex-disjoint alternating cycles C_1, C_2 in K_n^c covering all its vertices. Furthermore, assume that there are at least two X_1X_2- (or X_1Y_2- or Y_1X_2- or Y_1Y_2-) edges with different colors. Then K_n^c contains an alternating Hamiltonian cycle, which can be obtained in time $O(|V(C_1)||V(C_2)|)$.*

Proof: Assume first that there exist at least two X_1Y_2- or Y_1X_2-edges having different colors.

Let B be a bipartite tournament obtained from K_n^c as follows:

• its vertex set is defined as $V(B) = X' \cup Y'$ (X' and Y' are the bipartition classes of B), where $X' = X_1 \cup X_2$ and $Y' = Y_1 \cup Y_2$;

• the arc-set of B is defined from $E(K_n^c)$ as follows: (i) delete each colored edge inside the classes X' and Y' of B; (ii) let now x, y be two vertices of B, $x \in X'$ and $y \in Y'$; if the edge xy is a *red* one in K_n^c, then replace it by an arc oriented from x to y in B; otherwise, if xy is *blue*, replace it by an arc oriented from y to x in B.

Now, if there exist two X_1Y_2 (or Y_1X_2)-edges with different colors in K_n^c, then, clearly, B satisfies the conditions of lemma 1 (given in the introduction) and therefore it admits a Hamiltonian cycle. This Hamiltonian cycle corresponds to an alternating Hamiltonian cycle of K_n^c.

Assume next that there are at least two X_1X_2- or Y_1Y_2-edges with different colors in K_n^c . In this case, define a bipartite tournament with classes $X' = X_1 \cup X_2$ and $Y' = Y_1 \cup Y_2$ and edge-set as previously and complete the argument as above. ■

Let us now establish procedure 1 which, given a 2-edge-colored graph K_n^c and two alternating cycles C_1, C_2 of orders t and s, respectively, such that C_1 dominates C_2, it outputs either an alternating Hamiltonian cycle or else a statement that all X_1X_1 and X_1C_2 edges are monochromatic. It is easy to see that the complexity of procedure 1 is of $O(t^2)$.

Input: a 2-edge-colored graph K_n^c and two alternating cycles C_1, C_2 such that C_1 dominates C_2.

Output: Either an alternating Hamiltonian cycle or else that all X_1X_1 and X_1C_2 edges are monochromatic.

Assume w.l.ofg. that all X_1C_2 edges are *red*; we look now if there exists a *blue* edge $x_i\,x_j$, where $i \neq j$ and x_i, $x_j \in X_1$; if this is the case, then an alternating cycle of K_n^c is the cycle $y_{i-1}\,x_{i-1} \ldots x_j\,x_i\,y_i\,x_{i+1} \ldots y_{j-1}\,y'_h\,x'_h\,y'_{h-1} \cdots y'_{h+1}\,x'_{h+1}\,y_{i-1}$, where y'_h is an appropriate vertex of Y_2.

Similarly, if there exists a *red* edge $y_i\,y_i$, where $i \neq j$ and y_i, $y_j \in Y_1$, then by using the same arguments, one can find, once more, an alternating Hamiltonien cycle of K_n^c.

Procedure 1.

The following lemma 3 will be used in algorithm 1 given later.

LEMMA 3: *Let K_n^c be a 2-edge-colored complete graph containing an alternating factor F consisting of cycles C_1C_2, ..., C_m, $m \geq 2$. Assume that C_i dominates C_{i+1} for each $i = 1, 2, \ldots, m-1$. Assume, without loss of generality, that all X_1C_2 edges are* red. *Then,* (i) *there exists an alternating Hamiltonian path with begin in Y_1 and terminus in C_m such that both first and last edges of this path are* blue; (ii) *if C_1 dominates C_m and the edges $X_1\,C_m$ are* blue, *then K_n^c admits an alternating Hamiltonian cycle.*

Proof: (i) Since C_1 dominates C_2 and all X_1C_2 edges are *red*, then all Y_1C_2 edges are *blue*. We first find a path P_1 with vertex set $\{x_{11}\} \cup V(C_2)$ such that its first edge is *red* and its last one is *blue* as follows: if all X_2C_3 edges are *red*, then P_1 has begin x_{11} and terminus x_{2i}, where the vertex x_{2i} is appropriately chosen in X_2. On the other hand, if all X_2C_3 edges are *blue*, then P_1 has begin x_{11} and terminus y_{2i}, where in this case $y_{2i} \in Y_2$. Assume w.l.olg. that P_1 has begin x_{11} and terminus y_{2i}, where in this case $y_{2i} \in Y_2$. Assume w.l.ofg. that P_1 has being x_{11} and terminus x_{2i}. We next find a path P_2 with vertex-set $\{x_{2i}\} \cup V(C_3)$ such that its first edge is *red* and its last edge is *blue* as follows: if the $X_3\,C_4$ edges are *blue*, then P_2 begins at x_{2i} and determinates at y_{3j}, where y_{3j} is appropriately chosen in Y_3. On the other hand, if the $X_3\,C_4$ edges are *red*, then P_2 begins at x_{2i} and determinates at x_{3j}, where $x_{3j} \in X_3$. Continuing in this way, that is trying to pass from a cycle C_i to a cycle C_{i+1}, $1 \leq i \leq m-1$, through a *red* edge, we find paths P_3, P_4 and so on, until the last path P_{m-1} is found. We complete the argument by setting $P = (V(C_1) \backslash \{x_{11}\}) \cup P_1 \cup P_2 \cup \ldots \cup P_{m-1}$.

(ii) The alternating path of (i) together with a *red* edge between Y_1 and C_m define an alternating Hamiltonian cycle of K_n^c. ■

DEFINITION 1: From a given factor $F = \{C_1, \ldots, C_m\}$ of K_n^c, one can define a new graph D as follows: replace each cycle C_i of F by a new vertex c_i and then add the arc c_ic_j, $i \neq j$, $i, j = 1, 2, \ldots, m$, in D, if and only if C_i dominates C_j in K_n^c. Otherwise, one can add both arcs c_ic_j and c_jc_i. Clearly, D is a semi-complete digraph. D is said to be the *underlying graph* of K_n^c, while C_i is called the *underlying cycle* of c_i. ■

The following lemma is crucial, since it mathematically justifies the first three steps of algorithm 1.

LEMMA 4: *If D is strongly connected, then K_n^c admits an alternating Hamiltonian cycle.*

Proof: By induction on the number m of vertices of D; the case $m = 2$ has been proved in lemma 2. Let D' be a proper strongly connected subgraph, if any, of D, *i.e.*, $|V(D')| < |V(D)|$. If D' exists, then the underlying cycles of D' can be contracted to one cycle in K_n^c by induction. Now, K_n^c admits a new factor with less than m cycles. Consequently, we complete the argument by applying again induction on the underlying graph of this new factor. Otherwise, if D' does not exist, then D consists of a Hamiltonian path, say c_1, $c_2, \ldots, c_m$, with all arcs directed from c_i to c_j, $i < j$, expect for the arc between c_1 and c_m which is directed from c_m to c_1. However, in this particular case, we can easily find an alternating Hamiltonian cycle of K_n^c by using appropriate edges from C_i to C_{i+1} in K_n^c, where i is considered modulo m. ■

[1] Find a *blue* maximum matching M_b and a *red* one M_r in K_n^c; if either $|M_b| < n/2$ or $|M_r| < n/2$, then stop; K_n^c has no alternating factor; otherwise, form an alternating factor F by considering the union of M_b and M_r.

[2] Let $C_1,, C_2, \ldots, C_m$, $m \geq 1$, be the alternating cycles of F (in what follows, we shall shortly write $F \to C_1,, C_2, \ldots, C_m$), if $m = 1$, then we stop by setting $\mu = 1$ and $R_1 = C_1$; assume that $m \geq 2$; if for some $i < j$, $i, j = 1, 2, \ldots, m$, neither C_i dominates C_j nor C_j dominates C_i, then by applying lemma 2 on the subgraph of K_n^c induced by $V(C_i) \cup V(C_j)$, we produce an alternating cycle, say C', with vertex set $V(C_i) \cup V(C_j)$; we set $C_i \leftarrow C'$, $C_h \leftarrow C_{h+1}$ for all h, $j \leq h \leq m-1$, $m \leftarrow m-1$, $F \leftarrow C_1, C_2, \ldots, C_m$ and then we go to the beginning of this step;
when this step terminates, if $m = 1$, then we set $\mu = 1$, $R_1 = C_1$ and then stop the algorithm; if $m > 3$, then we go to step 3, else we go to step 5.

[3] After step 2, the underlying graph D of K_n^c is clearly a tournament; now, we find the strongly connected components D_i, $i = 1, 2, \ldots, \ell$ of D by using the algorithm of [12]; set $|V(D_i)| = d_i$; if for each i, $d_i = 1$, *i.e.*, D is a transitive tournament, then go to step 4; otherwise, for each non trivial component D_i of D, the algorithm of [12] produces a Hamiltonian cycle denoted, say, by $c_1^i, c_2^i, \ldots, c_{d_i}^i$; by using appropriate colored edges between each pair of the underlying alternating cycles C_j^i and C_{j+1}^i, we define easily an alternating cycle C_i corresponding to each such component D_i of D; if $\ell = 1$, then we terminate the algorithm by setting $\mu = 1$ and $R_1 = C_1$; on the other hand, if $\ell \geq 2$, then we set $m \leftarrow \ell$, $F \leftarrow C_1, C_2, \ldots, C_m$; now, if $\ell = 2$ and neither C_1 dominates C_2 nor C_2 dominates C_1, then we go back to the beginning of step 2; otherwise, we go to step 5.

[4] By the actual structure of K_n^c, C_i dominates C_j for each $1 \leq i < j \leq m$, *i.e.*, by lemma 4 the underlying graph is a transitive tournament; let $C_1, C_2, \ldots, C_m$ be an ordering of the cycles such that C_i dominates C_{i+1} for each $i = 1, 2, \ldots, m-1$; assume, without loss of generality, that the $X_1 C_2$ edges are *red*; if for some cycle C_i, ≥ 2, the edges $X_i C_{i+1}$ are *blue*, then we may interchange the X_i class by the Y_i class in C_i without modifying the ordering of the cycles; therefore, in the sequel, we assume that the edges $X_i C_{i+1}$ are *red* for each $i = 1, 2, \ldots, m-1$;
we determine, if any, the smallest integer h_m, $1 \leq h_m \leq m-2$, such that $X_{h_m} C_m$ and $X_{h_m} C_{h+1}$ (or, equivalently, $Y_{h_m} C_m$ and $Y_{h_m} C_{h_m+1}$) are not monochromatic; by using lemma 3, we find an alternating cycle C with vertex set

$$V(C_{h_m}) \cup V(C_{h_m+1}) \cup \ldots \cup V(C_m);$$

set $m \leftarrow m - h_m + 1$, $C_m \leftarrow C$, $C_m \leftarrow C_{h_m+i-1}$ and go back to the beginning of this step; if h_m does not exist, then we try to find the minimum number h_{m-1} corresponding to the cycle C_{m-1} with the above property and then repeat this step, and so on.

[5] At the end of the previous step, clearly the edges $X_1 C_i$ are *red*, for all $i = 2, \ldots, m$; we look now if there is a *blue* edge e inside $X_1 X_1$ (or a *red* edge inside $Y_1 Y_1$);
if e does not exist, then set $\mu \leftarrow \mu + 1$, $R_\mu \leftarrow C_1$; next, set $m \leftarrow m-1$, $C_i \leftarrow C_{i+1}$, $i = 2, \ldots, m-1$ and go back to the begining of this step;
if e exists, say in $X_1 X_1$, set $e \leftarrow x_i x_j$; by using the arguments of lemma 3, find an alternating path in $\{y_{i-1}\} \cup V(C_2) \cup \ldots \cup V(C_m)$ with begin y_{i-1}, terminus in C_m, and such that its first and last edges are both *blue*; then, by using the segment $y_{i-1}\, x_{i-1} \ldots x_j\, x_i\, y_i\, x_{i+1} \ldots y_{j-1}$ of C_1, define an alternating cycle with vertex set $V(C_1) \cup V(C_2) \cup \ldots \cup V(C_m)$.

Algorithm 1.

Algorithm 1 finds an alternating factor with a minimum number of alternating cycles in 2-edge-colored complete graphs in $O(n^3)$ steps. Its input consists of a complete graph K_n^c on n vertices whose edges are colored *red* and *blue* and the output is either an alternating factor F_μ of K_n^c with a minimum number of alternating cycles $R_1, \ldots, R_\mu$, $\mu \geq 1$, or else an answer that K_n^c has no alternating factor at all. We suppose that K_n^c has an even number of vertices, since otherwise it has no alternating factor. Furthermore, in the beginning, we initialize μ to zero.

Algorithm 1 terminates within at most $O(n^3)$ operations. Namely, finding perfect matchings in step 1 needs no more than $O(n^{2.5})$ operations [7]. Each call of step 2 terminates within $O(n^2)$ operations. In fact, we have to check the domination relation of each pair of cycles C_i and C_j. Since the cost for each pair is $O(|V(C_i)|\,|V(C_j)|)$, the whole cost is bounded by $O(\sum_{i \neq j} |V(C_i)|\,|V(C_j)|) \leq O(n^2)$. Also, step 3 costs $O(n^2)$ operations, *i.e.*, the complexity for finding the minimum number of cycles covering the vertices of a tournament of order $m = O(n)$ [12]. Since steps 2 and 3 are called $O(n)$ times, it follows that the whole of executions of these steps requires a total amount of $O(n^3)$ operations. Finally, steps 4 and 5 terminate with at most $O(n^2)$, since the edges of K_n^c are examined a constant number of times. It follows that the complexity of the whole algorithm is $O(n^3)$. Moreover, we prove that when our algorithm terminates, then K_n^c has no alternating factor with less than μ alternating cycles. This can be proved by showing that conditions of theorem 1 are satisfied. Namely, we define $k_i = \sum_{j \leq i} c_j$, $i = 1, 2, \ldots, \mu - 1$. It follows from the structure of K_n^c that the vertices with the k_i smallest *blue* degrees and the k_i smallest *red* degrees are these of $V(C_1) \cup V(C_2) \cup V(C_3) \cup \ldots \cup V(C_i)$. By considering the sum of these smallest degrees, we obtain

$$\sum_{1 \leq j \leq i} |X_i|\,|Y_j| + \sum_{1 \leq h < j \leq i} |X_i|\,|Y_h| + \sum_{1 \leq h < j \leq i} |X_j|\,|X_h|$$
$$+ \sum_{1 \leq h < j \leq i} |Y_j|\,|Y_h| + \sum_{1 \leq h < j \leq i} |Y_j|\,|X_h|$$
$$= \sum_{1 \leq j \leq i} c_j^2 + 2 \sum_{1 \leq j < h \leq i} c_j\, c_h = (c_1 + c_2 + \cdots + c_i)^2 = k_i^2$$

which is our assertion.

From algorithm 1, we obtain the following concluding theorem 2.

THEOREM 2: *There exists an* $O(n^3)$ *algorithm for finding Hamiltonian cycles in a 2-edge-colored complete graph* K_n^c.

We conclude this section by giving a characterization of 2-edge-colored complete graphs admitting alternating Hamiltonian paths.

THEOREM 3: *Any 2-edge-colored complete graph* K_n^c *has a Hamiltonian path if an only if the graph* K_n^c *has:* (i) *an alternating factor or,* (ii) *an "almost alternating factor", that is a spanning subgraph which differs from a factor by the color of exactly one edge e or, finally,* (iii) *an odd number of vertices and, furthermore,* K_n^c *has a* red *matching* ([5]) M_r *and a* blue *one* ([6]) M_b, *each one having cardinality* $\frac{n-1}{2}$.

Proof: The necessity is obvious. Let now G be a 2-edge-colored complete graph obtained from K_n^c depending upon the case (i), (ii) or (iii) as follows:

(i) define $G \equiv K_n^c$;

(ii) define $G \equiv K_n^c$, but change the color of the edge e, that is color e *blue* in G, if its color was *red* in K_n^c and vice versa;

(iii) let x be the vertex of K_n^c which is not saturated by M_b; in this case, define $V(G) = V(K_n^c) \cup \{z\}$, where z is a new vertex and $E(G) = E(K_n^c) \cup \{zw | w \in V(K_n^c)\}$; the edge zx is colored *blue* and any other edge zw, $(w \in V(K_n^c) \setminus \{x\})$, is colored *red* in G.

Let now F be an alternating factor of G consisting of alternating cycles $C_1, \ldots, C_m, m \geq 1$. If $m = 1$, then G has an alternating Hamiltonian cycle, and therefore, in an obvious way, we can find an alternating Hamiltonian path in K_n^c. In what follows, assume that $m \geq 2$. Furthermore, by using the arguments of algorithm 1, we can suppose that C_i dominates C_j for i, $j = 1, 2, \ldots, m$ and $i < j$. Now, if G is obtained as described in (i) (resp., in (ii)), then by using the arguments of lemma 3, we may find an alternating Hamiltonian path (resp., an alternating Hamiltonian path avoiding e) in K_n^c. On the other hand, if G is obtained as described in (iii), then we can see that z belongs to C_1 since $b(z) = 1$. Consequently, once more, we may complete the proof by using the arguments of lemma 3. ■

([5]) A matching all of edges of which are *red*.
([6]) A matching all of edges of which are *blue*.

Relying on lemmas 2, 3 and 3, theorem 3, we deduce an $O(n^3)$ algorithm for finding alternating Hamiltonian paths. The techniques used for this algorithm are pretty much similar than the ones of algorithm 1 and thus it (the algorithm) is omitted.

3. SOME NP-COMPLETENESS RESULTS

In this section, we consider the problem of finding Hamiltonian configurations with specified edge-colorings in k-edge-colored graphs, $k \geq 3$. We prove that some of these problems are NP-complete.

Notation: Let p be an inter and $\Psi = \{\chi_1 \chi_2, \ldots, \chi_k\}$ be the set of used colors. A $(\chi_1 \chi_2 \ldots \chi_k)$ cycle (path) is a cycle (path) of length pk such that the sequence of colors $\langle \chi_1 \chi_2 \ldots \chi_k \rangle$ is a repeated p times.

THEOREM 4: *The problem* Π*: "given a* 3*-edge colored complete graph* K_n^c*, does there exist a Hamiltonian* $(\chi_1 \chi_2 \chi_3)$ *cycle in* K_n^c*?" is NP-complete?*

Proof: Π is trivially in NP.

The reduction is from the *directed Hamiltonian cycle problem* (DHC, [10]).

Let us consider an instance $D = (V, A)$ of DHC. We first split each vertex v_i, $i = 1, \ldots, n$, of D into three vertices v_{i_1}, v_{i_2}, v_{i_3}. We color the edge $v_{i_1} v_{i_2}$ by χ_1, the edge $v_{i_2} v_{i_3}$ by χ_2 and the edge $v_{i_3} v_{j_1}$ by χ_3, only if $v_i v_j \in D$ (of course, v_j is also splitted into v_{j_1}, v_{j_2}, v_{j_3}). We complete the graph by adding edges of color χ_1. Let us denote by K_n^c the so-obtained complete edge-colored graph.

If a Hamiltonian cycle H is given for D, then it is easy to construct a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle for K_n^c as follows: first, we order, arbitrary, the vertices of H in such a way that v_i precedes v_j in the ordering, if and only if v_i is the predecessor of v_j in H; next, we consider the cycle H^c of K_n^c where we have replaced every vertex v_i of H by the path $v_{i_1} v_{i_2} v_{i_3}$.

Let us now suppose that a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle H^c is given for K_n^c.

If the sequence of vertices in H^c is $v_{i_1} v_{i_2} v_{i_3}, \ldots, i = 1, \ldots, n$, then it is easy to construct a Hamiltonian cycle H for D by simply replacing the sequence $v_{i_1} v_{i_2} v_{i_3}$ by v_i.

Let us now suppose that the sequence of vertices of H^c has not the form just described. Then, it is easy to see that, for every i, the segment of H^c colored by χ_2 and χ_3 is of the form $v_{i_2} v_{i_3} v_{j_1}$, since, for every i and j such that v_i is predecessor of v_j in D, $v_{i_2} v_{i_3}$ and $v_{i_3} v_{j_1}$ are the only edges colored by χ_2 and χ_3, respectively. Let us now see which can be the vertex x "sending" an edge of color χ_1 to v_{i_2} (we have already examined the case where this vertex is v_{i_1});

(i) if $x = v_{k_2}$, $k \neq i$, then there must be an edge of H^c colored by χ_3 incident to v_{k_2}, and this is impossible by the construction of K_n^c;

(ii) if $x = v_{k_3}$, $k \neq i$, then we must suppose that, in H^c, there exists an edge, colored by χ_3, incident to v_{k_3}; by the construction of K_n^c, this edge has to be of the form $v_{l_1} v_{k_3}$; consequently, there must be an edge of H^c, colored by χ_2, incident to v_{l_1}, impossible by the way the edges of K_n^c are colored.

So, the only possibility, in view of the hypothesis on the feasibility of H^c, is that $x = v_{m_1}$, $m \neq i$.

On the other hand, let us suppose that there exists an edge of H^c, colored by χ_1, of the form $v_{i_3} v_{j_1}$ (7); then, by the way the coloring of the edges of K_n^c has been performed, v_{i_3} has to be adjacent, in H^c, to the edge $v_{i_2} v_{i_3}$ colored by χ_2; but then, v_{i_2} has to be adjacent, in H^c, to an edge colored by χ_3, impossible given the way the coloring of the edges of K_n^c has been constructed.

The above remarks indicate that once a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle H^c has been found in K_n^c, one can, in any case, reconstitute a sequence, H, of all of the vertices of D such that every two consecutive vertices are in relation predecessor-successor, every vertex appearing once and only once in H; this can be done by simply examining every segment $v_{i_2} v_{i_3} v_{j_1}$ of H^c and putting v_i and v_j aside in H; then, H constitutes a Hamiltonian cycle for D. ■

THEOREM 5: *The problem: "given positive integers p and k, $k \geq 4$, and a k-edge-colored complete graph K_n^c such that $n = kp$, does K_n^c contain a $(\chi_1 \chi_2 \ldots \chi_k)$ Hamiltonian cycle C? is NP-complete.*

Proof: Our problem is in NP, since for a given cycle, we may deduce in polynomial time if it has the required properties.

For the proof of the completeness, we transform DHC to our problem.

Consider any arbitrary instance of DHC by taking a directed graph $D(V, A)$ with vertex set $V(D)$ and arc set $A(D)$. We have to construct a k-edge-colored complete graph K_n^csuch that D has a Hamiltonian cycle C'

(7) We recall that such an edge indicates that v_j is not successor of v_i in D.

if and only if K_n^c has a $(\chi_1\,\chi_2\ldots\chi_k)$ Hamiltonian cycle C such that the sequence of colors $\langle\chi_1\,\chi_2\ldots\chi_k\rangle$ appears n/k times on C.

Set $|V(D)| = n'$, $V(D) = \{v_1, v_2, \ldots, v_{n'}\}$ and $n = kn'$. The graph K_n^c has vertex-set $V(K_n^c) = \bigcup_{1\le i\le n'} \{v_{i,j} | 1 \le j \le k\}$, where the k vertices $\{v_{i,j} | 1 \le j \le k\}$ of K_n^c are associated to each vertex v_i of D; $E(K_n^c) = \{xy | x, y \in V(K_n^c)\}$.

Every edge $v_{i,k}\,v_{j,1}$, $i \ne j$, $1 \le j, i \le n'$, is colored by χ_k if and only if $v_i\,v_j$ is an arc of D; otherwise, $v_{i,k}\,v_{j,1}$ is colored χ_1. In addition, the edges $v_{i,1}\,v_{j,2}$, $i \ne j$, are colored χ_k, each edge $v_{i,j}\,v_{i,j+1}$, $j = 1, 2, \ldots, k-1$, is colored χ_j, and any other edge of K_n^c is colored χ_1. So, we have constructed a complete graph K_n^c on $n'\,k$ vertices, its edges being colored by k colors. Clearly, this construction is obtained in polynomial time.

Let us suppose now that a Hamiltonian cycle C' is found in D. A cycle C is easily constructed in K_n^c by replacing each vertex v_i (resp., each arc $v_i v_{i+1}$) of C' by the corresponding path $v_{i,1}\,v_{i,2}\ldots v_{i,k-1}\,v_{i,k}$ (resp. by the corresponding edge $v_{i,k}\,v_{i+1,1}$) of K_n^c. Clearly, C satisfies all requirements.

Conversely, assume that K_n^c contains a Hamiltonian cycle C such that the sequence of colors $\langle\chi_1\,\chi_2\ldots\chi_k\rangle$ appears n' times on C. We claim that we may replace any ordered sequence of edges $\langle\varepsilon_1\,\varepsilon_2\ldots\varepsilon_k\rangle$ on C with colors $\langle\chi_1, \chi_2, \ldots, \chi_k\rangle$, respectively, by an arc $v_i\,v_j$, $i \ne j$, of D and obtain thereby a Hamiltonian cycle in D.

To prove this, we have to show by contradiction that the sequence $\langle\varepsilon_1\,\varepsilon_2\ldots\varepsilon_k\rangle$ of edges is identified by the path $v_{i,1}\,v_{i,2}\ldots v_{i,k-1}\,v_{i,k}\,v_{j,1}$ of vertices of K_n^c. By the way K_n^c is colored, if this identity breaks off somewhere, this must arise either on ε_1 or on ε_k.

Suppose that ε_1 is not the edge $v_{i,1}\,v_{i,2}$. We then distinguish five subcases depending upon ε_1.

(*a*) $\varepsilon_1 = v_{j,p}\,v_{i,2}$, $i \ne j$, $1 \le i, j \le n'$, $3 \le p \le k-1$; since the vertex $v_{j,p}$ is non-adjacent to an edge colored by χ_k in K_n^c, we obtain a contradiction;

(*b*) $\varepsilon_1 = v_{j,k}\,v_{i,2}$, $i \ne j$, $1 \le i, j \le n'$; since ε_1 precedes an edge, say ε'_k, with color χ_k on C, by our construction we have $\varepsilon'_k = v_{j,k}\,v_{p,1}$, $1 \le p \le n'$; however, $v_{p,1}$ is non-adjacent to an edge with color χ_{k-1}, a contradiction;

(*c*) $\varepsilon_1 = v_{j,2}\,v_{i,2}$, $i \ne j$, $1 \le i, j \le n'$; since ε_1 precedes an edge ε'_k, with color χ_k on C, we have $\varepsilon'_k = v_{j,2}\,v_{p,1}$, $j \ne p$, $1 \le p \le n'$; however, as in case (*b*), we can see that $v_{p,1}$ is non adjacent to an edge with color χ_{k-1}, once more a contradiction;

(*d*) $\varepsilon_1 = xv_{i,3}$, $1 \le i \le n'$, where $x \in \{v_{j,p} | i \neq j,\ 1 \le j \le n',\ 1 \le p \le k-1\}$; in this case, we have a contradiction, since in order to complete our colored cycle, we have to go through the edge $v_{i,3}\, v_{i,2}$, while $v_{i,2}$ is non-adjacent to an edge with color χ_3;

(*e*) $\varepsilon_1 = v_{i,2}\, v_{i,p}$, $3 \le p \le k$; then, necessarily, $v_{i,p}\, v_{i,2} \ldots v_{i,p-1} \in C$; then, the other edge of C adjacent to $v_{i,p-1}$ has to be colored by χ_{p-1}, and such an edge must not be edge $v_{i,p-1}\, v_{i,p}$ (otherwise, the Hamiltonicity of C collapses); on the other hand, by the construction K_n^c, there is no edge, other than $v_{i,p-1}\, v_{i,p}$, incident to $v_{i,p-1}$ and colored by χ_{p-1}, a contradiction.

Let us now suppose that the identity breaks off on ε_k; so, the edge ε_k is an edge $v_{i,1}\, v_{j,2}$, $i \neq j$; then, it is easy to see that, since $v_{j,2}$ is not adjacent to an edge colored by χ_{k-1}, this case can never occur on the hypothesis that C is $(\chi_1\, \chi_2 \ldots \chi_k)$ Hamiltonian. ■

From theorems 4 and 5, we get the following concluding theorem.

THEOREM 6: *Deciding if an edge-colored complete graph K_n^c admits a $(\chi_1\, \chi_2 \ldots \chi_k)$ Hamiltonian cycle is NP-complete for $k > 2$ (and n a multiple of k).*

Also, an immediate consequence of theorem 5 is the following corollary.

COROLLARY 1: *The problem of finding a longest alternating cycle or path with prescribed order in an edge-colored complete graph is NP-hard.*

By using arguments similar to those of the proof of theorem 5, we may prove the following result on Hamiltonian paths.

THEOREM 7: *The problem: "given two positive integers p and k, $k \ge 4$ and a k-edge colored complete graph K_n^c such that $n = kp + 1$, does K_n^c contain a Hamiltonian $(\chi_1\, \chi_2 \ldots \chi_k)$ path P (resp. a Hamiltonian $(\chi_1\, \chi_2 \ldots \chi_k)$ path P' with specified extremities)?" is NP-complete.*

In fact, even if the *ordering prerequisite* is relaxed, the Hamiltonicity problem remains NP-hard, provided that a *frequency* on the occurrence of the colors is maintained.

THEOREM 8: *The problem PF: "given two positive integers p and k, $k \ge 3$ and a k-edge colored complete graph K_n^c such that $n = kp$, does K_n^c contain an alternating Hamiltonian cycle C such that each color appears at least p times on C?" is NP-hard.*

Proof: Let us suppose that a polynomial algorithm $\mathcal{A}$ solves PF. We can deduce that DHP (where P stands for path), restricted to assymetric digraphs, can be solved in polynomial time (we note DHP, even restricted to assymmetric digraphs, is NP-complete).

Consider an instance $G = (V, E)$ of DHP and label its vertices by $1, 2, \ldots, n$. If ij is an arc of G, then color the edge ij of the complete graph under construction by j. For each color $k \in \{1, 2, \ldots, n\}$, color the edges of $\overline{G} = (V, (V \times V \backslash E))$ by k and apply $\mathcal{A}$ on the so produced instance $G \cup \overline{G}$ (the complete graph on $|V|$ vertices), of PF (denoted by K_n^c) with $p = 1$.

Suppose that an alternating Hamiltonian cycle C is found by $\mathcal{A}$ on K_n^k. Then, since C uses at most one edge of $\overline{G}$, C corresponds exactly to a Hamiltonian path of G.

Conversely, assume that G contains a Hamiltonian path H and let k be the first vertex of H. Then, by coloring $\overline{G}$ by k, $\mathcal{A}$ yields a alternating Hamiltonian cycle for K_n^k such that each color appears exactly once. ■

LEMMA 5: *Consider the problem Π_1: "given a 3-edge-colored complete graph K_n^c and $e = xy$ an edge of color χ_1 in K_n^c, does there exist a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle C in K_n^c, such that e appears in C with x adjacent to an edge of color χ_3 in C?"; Π_1 reduces to Π.*

Proof: We show that starting from an instance K_n^c of Π_1, we can construct an instance $\hat{K}_n^c$ of Π such that, if $\hat{K}_n^c$ admits a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle $\hat{C}$, then K_n^c admits a Hamiltonian $(\chi_1 \chi_2 \chi_3)$ cycle C such that $e = xy$ (colored by χ_1) appears in C with x adjacent (in C) to an edge of color χ_3.

Given the graph K_n^c, we replace the edge $e = xy$ by five new vertices x, x_1, x_2, x_3 and y, then, complete the graph by adding all the missing edges.

We color the new edges as follows:

$$\chi(x_1 z) = \chi_2, \qquad \forall z \in V(K_n^c) \backslash \{x, y, x_1, x_2, x_3\},$$

$$\chi(x_2 z) = \chi_1, \qquad \forall z \in V(K_n^c) \backslash \{x, y, x_1, x_2, x_3\},$$

$$\chi(x_3 z) = \chi_3, \qquad \forall z \in V(K_n^c) \backslash \{x, y, x_1, x_2, x_3\},$$

$$\chi(x x_1) = \chi_1, \quad \chi(x x_2) = \chi_1, \quad \chi(x x_3) = \chi_2, \quad \chi(x y) = \chi_3,$$

$$\chi(x_1 x_2) = \chi_2, \quad \chi(x_1 x_3) = \chi_1, \quad \chi(x_1 y) = \chi_2,$$

$$\chi(x_2 x_3) = \chi_3, \quad \chi(x_2 y) = \chi_1, \quad \chi(x_3 y) = \chi_1.$$

In the so constructed graph $\hat{K}_n^c$, it suffices to show that any $(\chi_1\,\chi_2\,\chi_3)$ Hamiltonian cycle $\hat{C}$ contains the vertex-sequence $x\,x_1\,x_2\,x_3\,y$; this property means that, starting from $\hat{C}$, we can obtain a $(\chi_1\,\chi_2\,\chi_3)$ Hamiltonian cycle C in K_n^c by just replacing the vertex-sequence $x\,x_1\,x_2\,x_3\,y$ of $\hat{C}$ by the edge $e = xy$ (colored by χ_1).

Let us suppose that $xx_1 \notin \hat{C}$. Then, given that x_1 has only two adjacent colors χ_1 and χ_2 and, moreover, the only edge colored by χ_1 adjacent to x_1, is the edge $x_1\,x_3$ (the edge xx_1 being excluded), one can conclude that $x_1\,x_3 \in \hat{C}$. So, let us suppose that $zx_1\,x_3\,z' \in \hat{C}$, where $\chi\,(zx_1) = \chi_2$, $\chi\,(x_1\,x_3) = \chi_1$ and $\chi\,(x_3\,z') = \chi_3$, and $\{z,\,z'\} \subseteq V\,(\hat{K}_n^c)\backslash\{x,\,x_2\}$. If this is the case, then all of the edges adjacent to x_2 are colored by χ_1 and, consequently, vertex x_2 cannot make part of any alternating Hamiltonian cycle of $\hat{K}_n^c$; consequently, since $\chi\,(x_1\,x_2) = \chi_2$ and $\chi\,(x_2\,x_3) = \chi_3$, one of the z, z' must be x_2; (*a*) suppose that $z' = x_2$, so, $zx_1\,x_3\,x_2 \in \hat{C}$; then, the other edge of $\hat{C}$ incident to x_2 has to be colored by χ_2, and the only edge so colored is $x_2\,x_1$; but, in this case, $zx_1\,x_3\,x_2\,x_1 \in \hat{C}$, a contradiction since $\hat{C}$ is supposed Hamiltonian; (*b*) on the other hand, if we suppose that $z = x_2$, then with arguments exactly similar to the ones of case (*a*), we can conclude that $x_3\,x_2\,x_1\,x_3\,z' \in \hat{C}$, another contradiction. So, $xx_1 \in \hat{C}$.

Let us now suppose that $x_1\,x_2 \notin \hat{C}$. Then, since the remaining (except $x_1\,x_2$) edges adjacent to x_2 are colored either by χ_1 or by χ_3, and, moreover, the only edge colored by χ_3 is the edge $x_2\,x_3$, we can suppose that $zx_2x_3 \in \hat{C}$, where $\chi\,(zx_2) = \chi_1$, $\chi\,(x_2\,x_3) = \chi_3$, and $z \in V\,(\hat{K}_n^c)$; then, the only edge adjacent to x_3 and colored by χ_2 is edge $x_3\,x$; so, $zx_2\,x_3\,x \in \hat{C}$ and, by the previous discussion, $zx_2\,x_3\,xx_1 \in \hat{C}$; consequently, in $\hat{C}$, the other edge adjacent to x_1 has to be colored by χ_3 and such an edge does not exist; we conclude then that $x_1\,x_2 \in \hat{C}$.

Now, it is easy to see that, since $xx_1\,x_2 \in \hat{C}$, the edge of $\hat{C}$ adjacent to x_2 has to be colored by χ_3 and the only feasible (from Hamiltonicity point of view) edge adjacent to x_2 in $\hat{K}_n^c$ and colored by χ_3 is the edge $x_2\,x_3$; so, $xx_1\,x_2\,x_3 \in \hat{C}$.

With the same arguments, the edge $x_3\,y \in \hat{C}$.

So, $xx_1\,x_2\,x_3\,y \in \hat{C}$ and this concludes the proof of lemma 5. ■

The above lemma 5 shows that we can force the $(\chi_1\,\chi_2\,\chi_3)$ Hamiltonian cycle to go through a given set A of edges of an edge-colored complete graph in a certain order.

THEOREM 9: *Consider the problem Π': "given a 3-edge-colored complete graph K_n^c and a subset $S \subset V(K_n^c)$ of six vertices of K_n^c, does there exist a $(\chi_1\,\chi_2\,\chi_3)$ cycle in K_n^c containing the vertices of S?"; Π' is NP-hard.*

Proof: Consider first the following *local cycle problem* (LC, [9]) where, *given a directed graph G and two specified vertices a and b of G, we search if there exists a cycle through the vertives a and b in G.*

We know that LC is NP-complete. Moreover, it is easy to see that LC is NP-complete even if G is bipartite (it suffices to add an intermediate vertex on each arc of G).

We consider now the following decision problem LC′:

Instance: A directed bipartite graph G, two vertices a and b of G, and an arc au of G.

Question: Does there exist a cycle of G through a and b using arc au?

Clearly, LC′ is NP-hard because if we have a polynomial algorithm $\mathcal{A}$ for LC′, then applying $\mathcal{A}$ on every instance $(G,\ a,\ b,\ au)$ (for every arc au of G), we can solve LC for every instance $(G,\ a,\ b)$ where G is bipartite.

We are going now to reduce LC′ to Π'. Let $G = ((X,\ Y,\ A),\ a,\ b,\ au)$ be an instance of LC′ (X and Y are the color classes and A is the arc-set of G). We can suppose without loss of generality that $a \in X$.

Construct a 3-edge-colored complete graph K_n^c in the following way:

1) $V K_n^c = X \cup Y \cup Y'$, where $|Y'| = |Y|$;

2) all edges in X, Y, Y' are colored by χ_3, χ_1 and χ_2, respectively;

3) for every arc xy in $A(G)$ (where $x \in X$ and $y \in Y$), color the corresponding edge xy of K_n^c by χ_1;

4) for every pair $(x,\ y)$ of vertices of G, such that $x \in X$, $y \in Y$ and $xy \notin A(G)$, color the edge xy of K_n^c by χ_3;

5) add a perfect matching M in $(Y,\ Y')$ and color its edges by χ_2; color all the other edges between $(Y,\ Y')$ by χ_1; for every vertex y of Y, we denote by y' its mate ([8]) with respect to the matching M;

6) for every vertex $y \in Y$, if arc $yx \in A(G)$ $(x \in X)$, then color the edge $y'x$ by χ_3; the rest of the (non-colored) edges incident to y' are colored by χ_2;

([8]) Given a matching M and an edge $xy \in M$, we consider that x (resp., y) is the mate of y (resp., x).

7) replace the particular arc au of G by the component D of 5 vertices designed in the proof of lemma 5 and set $S = V(D) \cup \{b\}$. More precisely, we add three vertices x_1, x_2, x_3 on the edge au with the colors indicated in lemma 5, namely: for every $z \in V(K_n^c)\setminus\{a, u, x_1, x_2, x_3\}$, $\chi(x_1 z) = \chi_2$, $\chi(x_2 z) = \chi_1$, $\chi(x_3 z) = \chi_3$ and $\chi(ax_1) = \chi(ax_2) = \chi_1$, $\chi(ax_3) = \chi_2$, $\chi(au) = \chi_3$, $\chi(x_1 x_2) = \chi(x_1 u) = \chi_2$, $\chi(x_1 x_3) = \chi_1$, $\chi(x_2 x_3) = \chi_3$ and $\chi(x_2 u) = \chi(x_3 u) = \chi_1$ and we set $S = \{a, x_1, x_2, x_3, u, b\}$. This completes the description of the instance of Π'.

Now, we claim that G admits a cycle containing a and b and passing through the arc au if and only if K_n^c admits a $(\chi_1 \chi_2 \chi_3)$ cycle containing S.

In fact, let C be a cycle of G containing a and b and passing through the arc au. Then, replacing every sequence xyx' of C (where $x, x' \in X$ and $y \in Y$) by the sequence $xyy'x'$ (where y' is the mate of y with respect to M), and replacing the particular edge au of C by the sequence $ax_1 x_2 x_3 u$, we obtain a $(\chi_1 \chi_2 \chi_3)$ cycle of K_n^c containing S.

Conversely, let C' be a $(\chi_1 \chi_2 \chi_3)$ cycle of K_n^c containing S. From the proof of lemma 5, we know that C' contains necessarily the sequence $ax_1 x_2 x_3 u$ (because the only property used in the proof of lemma 5, is that the involved cycle passes through all of the vertices of the component of the lemma, here denoted by a, x_1, x_2, x_3 and u). Let us write $C' = (ax_1 x_2 x_3 u) z_1 z_2 \dots z_m$; then $\chi(uz_1) = \chi_2$ and $z_1 = u'$ $(u' \in Y')$ because uu' is the only edge of color χ_2 incident to u; therefore, $uz_2 \in A(G)$ because $\chi(z_1 z_2) = \chi_3$. Now, repeating this argument m times, we find that C' has the form $C' = (ax_1 x_2 x_3 u) u' a_1 b_1 b_1' \dots a_k b_k b_k'$, where $a_i \in X$, $b_i \in Y$ and b_i' is the counterpart of b_i in Y'). Hence, by definition of the colors in K_n^c, the cycle $C = aua_1 b_1 \dots a_k b_k$ is a cycle of G containing a and b and passing through the arc au as claimed. ■

We shall conclude this section with the two following open problems.

PROBLEM 1: *What is the complexity of finding an alternating Hamiltonian cycle in a k-edge-colored complete graph, $k \geq 3$?*

PROBLEM 2: *Let x and y be two specified vertices in a k-edge-colored complete graph, $k \geq 2$. What is the complexity of finding an alternating Hamiltonian path between x, y in such a graph?*

Input: a complete k-partite graph $G = (V, E)$ with an even number of vertices and with vertex classes $G_1, G_2, \ldots, G_k$ satisfying condition (**a**): $|G_i| \leq \sum_{j \neq i} |G_j|$, $1 \leq i, j \leq k$.

Output: a perfect matching M of G.

1. order the classes $G_1, G_2, \ldots, G_k$ in decreasing order;
2. $M \leftarrow \emptyset$;
3. **while** M does not saturate all vertices of G **do**
   put $M \leftarrow M \cup e$ where $e = xy$ is an edge between the two first classes;
   delete the vertices x and y, as well as all of their incident edges;
   define appropriately a new decreasing ordering of the classes of the obtained graph
   endwhile

Procedure 2. Matching procedure.

4. EULERIAN ALTERNATING CYCLES AND PATHS

In this section, we study the existence of alternating Eulerian cycles in edge-colored graphs. In what follows, a cycle (resp., a path) is not necessarily elementary, *i.e.,* it goes through an edge once, but it can go through a vertex many times.

In view of theorem 10 and algorithm 2, we establish procedure 2 that finds a perfect matching in a specified family of complete k-partite graphs.

Concerning the completeness of procedure 2, we first notice that G admits a perfect matching since condition (**a**) guarantees that G satisfies Tutte's well known condition ([5] page 76, theorem 5.4). Now, in order to prove the correctness of the procedure, it suffices to show that after each step the new obtained graph has always a perfect matching, *i.e.*, it satisfies (**a**).

The proof is by induction on n. It is clear that, for $n = k$, the procedure is correct.

Suppose that it is correct for $n - 2$; we shall prove its correctness for n.

Assume that when we delete an edge xy, we find a new graph which admits a class G'_r satisfying $|G'_r| > \sum_{i \neq r} |G'_i|$. Then, we had either $|G_r| = \sum_{i \neq r} |G_i|$, or $|G_r| = \sum_{i \neq r} |G_i| - 1$. Now, if $|G_r| = \sum_{i \neq r} |G_i|$, then x belongs to $|G_r|$ and y belongs to another class, a contradiction to our assumption that $|G'_r| > \sum_{i \neq r} |G'_i|$. On the other hand, if $|G_r| = \sum_{i \neq r} |G_i| - 1$, we have a contradiction since n is even.

The sorting in step 1 can be performed in $O(|V| \log |V|)$. In step 3, the deletion of an edge entails the decrease of the cardinalities of only two classes by one. The new sorting can be performed within $O(\log |V|)$ by using a heap (in fact we have, eventually, to change the place of the two classes, the cardinalities of which have been changed), and this reordering will be performed at most $\frac{|V|}{2}$ times, so the complexity of this operation for the total of the executions of the **while** loop calls will be of $O(|V| \log |V|)$. On the other hand, the deletion of the edges incident to the selected one, takes time $O(|E|)$, once more time, for the whole of the iterations of the **while** loop. Thus, the total time is of $O(\max\{|E|, |V| \log |V|\})$.

Input: an edge colored graph G^c satisfying the hypotheses of theorem 10.
Output: an alternating Eulerian cycle.

1. **for** every vertex v of G^c **do** apply procedure 2 to G_v **endfor**
   $i \leftarrow 1$;
   $m \leftarrow 1$;
2. $P \leftarrow y_0 y_1$
   while there exists $M_{y_m}(y_{m-1} y_m) \notin E(P)$ **do**
   $P \leftarrow P \cup \{M_{y_m}(y_{m-1} y_m)\}$;
   $m \leftarrow m + 1$;
   mark that y_m belongs to P
   endwhile
   $C_i \leftarrow P$;
   $G^c \leftarrow G^c \backslash E(C_i)$ (*i.e.*, we delete the edges but not their extremities)
3. **if** $E(G^c)$ is not empty, **then** find an edge wz in $E(G^c)$ **endif**
   $i \leftarrow i + 1$;
   $y_0 \leftarrow w$;
   $y_1 \leftarrow z$;
   go to step 2;
4. we stack a cycle and we start walking around it until a vertex that is an intersection point with another cycle is found;
   we stack the new cycle and we start walking now around it by preserving the alternance of colors on the point we have changed the cycle we are walking (we notice here that this preservation is always possible);
   we continue this procedure until a cycle is entirely walked out in which case is unstacked;
   we continue in this way until the stack becomes empty;
   The above walk determines an Eulerian cycle.

Algorithm 2.

THEOREM 10: *Let G^c be an edge colored graph of order n. Then, there exists an alternating Eulerian cycle in G^c if and only if it is connected, for each vertex x and for each color i, the total degree of x is even,* *and,* $\chi_i(x) \leq \sum_{j \neq i} \chi_j(x)$. *Moreover, algorithm* 2 *finds such a cycle in* $O(\max\{n|E|, n^2 \log n\})$.

Proof: Let us notice that the necessary condition is obvious.

Let us prove the sufficient one.

For every vertex v and every edge e incident to v, we will associate an edge denoted by $M_v(e)$ incident to v such that $\chi(e) \neq \chi(M_v(e))$. This association guarantees that each time we visit v through the edge e, we can leave v through $M_v(e)$. In order to determine such an association, for each vertex v, we define a new graph G_v such that the vertices of G_v are the edges adjacent to v. Furthermore, two vertices are connected in G_v if their corresponding edges in G^c have different colors. It is clear that associating e to $M_v(e)$ is the same as finding a *perfect matching* in G_v. We remark that G_v verifies condition (**a**) and that G_v is a complete k-partite graph. Consequently, procedure 2 produces always a perfect matching in G_v.

It is easy to see that algorithm 2 is correct for small values of $|E(G^c)|$. Let us now prove, by induction on $|E(G^c)|$, that the algorithm comes up with an Eulerian cycle. Applying steps 1 and 2 (in step 2, we suppose that $y_0 y_1$ is an edge of G^c) of the algorithm, we obtain an alternating cycle C. If $E(C) = E(G^c)$, we have the desired walk. If not, then it can easily be seen that the induction hypothesis is preserved in each connected component of $G = (V, (E \backslash E(C))$. So, each connected connected component of G^c admits an alternating Eulerian cycle. Consequently, after step 3, the cycles $C_1, C_2, \ldots, C_i$ represent an edge-decomposition of G^c. In step 4, we clearly visit all cycles $C_1, C_2, \ldots, C_i$, since G^c is connected. Finally, at the end of step 4, we find an Eulerian cycle, since by stacking-unstacking a cycle, we preserve that its edges are visited only once.

Concerning the complexity of the algorithm, step 1 uses $n = |V|$ times procedure 2, requiring thus a total time of $O(\max\{n|E|, n^2 \log n\})$. Steps 2 and 3 are both performed in $O(|E|)$ steps, since each edge is visited once. In step 4, since the walk of a cycle is performed following its edges, the total time for all of the walks will take $O(|E|)$; on the other hand, each cycle will be treated at most 2 times by the stacking-unstacking operation; so, the total time complexity of step 4 will be of $O(|E|)$. It follows that the whole time complexity of algorithm is of $O(\max\{n|E|, n^2 \log n\})$. ■

A similar algorithm can be used to obtain the following theorem in the case of directed edge-colored graphs.

THEOREM 11: *Let D^c be an edge-colored digraph. Then there exists an alternating Eulerian walk in D^c if and only if* (i) *D^c is strongly connected,* (ii) *for each vertex x and for each color i, $d^+(x) = d^-(x)$ and $d_i^+(x) \leq \sum_{j \neq i} d_j^-(x)$, where for every vertex x of D^c, $d^+(x)$ (resp., $d^-(x)$) denotes the external (resp., internal) degree of x and $d_i^+(x)$ (resp., $d^-(x)$) denotes the external (resp., internal) degree of color i of x.*

5. CONCLUSIONS

The starting point of our work was Bánkfalvis' theorem [1] mentioned in the introduction. In fact, in [1] the given characterization is not algorithmic. In this paper, we have shown how to exploit their results to obtain polynomial algorithms for finding alternating Hamiltonian cycles and paths in 2-edge-colored complete graphs. As a byproduct, we obtain an efficient algorithm for the Hamiltonian circuit problem in bipartite tournaments. Moreover, we have studied the case of k-edge-colored complete graphs ($k \geq 3$) and we have established a number of NP-completeness results when additional conditions on the frequency of occurrence of the colors in the Hamiltonian cycles and paths are imposed. The general problem (when no frequency constraints are imposed) remains open. However, our feeling is that this latter problem is computationally "easy".

Finally, in the last section of the paper, we have studied the problem of the existence of alternating Eulerian cycles in edge-colored graphs. We have given a polynomial characterization of the existence of such cycles and, moreover, a constructive proof for this characterization.

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