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GENERALIZED OPTIMAL SEARCH PATHS FOR CONTINUOUS UNIVARIATE RANDOM VARIABLES (*)

by Zaid T. BALKHI ([1])

Abstract. — *The purpose of this paper is to solve the Generalized Linear Search Problem for continuous random variables. This problem is concerned with finding a target located on a line. The position of the target is given by the value of a random variable which has a prior distribution. A searcher starts looking for the target from some point, moving along with an upper bound on his speed. The target being sought for might be in either direction from the starting point, so the searcher needs to change his direction many times before he attains his goal. with minimality of average time to target detection as the measure of optimality of search paths, we have obtained algorithms that find such paths for those targets which have absolutely continuous distributions. More detailed properties of optimal search paths are, also, studied. One of the main results is that: these search paths are not minimal, in some cases, for some types of target distributions.*

Keywords : Linear search; Optimization; Normal and bimodal normal distributions.

Résumé. — *Dans ce papier nous étudions le problème de la recherche linéaire généralisée dans le cas des variables aléatoires continues. Ce problème consiste à trouver un objet localisé sur une ligne. La position de l'objet est donnée par la valeur de la variable aléatoire qui répond à la loi de probabilité. Un chercheur commence à chercher son objet à partir d'un certain point en se déplaçant sur la ligne avec une vitesse ne dépassant pas une certaine borne. Vu que l'objet peut être situé à droite ou à gauche du point de commencement, le chercheur a besoin de changer sa direction plusieurs fois avant de détecter l'objet. En considérant le temps moyen minimum pour détecter l'objet comme mesure de l'optimalité du chemin de recherche, nous avons obtenu des algorithmes qui permettent de trouver tels chemins pour les objets qui ont des lois de probabilité absolument continues. Des autres propriétés des chemins optimaux sont également étudiées. Un des résultats fondamentaux est: ces chemins ne sont pas minimaux, dans certains cas, pour quelques types de lois de probabilité.*

1. INTRODUCTION

The following problem has been considered in the literature. A target is assumed to be located on a line. Its position x is given by the value of a random variable X, which has a known (or unknown) distribution F.

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A searcher starts looking for the target from some point a_0 on the line ($|a_0| < \infty$), moving along the line with an upper bound on his speed. The target being sought for might be in either direction from the starting point a_0, so the searcher would conduct his search in the following manner: Start at a_0 go to the left (right) as far as a_1. If the target is not found there, turn back and explore the right (left) part of a_0 as far as a_2. If the target is still not found, retrace the steps again to explore the left (right) part of a_1 as far as a_3, and so fourth until the target be detected. Let us define c and d as follows

$$c = \inf\{x: F(x) > 0\}, \qquad d = \sup\{x: F(x) < 1\}.$$

Then a search path may, in general, be represented by a sequence $A = \{a_i;\ i \geqq 0\}$ with $a_{2i} \to c$ and $a_{2i-1} \to d$ as $i \to \infty$, or vice versa. Figure 1 gives an illustration of such search paths. Observe that the two search paths depicted in Figure 1 are duals and of sequential type. Moreover it is to be noted that these two search paths will give us several possible cases of search when we consider all relative positions, of the starting point a_0, to the origin (*see* [2]).

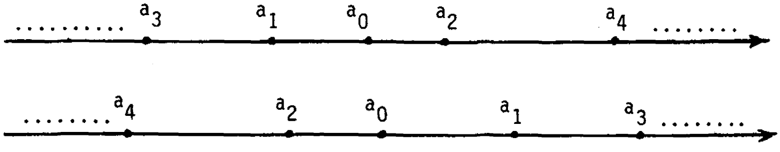

Figure 1.

The problem is of interest because it may arise in many real world situations such as:

(i) Searching for lost persons or objects on roads (Beck [5], Beck and Newman [7], and Rousseeuw [13]).

(ii) Searching for a faulty unit in a large linear system such as electrical power lines, telephone lines, petrol or gas supply lines, and mining systems (Balkhi [2]).

(iii) Estimating a distribution parameter whose probability locations are given. The parameter, here, may be regarded as a target to be searched for.

In the above examples, and in many others of this type, (*see* [2]) the target distribution is given or to be estimated. It is possible, however, to study this problem as a game between the searcher and the target (*see* [7] and [11]).

For any of such problems the path length of some search path $A=\{a_i;\ i\geqq 0\}$, from the starting point a_0 until reaching the target x, is considered as the cost of the search. By virtue of the randomness of the position of x, it is clear that the cost of the search is, also, a random variable. The aim of the search is, then, to minimize its expected cost. Any search path that fulfils this aim is referred to as an optimal search path (O.S.P.). For all possible cases of search, the solution of this problem consists of two stages. The first is the establishment of the existence of (O.S.P.)'s. This stage has been, in fact, completed by many authors. A review of their results will be the subject of the next section. The second stage is the construction of (O.S.P.)'s. Concerning the case $a_0=0$ and the second stage Beck [6] and Franck [10] have indicated that a recursive formula for the entries a_i's of a minimizing search path is available under proper differentiability conditions on the expected cost. But the solution there has not been given in a useful sense. Rousseeuw [13] has done some investigations about (O.S.P.)'s for the case $a_0=0$. But his wotk was concentrated on the Normal distribution and its analogous symmetric distributions only. Besides to the case $a_0=0$, there are, however, many other cases of possible search. Some of these cases have been previously considered in Balkhi [1]. Later Balkhi [2] has shown that there are only five cases of possible search one of which is the case $a_0=0$. The other four cover all possible cases of search for which $a_0\neq 0$. The work of [2], in fact, has focussed on giving sufficient conditions under which there exists an (O.S.P.) for each possible case of search.

In this paper the construction of (O.S.P.)'s, for the only five possible cases of search considered in [2] and for regular target distributions (*see* definition 2.2), will be introduced in a unified way. The main properties of (O.S.P.)'s will be given some emphasis. An algorithm by which we can calculate (O.S.P.)'s together with an illustrative examples are also introduced. The numerical results of these examples will then show that some of the possible cases of search is better than some others in the sense that they give less expected cost. Justifying, thus, the generalization of this problem that have been previously considered by this author.

2. LITERATURE REVIEW

Authors in [5] to [8], [10], [11] and [13] have dealt with the case $a_0=0$ only. Under the name "The Generalized Linear Search Problem" (GLSP) Balkhi [2] has introduced this problem in more general approach by considering any starting point $a_0\,(|a_0|<\infty)$ other than the origin. The additional assumption that the number of elements, of a search path $A=\{a_i;\ i\geqq 0\}$, between the origin and a_0, is finite, is also presumed in [2] (This assumption may be justified by [2] Lemma 3.8). It is shown, then, that we have only five possible cases of search, one of which is the case $a_0=0$. These cases are referred to as case (k); $k=0$, 1, 2, 3 and 4 [case (0) is the case for which $a_0=0$]. The class of search paths in case (k) is denoted by Q_k; $k=0$, 1, 2, 3 and 4. With the conventions that $a_0\neq 0$ for $k=1$, 2, 3 and 4, $a_{-1}=0$ for $k=0$, and $a_0\neq a_1$ for all k (The last assumption is justified by the fact that the searcher needs to move from a_0 to a new point namely a_1, at the outset of his search). Then class Q_k consists of all search paths of the following type

$$\ldots<a_4<a_2<0=a_0<a_1<a_3<a_5<\ldots;\qquad k=0 \tag{2.1}$$

$$\ldots<a_5<a_3<a_1\leqq 0<a_0<a_2<a_4<\ldots;\qquad k=1 \tag{2.2}$$

$$\ldots<a_4<a_2\leqq 0<a_0<a_1<a_3<a_5<\ldots;\qquad k=2 \tag{2.3}$$

$$\ldots<a_5<a_3\leqq 0\leqq a_1<a_0<a_2<a_4<\ldots;\qquad k=3 \tag{2.4}$$

$$\ldots<a_4\leqq 0\leqq a_2\leqq a_0<a_1\leqq a_3<a_5<\ldots;\qquad k=4 \tag{2.5}$$

and their duals which can be obtained by reversing the inequalities in (2.1) through (2.5) (*See* [2] for more details). For a search path $A=\{a_i;\ i\geqq 0\}$ form class Q_k, the expected cost is denoted by $D_k(A, F)$. As has been shown in [2] we have

$$D_k(A, F)=M(F)+\Delta_k(A, F);\qquad k=0,\ 1,\ 2,\ 3 \text{ and } 4 \tag{2.6}$$

where $M(F)=\int_c^d |x|\,dF(x)$ (The first absolute moment of F).

$$\Delta_0(A, F)=2\sum_{i=1}^{\infty}|a_i|\{1-\operatorname{sign}(a_i)[F(a_i)-F(a_{i-1})]\}. \tag{2.7}$$

$$\Delta_k(A, F)=-2\left|\int_0^{a_0}|x|\,dF(x)\right|-(-1)^k.\ |a_0|+2\sum_{i=1}^{\infty}|a_i| \times\{1-\operatorname{sign}(a_i)[F(a_i)-F(a_{i-1})]\};\qquad k=1,\ 2 \tag{2.8}$$

$$\Delta_k(A, F)=-2\left|\int_0^{a_0}|x|\,dF(x)\right|-(-1)^k.\ |a_0|-4|a_{k-2}| +2\sum_{i=1}^{\infty}|a_i|\{1-\operatorname{sign}(a_i)[F(a_i)-F(a_{i-1})]\};\qquad k=3,\ 4 \tag{2.9}$$

(*see* Balkhi [2], theorems 2.1, 2.3 and remark 2.4). An (O.S.P.) from class Q_k is given formally by the following definition:

DEFINITION 2.1: Let

$$m_k = \inf\{D_k(A, F): A \in Q_k\}; \qquad k=0, 1, 2, 3 \text{ and } 4 \tag{2.10}$$

If $A^* \in Q_k$ is such that $m_k = D(A^*, F)$, then A^* is said to be an (O.S.P.) from class Q_k; $k=0, 1, 2, 3$ and 4.

The existence of (O.S.P.)'s in class Q_0 has been established in Beck [5] and Franck [10] by assuming different (but not equivalent) conditions on the underlying distribution F, that give necessary and sufficient conditions for such existence. For the (GLSP) considered here, Balkhi [2] proved the following two theorems.

THEOREM 2.1: *There exists a search path from class Q_k; $k=0, 1, 3$ and 4, with finite expected cost if and only if $M(F) < \infty$.*

THEOREM 2.2: *Let $F^-(0)$, $F^+(0)$ denote the left hand and right hand derivatives of F at zero respectively. If $M(F) < \infty$, then there exists an (O.S.P.) from class Q_k if*

(i) *For $k=0, 1$, at least one of $F^-(0)$, $F^+(0)$ is finite.*

(ii) *For $k=2, 3$ and 4, both $F^-(0)$, $F^+(0)$ are finite.*

Thus the existence of (O.S.P.)'s for $k=0, 1$ ($k=2, 3$ and 4) is guaranteed under the finiteness of $M(F)$ and $F^-(0)$ or $F^+(0)$ ($M(F)$, $F^-(0)$, and $F^+(0)$). Under some special assumptions which include the above ones Fristedt and Heath [11] proved the following theorem.

THEOREM 2.3: *If $M(F) < \infty$, then there exists an (O.S.P.) from class Q_0 with constant speed equal to 1.*

Theorem 2.3 does not have special assumptions concerning class Q_0 per se, so this theorem holds for any of the classes Q_k; $k=0, 1, 2, 3$ and 4. Thus, for all classes Q_k we might consider the expected cost of the search to be either $D_k(A, F)$ or $T_k(A, F)$, where $T_k(A, F)$ denotes the expected searching time using the search path $A=\{a_i; i \geq 0\}$ from class Q_k *i. e.*

$$D_k(A, F) = T_k(A, F) = M(F) + \Delta_k(A, F); \qquad k=0, 1, 2, 3 \text{ and } 4. \tag{2.11}$$

The following definition is often needed in the sequal.

DEFINITION 2.2: If the target distribution F is absolutely continuous with strictly positive density f, then F is said to be regular.

Of special interest are symmetric target distribution *i. e.*

$$F(-x)=1-F(x), \qquad \forall x \in \mathbb{R}. \tag{2.12}$$

For this type of distributions (*i. e.* symmetric) then more appropriate formulas, for theoritical and computational purposes, are available for the expected cost. To see this let $A=\{a_i;\ i \geqq 0\} \in Q_k$; $y_i \equiv |a_i|$, $i \geqq 0$. If F is symmetric then

$$1-\operatorname{sign}(a_i)[F(a_i)-F(a_{i-1})]=2-[F(y_i)-F(y_{i-1})]; \tag{2.13}$$
$$i \geqq k \quad \text{for} \quad k=1,\, 2,\, 3 \text{ and } 4 \qquad \text{and } i \geqq 1 \quad \text{for} \quad k=0.$$

Let $Y=\{y_i;\ i \geqq 0\}$, then from (2.13) and our hypotheses we can easily see that $\Delta_k(Y, F)=\Delta_k(A, F)$; $k=0$, 1, 2, 3 and 4, where

$$\Delta_0(Y, F)=2\sum_{i=0}^{\infty}[1-F(y_i)](y_i+y_{i+1})=y_1+2\sum_{i=1}^{\infty}[1-F(y_i)](y_i+y_{i+1}) \tag{2.14}$$

$$\begin{aligned}\Delta_k(Y, F)=&-2\int_0^{y_0}|x|\,dF(x)\\ &-(-1)^k y_0+2y_1[1-F((-1)^{k+1}y_0)]\\ &+2\sum_{i=1}^{\infty}[1-F(y_i)](y_i+y_{i+1}); \qquad k=1,\, 2\end{aligned} \tag{2.15}$$

$$\begin{aligned}\Delta_k(Y, F)=&-2\int_0^{y_0}|x|\,dF(x)\\ &-(-1)^k y_0-2y_{k-2}[1+F(y_{k-2})-F(y_{k-3})]\\ &+2(k-3)y_1[1+F(y_0)-F(y_1)]+2y_{k-1}F(y_{k-2})\\ &+2\sum_{i=k-1}^{\infty}[1-F(y_i)](y_i+y_{i+1}); \qquad k=3,\, 4\end{aligned} \tag{2.16}$$

Formulas (2.14) through (2.16) make it possible to disregard the signs of the entries of a search path $A=\{a_i;\ i \geqq 0\}$ by using the equivalent search path $Y=\{y_i;\ i \geqq 0\}$. This, in fact, results in more efficient computational algorithms that calculate the entries of (O.S.P.)'s and the corresponding optimal costs. Moreover, by using (2.14) and (2.15), Balkhi [1] proved the following interesting result (*see* [1] pp. 173-174).

THEOREM 2.4: *If the underlying distribution F is symmetric, and if A is an (O.S.P.) from class Q_k, then*

$$|a_{i+1}| > |a_i| \quad \textit{for all } i \geqq 0; \quad k=0,\ 1,\ 2.$$

Using similar techniques as those used in [1] we can easily show that this theorem holds, also, for $k=3$ and 4 with $i \geqq k-2$. Thus we have

$$|a_{i+1}| > |a_i|;$$

$$i \geqq 0 \quad \text{for } k=0,\ 1 \text{ and } 2 \qquad \text{and} \qquad i \geqq k-2 \quad \text{for } k=3 \text{ and } 4.$$

Thus for symmetric target distributions we can restrict our attention to the search paths $Y=\{y;\ i \geqq 0\}$ for which

$$\text{(2.17)} \quad \begin{cases} & y_{i+1} > y_i; \\ i \geqq 0 \quad \text{for } k=0,\ 1 \text{ and } 2, & \text{and} \qquad i \geqq k-2 \quad \text{for } k=3 \text{ and } 4. \end{cases}$$

Remark 2.1. – There is a kind of scale invariance on the expected cost. For if $A=\{a_i;\ i \geqq 0\}$ is a search path from class Q_k; $k=0$, 1, 2, 3 and 4. And if we define $\lambda A=\{\lambda a_i;\ i \geqq 0\}$, $F_\lambda(x)=F(x/\lambda)$, so that the support of F_λ is $(\lambda c,\ \lambda d)$, then

$$\text{(2.19)} \qquad D_k(\lambda A,\ F_\lambda)=\lambda D_k(A,\ F); \qquad k=0,\ 1,\ 2,\ 3 \text{ and } 4$$

which can be easily seen from (2.6) through (2.9) (*see* also [13] remark 1.1).

Remark 2.1 means that the expected cost of the search does not depend on the type of distribution, but it depends also on its scale parameter. It is, therefore, meaningful to standardize the expected cost by other parameter of the same scale, say by $M(F)$. Relating to this fact Rousseeuw [13] has proved the following theorem.

THEOREM 2.5: *If the underlying distribution F is symmetric and regular, and if A is an (O.S.P.) from class Q_0, then*

$$2 < T_0(A, F)/M(F) < 4.591. \tag{2.18}$$

3. OPTIMAL SEARCH PATHS

(*a*) Critical search paths

As it can be seen from (2.6) through (2.9) the (GLSP) depends on two unknown factors. Those are the target distribution F, and the search path $A = \{a_i;\ i \geqq 0\}$ used by the searcher. Let us assume, from now on, that the target distribution is known. Nevertheless we still face a difficult optimization problem. Because this problem has an infinite number of variables; that is $A = \{a_i;\ i \geqq 0\}$. However, if we assume (from now on) that the target distribution F is also regular and that $M(F)$ is finite. Then the structure of the (GLSP) becomes easy and even simple as we shall see below. But let us first give a pertinent definition and remark.

DEFINITION 3.1: *If* $A = \{a_i;\ i \geqq 0\}$ is a search path from class Q_k such that tha derivative of $\Delta_k(A, F)$ with respect to A does exist and all partial derivatives of $\Delta_k(A, F)$ with respect to rhe a_i's vanish, then A is said to be a critical search path (C.S.P.) from class Q_k; $k = 0, 1, 2, 3$ and 4.

Remark 3.1. – We infer that if $\Delta_k(A, F)$ is differentiable on Q_k then the set of critical search paths from Q_k will contain all of the relative minimal and relative maximal search paths. Of course this set may also contain search paths at which $\Delta_k(A, F)$ does not have relative minimal or maximal search paths. In addition the function $\Delta_k(A, F)$ may have relative extremum at a search path from Q_k at which the derivative of $\Delta_k(A, F)$ with respect to A does not exist or $\Delta_k(A, F)$ may have a relative extremum at a search path which is not an interior point from Q_k. □

Now by the regularity condition on F and the finiteness of $M(F)$, then Theorem 2.2 guarantees the existence of (O.S.P.)'s in each of the classes Q_k; $k = 0, 1, 2, 3$ and 4. If $A = \{a_i;\ i \geqq 0\}$ is a (C.S.P.) from class Q_k, then $\partial \Delta_k(A, F)/\partial a_i$ exist for all pertinent values of i and k, and then

$$\frac{\partial \Delta_k(A, F)}{\partial a_i} = 0 \tag{3.1}$$

$$i \geqq 0 \quad \text{for } k = 1, 2, 3 \text{ and } 4, \qquad \text{and} \qquad i \geqq 1 \quad \text{for } k = 0.$$

Moreover, for the following tupled values of i and k

(3.2) $(k=0,\ 1 \text{ and } 2;\ i \geqq 1)$, $(k=3,\ i \geqq 2)$ and $(k=4;\ i \geqq 3)$.

Then relations (2.7) through (2.9) together with (3.1) give the following results.

$$\frac{\partial \Delta_k(A, F)}{\partial a_i} = 2 \operatorname{sign}(a_i)\{1 - \operatorname{sign}(a_i)[F(a_i) - F(a_{i-1})] - f(a_i)(|a_i| + |a_{i+1}|)\} \tag{3.3}$$

$$|a_{i+1}| = \frac{1 - \operatorname{sign}(a_i)[F(a_i) - F(a_{i-1})]}{f(a_i)} - |a_i|. \tag{3.4}$$

And from our hypotheses we have

$$a_{i+1} = -\operatorname{sign}(a_i).\,|a_{i+1}|. \tag{3.5}$$

Using the same reasoning as applied for the tupeled values of k and i in (3.2), the rest a_i's of a (C.S.P.) $A = \{a_i;\ i \geqq 0\} \in Q_k$ are given by the following relations:

$$|a_1| = 1/2f(a_0) - |a_0|; \qquad k=1 \tag{3.6}$$

$$|a_1| = |a_0| - 1/2f(a_0); \qquad k=3 \tag{3.7}$$

$$|a_1| = 1/2f(a_0) + |a_0|; \qquad k=2,\ 4 \tag{3.8}$$

$$|a_2| = \frac{1 + \operatorname{sign}(a_1)[F(a_1) - F(a_0)]}{f(a_1)} + |a_1|; \qquad k=3 \tag{3.9}$$

$$|a_2| = |a_1| - \frac{1 - \operatorname{sign}(a_1)[F(a_1) - F(a_0)]}{f(a_1)}; \qquad k=4 \tag{3.10}$$

$$|a_3| = \frac{1 + \operatorname{sign}(a_2)[F(a_2) - (a_1)]}{f(a_2)} + |a_2|; \qquad k=4. \tag{3.11}$$

For the signs of these entries we recall, from the hypotheses, that: For $k=1$, a_0 and a_1 have different signs. Whereas for $k=2$, 3 and 4, all the a_i's for which $i \leqq k-1$ have the same sign. Now, as it can be noticed from relations (3.4) through (3.11) and the signs of the a_i's indicated above, then for $k=1$, 2, 3 and 4 we have that a_1 is a function of a_0, and a_{i+1} is a function of a_{i-1} and a_i for all $i \geqq 1$. Hence a_{i+1} is a function of a_0. Thus if we assume that $a_0 = r$, then there exists a function ψ_i such that

$$a_i = \psi_i(r) \quad \text{for all } i \geqq 0, \text{ and } k=1,\ 2,\ 3 \text{ and } 4 \tag{3.12}$$

where $\psi_0(r)=r$. But for the case $k=0$ we have to take $r=a_1$ since then $a_0=0$. With the convention that $\psi_0(0)=0$ for $k=0$, then a (C.S.P.) $A=\{a_i;\ i\geqq 0\}$ from class Q_k; $k=0$, 1, 2, 3 and 4 is of the form

$$A=\{\psi_i(r);\ i\geqq 0\} \tag{3.13}$$

Therefore, if the set of (C.S.P.)'s from class Q_k is not empty (*see* Remark 4.1 in the next section) then we have

$$\inf\{\Delta_k(A, F);\ A=\{a_i;\ i\geqq 0\}\in Q_k\}=\inf\{\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F);\ r\in\mathbb{R}\}. \tag{3.14}$$

Thus under regularity condition on F, the (GLSP) problem has been reduced from a problem with an infinite number of variables $\{a_i;\ i\geqq 0\}$ to a problem with only one single variable, namely $r=a_0$ for $k=1$, 2, 3 and 4, and $r=a_1$ for $k=0$.

(*b*) Optimal search paths

Let us assume that the set of (C.S.P.)'s from class Q_k is not empty, and let

$$\Delta_k^*(r^*, F)=\inf_{r\in\mathbb{R}}\{\Delta_k(\{\psi_i(r); i\geqq 0\}, F)\};\quad k=0,\ 1,\ 2,\ 3 \text{ and } 4. \tag{3.15}$$

We then can address ourselves to solving (3.15) under the side condition [recall the conditions (2.1) through (2.5)].

$$|\psi_{i+2}(r)|>|\psi_i(r)| \quad \text{for all } i\geqq k-1;\quad k=0,\ 1,\ 2,\ 3 \text{ and } 4 \tag{3.16}$$

at any distribution. And the side condition [recall (2.17)].

$$|\psi_{i+1}(r)|>|\psi_i(r)|,\quad i\geqq 0 \ \text{ for } k=0,\ 1 \text{ and } 2 \quad\text{and}\quad i\geqq k-2 \ \text{ for } k=3 \text{ and } 4 \tag{3.17}$$

at the symmetric distributions. Whenever these side conditions are not satisfied, we shall consider that the corresponding $\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F)$ is not defined. The search path $\{\psi_i(r^*);\ i\geqq 0\}$ that defined by (3.15) and that satisfies these side conditions is an (O.S.P) from class Q_k; $k=0$, 1, 2, 3 and 4.

The procedure of finding an (O.S.P) from (3.15) would be as follows: For each $r\in\mathbb{R}$ we construct all $a_i=\psi_i(r)$ from the relevant relations of (3.4) through (3.11). And then we calculate the corresponding $\Delta_k(\{\psi_i(r)\}, F)$ from (2.7) through (2.9). From those values of r that satisfy the pertinent side condition, we choose the value r^* that satisfy (3.15). Another equivalent procedure of finding an (O.S.P.) from (3.15) is as follows: From all (C.S.P.)'s of the form (3.13) we find the minimal search paths. Then we take the overall minimum of all minimizing search paths. However, there are some difficulties that arise when applying such procedures. One of the main difficulties for instance, is to consider all values of r from $\mathbb{R}$. Another one is that; it is not known as to whether the relevant side conditions, indicated above, are fulfilled everywhere. A third one is that; though our optimization problem has been reduced from a problem with an infinite number of variables to a problem with only one single variable. But it is still one difficult variable. This is so since each (C.S.P.) has an infinite number of entries. It would be therefore, rather difficult to verify that a given (C.S.P.) is of minimal type. Unfortunately overcoming such difficulties is not always possible as we shall see in the next section. Nevertheless, the properties of (O.S.P.)'s which will be studied in the next section will provide us with valuable information that will, at least, be a helpful tool for verifying and facilitating the numerical calculations of (O.S.P.)'s.

4. PROPERTIES OF OPTIMAL SEARCH PATHS

Some properties of (O.S.P.)'s have already been established, and being held at any distribution F (*see* theorem 2.5 in [2] for the nonsymmetric distributions, and recall relation (2.17) for the symmetric ones). For regular distributions, however some other properties do, in fact, hold and are helpful in facilitating the solution of the (GLSP). In order to help the flow of our ideas we start with the following property of (O.S.P.)'s.

1. Nonminimality of some classes for certain type of distributions

Though the function $\Delta_k(A, F)$ has an infinite number of variables, the structure of our problem makes it possible to take $\Delta_k(A, F)$, with finite number of variables, as an approximation of its exact value. Such an approximation is justified by the fact that $|a_i|\{1\text{-sign}(a_i)[F(a_i)-F(a_{i-1})]\}$ approaches 0 as $i\to\infty$ (recall that $a_i\to-\infty$ and $a_{i-1}\to\infty$ as $i\to\infty$ or vice versa). Denote by n the number of entries from $A=\{a_i;\ i\geqq 0\}$ for which the indicated approximation is fulfilled for any desired level of precision. If $A=\{a_i;\ 0\leqq i\leqq n\}$ is an (approximated) search path, then A can not be minimal unless the Hessian matrix evaluated at A is positive definite (*see* theorems 42.4, 42.5 in [3]). For $k=1, 2, 3$ and 4, let $\delta_i=\partial^2\Delta_k(A, F)/\partial a_i^2;\ i\geqq 0$.

Simple calculations on (2.8) and (2.9) have shown that the Hessian is symmetric (provided that the derivative f' of f does exist and is continuous). And that the matrix H has the following form:

$$(4.1)\quad H=\begin{bmatrix} \delta_0 & 2f(a_0) & 0 & 0 & \dots & 0 & 0 \\ 2f(a_0) & \delta_1 & 2f(a_1) & 0 & \dots & 0 & 0 \\ 0 & 2f(a_1) & \delta_2 & 2f(a_2) & \dots & 0 & 0 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots & \dots\dots \\ 0 & 0 & 0 & 0 & \dots & \delta_{n-1} & 2f(a_{n-1}) \\ 0 & 0 & 0 & 0 & \dots & 2f(a_{n-1}) & \delta_n \end{bmatrix}$$

[For the case $k=0$ the resulting matrix H has a similar form as (4.1) with replacement of a_i by a_{i+1} and δ_i by δ_{i+1}; $i\geqq 0$]. But H is positive definite if and only if the determinants of its principle submatrices are strictly positive. Thus $A=\{a_i;\ i\geqq 0\}$ can not be a minimal search path from class Q_k unless

$$(4.2)\qquad \frac{\partial^2 \Delta_k(A, F)}{\partial r^2}>0; \qquad k=0,\ 1,\ 2,\ 3 \text{ and } 4$$

And

$$(4.3)\qquad \frac{\partial^2 \Delta_k}{\partial r^2}\frac{\partial^2 \Delta_k}{\partial a_1^2}-4[f(r)]^2>0; \qquad k=1,\ 2,\ 3 \text{ and } 4$$

But when the derivative f' of f does exist, then (4.2) is equivalent to

$$(4.4)\quad h(r)=2f(r)+\operatorname{sign}(r)\frac{f'(r)}{f(r)}\{1-\operatorname{sign}(r)[F(r)-F(0)]\}<0; \qquad k=0$$

$$(4.5)\qquad h(r)=2f(r)+\operatorname{sign}(r)\frac{f'(r)}{f(r)}<0; \qquad k=1 \text{ and } 3$$

$$(4.6)\qquad h(r)=2f(r)-\operatorname{sign}(r)\frac{f'(r)}{f(r)}<0; \qquad k=2 \text{ and } 4.$$

Suppose now that the distribution F is of the following type:

(4.7) "F is regular and has unimodal density f with the mode occuring at zero and the derivative f' does exist and is continuous".

Then $f'(r)/f(r)>0$ for $r<0$ and $f'(r)/f(r)<0$ for $r>0$, which means that the necessary condition (4.6) can not hold for $k=2, 4$. On the other hand, simple calculations on (2.9) yield:

$$\frac{\partial^2 \Delta_3(A, F)}{\partial a_1^2} = -4f(a_1) + 2(|a_2| - |a_1|) f'(a_1) \operatorname{sign}(a_1).$$

From which we can easily see that $\partial^2 \Delta_3/\partial a_1^2 < 0$ whenever F satisfies (4.7). But then (4.3) can not hold. For if $\partial^2 \Delta_3/\partial r^2 \leqq 0$ we are through, otherwise $\partial^2 \Delta_3/\partial r^2 \, \partial^2 \Delta_3/\partial a_1^2 - 4[f(r)]^2 < 0$. Thus we have actually proved thee following result.

THEOREM 4.1: *If F satisfies* (4.7), *then for $k = 2, 3$ and* 4, *any critical search path is not of minimal type.*

An illustration of Theorem 4.1 is given by the following example

Example 4.1: Suppose that the target position follows the Normal law

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2\, t^2}\, dt; \qquad x \in \mathbb{R}. \tag{4.8}$$

which is symmetric and unimodal with the mode occuring at zero. The optimal value $\Delta_2^*(r^*, F)$ that has been obtained from the (C.S.P.)'s is found to be 2.112 821 45. However, some given search paths (Noncritical) for which $a_0 = r$ for $k = 1$, 2, 3 and 4 such as

$$\begin{gathered} \{a_i = 0.5[r^{i+1} + (i+1)r];\ i \geqq 0\}, \\ \{a_i = r^{i+1};\ i \geqq 0,\ |r| > 1\} \\ \{a_i = (i+1)r;\ i \geqq 0\}, \qquad \{a_i = r^{i+1} + i^r;\ i \geqq 0\} \\ \{a_i = r^{i+1} + ir;\ i \geqq 0\}, \qquad \{a_i = (i+1)r + r^i;\ i \geqq 0\} \\ \{a_i = (i+1)r + ir;\ i \geqq 0\}, \text{etc.} \end{gathered} \tag{4.9}$$

And for which $a_1 = r$ for $k = 0$ such as

$$\left\{ \begin{gathered} \{a_i = ir;\ i \geqq 1\},\ \{a_i = r^i;\ i \geqq 1,\ |r| > 1\} \\ \{a_i = r^i + (i-1)r;\ i \geqq 1\}, \qquad \{a_i = 0.5(r^i + ir);\ i \geqq 1\}, \text{etc.} \end{gathered} \right. \tag{4.10}$$

have also been considered for comparison purposes (The value of Δ_k's at such search paths will be denoted by $\Delta_k(r, F)$ so that those values can be distinguished from $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ which we shall use for (C.S.P.)'s.) The minimal values of $\Delta_k(r, F)$ at the special search paths defined by (4.9) have been found to be 1.969 376 523, 1.848 029 33, 2.024 573 89, 2.123 231 33, 1.993 869 08, 2.372 076 21, 2.073 795 00 respectively. One can easily see that the value of $\Delta_2^*(r^*, F)$ is greater than the minimal values of $\Delta_2(r, F)$ at almost all special search paths defined by (4.9) giving thus an in sight to theorem 4.1 for $k=2$.

Remark 4.1: It has been found, by means of computers, that for the distribution (4.8), then

$$|a_1|<0 \text{ for } k=3 \text{ and } r\in\mathbb{R} \qquad \text{and} \qquad |a_2|<0 \text{ for } k=4 \text{ and } r\in\mathbb{R}$$

This means that the set of (C.S.P.)'s, from each of the classes Q_3, Q_4 and for the distribution (4.8), is empty, which seems to contradicts the result of Theorem 2.2. However, by reasons mentioned in Remark 3.1, one may construct many noncritical search paths like those defined by (4.9) and (4.10) and then use the trial and error process to extract (O.S.P.)'s from them.

2. Bounds on r

As indicated above, solving (3.15) for all $r\in\mathbb{R}$ is not an easy task. However some useful bounds on the only characteristic variable r are available. Since any (O.S.P.) is a minimal search path, some of these bounds come from the necessary condition (4.2) that have to hold for any minimal search path $A=\{a_i;\ i\geqq 0\}$ from class Q_k. When the inequalities (4.4), (4.5) and (4.6) have solutions they would be of special importance for obtaining significant bounds on r. An illustration is given in the following example.

Example 4.1 (Continued): Considering again that the target distribution is given by (4.8). Then for $k=1$, (4.5) gives.

$$h(r)=2f(r)-r<0 \tag{4.11}$$

which is equivalent to

$$r\in(-\infty, -\alpha)\cup(\alpha, \infty) \qquad \text{where} \quad \alpha\simeq 0.647\,142\,8. \tag{4.12}$$

Obtaining thus a lower (upper) bound $\alpha(-\alpha)$ on r when $r>0\,(r<0)$. □

However, the solution of each (4.4), (4.5) and (4.6) is highly dependent on the type of search (*i. e.* on k) and the type of target distribution F. For instance, equation (4.6) can not hold for any unimodal distribution with the mode occuring at zero as we have seen in the previous property.

Other bounds on r may be obtained from the forms of $\Delta_k(A, F)$ given by (2.7), (2.8) and (2.9). To see this, let δ_k be the value of $\Delta_k(A, F)$ at a given search path such as those given by (4.9) and (4.10). And denote by Q_k^m the set of minimal search paths from class Q_k, $k=0, 1, 2, 3$ and 4. Then some other significant bounds on r are given by the following theorem.

THEOREM 4.2: *Let* $B_1=\int_{-\infty}^{0}|x|\,dF(x)$, $B_2=\int_{0}^{\infty}|x|\,dF(x)$. *If* Q_k^m *is not empty, then*

$$r_1=-\left(\frac{1}{2}\delta_0+B_1\right)\leqq r\leqq\frac{1}{2}(\delta_0+B_2)=r_2;\qquad k=0 \tag{4.13}$$

$$r_1=-(\delta_1+2B_1)\leqq r\leqq\delta_1+2B_2=r_2;\qquad k=1 \tag{4.14}$$

$$r_1=-(\delta_k+4B_1)\leqq r\leqq\delta_k+4B_2=r_2;\qquad k=2 \text{ and } 3 \tag{4.15}$$

$$r_1=-(\delta_4+6B_1)\leqq r\leqq\delta_4+6B_2=r_2;\qquad k=4. \tag{4.16}$$

Proof: Let $M_k(r)$ be the subset from $\mathbb{R}$ for which the resulting (C.S.P)'s are minimal search paths. Let also $A_m=\{a_i=\psi_i(r);\ i\geqq 0\}$ be a minimal search path, and $\Delta_k(A_m, F)$ be the corresponding value of Δ_k at the search path A_m. Since, Q_k^m is not empty so for each $r\in M_k(r)$ we have

$$\delta_k\geqq\Delta_k(A_m, F);\qquad k=0, 1, 2, 3 \text{ and } 4. \tag{4.17}$$

The proof of (4.14) is direct from (2.8) and (4.17) with $k=1$. We shall now give the proof for $k=0$ and $k=4$. The proof for $k=2$ and 3 can be done by similar fashion.

(i) $k=0$: Let $r\in M_0(r)$, then from (2.7) and (4.17) we have

$$\delta_0\geqq\Delta_0(A_m, F)\geqq 2|r|\{1-\operatorname{sign}(r)[F(r)-F(0)]\}=2|r|-2\left|\int_0^r|r|\,dF(x)\right|.$$

which implies that $|r|\leqq(1/2)\delta_0+\left|\int_0^r|r|\,dF(x)\right|$. Since the integrand on the right side of the last inequality is a nonnegative function so by [4] Lemma 3.8 we have

$$\begin{cases}\displaystyle\int_r^0|r|\,dF(x)\leqq\int_{-\infty}^{0}|x|\,dF(x) & \text{for } r\leqq 0,\\ \text{and} & \\ \displaystyle\int_0^r|r|\,dF(x)\leqq\int_0^{\infty}|x|\,dF(x) & \text{for } r\geqq 0\end{cases} \tag{4.18}$$

which in turn implies (4.13).

(ii) $k=4$: Let $r \in M_4(r)$. Then from (2.9) and (4.17) with $k=4$ we have

$$\begin{aligned}\delta_4 \geqq -2\left|\int_0^r |x|\,dF(x)\right| - |r| - 4|a_2| + 2|a_1| \\ -2|a_1|\operatorname{sign}(a_1)[F(a_1)-F(a_0)] \\ +2|a_2|\{1-\operatorname{sign}(a_2)[F(a_2)-F(a_1)]\} \\ +2|a_3|\{1-\operatorname{sing}(a_3)[F(a_3)-F(a_2)]\}.\end{aligned}$$

Since for $k=4$, $|a_3| \geqq |a_1| > |a_0| = |r| \geqq |a_2|$ so we have

$$\begin{aligned}\delta_4 &\geqq -2\left|\int_0^r |x|\,dF(x)\right| - |r| + 2|r| \\ &\quad -2|a_1|\operatorname{sign}(a_1)[F(a_1)-F(a_0)] \\ &\quad -2|a_2|\{1-\operatorname{sign}(a_2)[F(a_2)-F(a_1)]\} \\ &\quad +2|a_3|\{1-\operatorname{sign}(a_3)[F(a_3)-F(a_2)]\} \\ &\geqq -2\left|\int_0^r |x|\,dF(x)\right| + |r| - 2|a_1|\operatorname{sign}(a_1)[F(a_1)-F(a_0) \\ &\quad -2|a_3|\{1-\operatorname{sign}(a_2)[F(a_2)-F(a_1)]\} \\ &\quad +2|a_3|\{1-\operatorname{sign}(a_3)[F(a_3)-F(a_2)]\}\end{aligned}$$

or

$$\delta_4 \geqq -2\left|\int_0^r |x|\,dF(x)\right| + |r| - 2\left|\int_{a_0}^{a_1} |a_1|\,dF(x)\right| - 2\left|\int_{a_1}^{a_3} |a_3|\,dF(x)\right|$$

which implies

$$|r| \leqq 2\left|\int_0^r |x|\,dF(x)\right| + 2\left|\int_{a_0}^{a_1} |a_1|\,dF(x)\right| + 2\left|\int_{a_1}^{a_3} |a_3|\,dF(x)\right| + \delta_4.$$

But, for $k=4$, the entries $a_0=r$, a_1, a_2 and a_3 have the same sign, so by similar arguments as those used in proving (4.18) we can easily see that: Each of $\int_r^0 |x|\,dF(x)$, $\int_{a_1}^{a_0} |a_1|\,dF(x)$ and $\int_{a_3}^{a_1} |a_3|\,dF(x)$ is less than or equal to $\int_{-\infty}^{0} |x| \, dF(x) = B_1$ for $r \leqq 0$. And each of

$$\int_0^r |x| \, dF(x), \qquad \int_{a_0}^{a_1} |a_1| \, dF(x) \qquad \text{and} \qquad \int_{a_1}^{a_3} |a_3| \, dF(x)$$

is less than or equal to $\int_0^{\infty} |x| \, dF(x) = B_2$ which in turn imply (4.16).

Remark 4.2: Examination of (2.14) through (2.16) show that, for symmetric distributions, the case $r > 0$ is equivalent to its dual $r < 0$ in the sense that both give the same value of $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ (*see* also [13]). For nonsymmetric distributions, however, we have to solve (3.15) for the two cases $r > 0$ and $r < 0$. And then choose the one with the least expected cost. Moreover, the bounds on r given by Theorem 4.2 will be relaxed in case of symmetric distributions. This is so, since then $B_1 = B_2 = (1/2) M(F)$. Thus for symmetric distributions we can content ourselves to the following bounds on r:

(4.19) $\qquad$ Either $r_1 \leqq r \leqq 0$, $\quad$ or $\quad 0 \leqq r \leqq r_2$; $\qquad k = 0, 1, 2, 3$ and 4

where $r_1 = -r_2$, and

(4.20)
$$r_2 = \frac{1}{2}[\delta_0 + M(F)] \text{ for } k = 0, \qquad r_2 = \delta_1 + M(F) \text{ for } k = 1$$
$$r_2 = \delta_k + 2M(F) \text{ for } k = 2 \text{ and } 3 \qquad \text{and} \qquad r_2 = \delta_4 + 3M(F) \text{ for } k = 4.$$

Example 4.1 (continued): For the distribution (4.8) and the fifth search path of (4.9) we obtain $\delta_1 \simeq 1.47$, $r_1 \simeq -2.26$, $r_2 \simeq 2.26$. Thus for the distribution (4.8), relation (3.15) is equivalent to

(4.21)
$$\Delta_1^*(r^*, F) = \inf_{r \in (-2.26,\ -0.647\,142\,8)} \{\Delta_1(\{\psi_i(r);\ i \geqq 0\}, F)\} \quad \text{for} \quad r < 0$$

and

(4.22)
$$\Delta_1^*(r^*, F) = \inf_{r \in (0.647\,142\,8,\ 2.26)} \{\Delta_1(\{\psi_i(r);\ i \geqq 0\}, F)\} \quad \text{for} \quad r > 0$$

But (4.8) is symmetric, so by remark 4.2 we consider either (4.21) or (4.22).

3. Fixed points

We have mentioned in section 3 (*b*) that the function $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ is not defined whenever the related side condition from (3.16) and (3.17) is not fulfilled. (recall that we are concerned with search paths of the form (3.13)). From our hypotheses, the relation

$$\psi_{i+1}(r) \neq \psi_i(r) \neq \psi_{i-1}(r) \tag{4.23}$$

for all pertinant i and k should also hold at any regular distribution. But it can happen that (4.23) does not hold everywhere for any distribution. Indeed if we assume that

$$\psi_{i+1}(r) = \psi_i(r) = \psi_{i-1}(r) = \gamma \tag{4.24}$$

for all pertinant i and k, then equation (3.12) is equivalent to

$$\gamma = \psi_i(\gamma)\ (\Leftrightarrow \psi_i(r) = r) \tag{4.25}$$

for all pertinant i and k.

In such cases, then by the bounds on r indicated above and by Brouwer Fixed Point Theorem (*see* [3] Theorem 23.8] the continuous function $\psi_i(r)$ has at least one fixed point. In this case equation (3.4) is equivalent to

$$\frac{1}{f(\gamma)} - 2|\gamma| = 0; \quad i \text{ and } k \text{ are given by (3.2)} \tag{4.26}$$

which in turn gives the fixed points of $\psi_i(r)$ (if any) at any regular distribution F. Moreover, it is also possible to obtain some other kinds of fixed points for the function $|\psi_i(r)|$. For if we assume that

$$|\psi_{i+1}(r)| = |\psi_i(r)| = |\psi_{i-1}(r)| = \beta \tag{4.27}$$

for all pertinant i and k.

Then (3.4) is equivalent to

$$\begin{cases} [1 + F(-\beta) - F(\beta)]/f(\pm\beta) - 2\beta = 0; \\ \beta \geqq 0;\ i \text{ and } k \text{ are given by (3.2)} \end{cases} \tag{4.28}$$

which gives the fixed points (if any) for the function $|\psi_i(r)|$ at any regular distribution F. When the F is also symmetric, then by (2.13), relation (3.4) will have the form

$$y_{i+1} = \frac{2 - [F(y_i) + F(y_{i-1})]}{f(y_i)} - y_i;\ i \geqq k;\ k = 0,\ 1,\ 2,\ 3 \text{ and } 4. \tag{4.29}$$

Substituting (4.27) in (4.29) we obtain

$$\text{(4.30)} \qquad \begin{cases} [1-F(\beta)]/f(\beta)-\beta=0; \\ i \text{ and } k \text{ are given as in (4.29)} \end{cases}$$

which gives the fixed points (if any) of $|\psi_i(r)|$ for regular and symmetric distributions. From the above discussion we observe that the existence of fixed points for the functions $\psi_i(r)$, $|\psi_i(r)|$ is not guaranteed. Because we cannot assure that each or both of (4.24) and (4.27) are realy fulfilled for all pertinent i and k. However when such points do exist the functions $\psi_i(r)$ and $|\psi_i(r)|$; $i\geqq 0$ change their values very slowly near them. Therefore the search paths $\{\psi_i(r);\ i\geqq 0\}$, $\{|\psi_i(r)|;\ i\geqq 0\}$, get trapped around these points. Then the side consition (3.16) [(3.17) for symmetric distributions] no longer holds which means that the corresponding $\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F)$, $(\Delta_k(\{|\psi_i(r)|;\ i\geqq 0\}, F))$ is not defined.

Example 4.1 (continued): For the distribution (4.8), the solution of (4.30) is $\beta\simeq 0.751\,791\,5$. The corresponding $\{|\psi_i(r)|;\ i\geqq 0\}$ for $k=0$ does in fact get trapped around this value of β (*see* [1] the table on pages 27-28 concerning with the case $k=0$ at the distribution (4.8)). This results in a gap on the plot of $\Delta_0(\{|\psi_i(r)|,\ i\geqq 0\}, F)$ as it can be seen from figure 2 below which shows the plot of $\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F)$, $k=0$, 1 and 2 as functions of r, at the distribution F that is given by (4.8). Each point from the plot of $\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F)$, in this figure, corresponds to a critical search path from class Q_k; $k=0$, 1 and 2. (The set of critical search paths from each of Q_3 and Q_4, for the distribution (4.8), is empty as has indicated before.)

4. The (GLSP) as a function of r only

So far we have shown that the (GLSP) is completely characterized by the first entry r. The question we address now is how the changes in r affect the values of the Δ_k's and the a_i's *i. e.* what about the derivatives of the Δ_k's and the a_i's with respect to r as the only significant variable. In fact we have

$$\text{(4.31)} \qquad \frac{d\Delta_k}{dr}=\sum_{i=0}^{\infty}\frac{\partial\Delta_k}{\partial a_i}\frac{da_i}{dr}; \qquad k=1,\ 2,\ 3 \text{ and } 4.$$

Since (3.1) holds at any critical search path, we obtain

$$\text{(4.32)} \qquad \frac{d^2\Delta_k}{dr^2}=\sum_{i=0}^{\infty}\frac{\partial^2\Delta_k}{\partial a_i^2}\left(\frac{da_i}{dr}\right)^2; \qquad k=1,\ 2,\ 3 \text{ and } 4$$

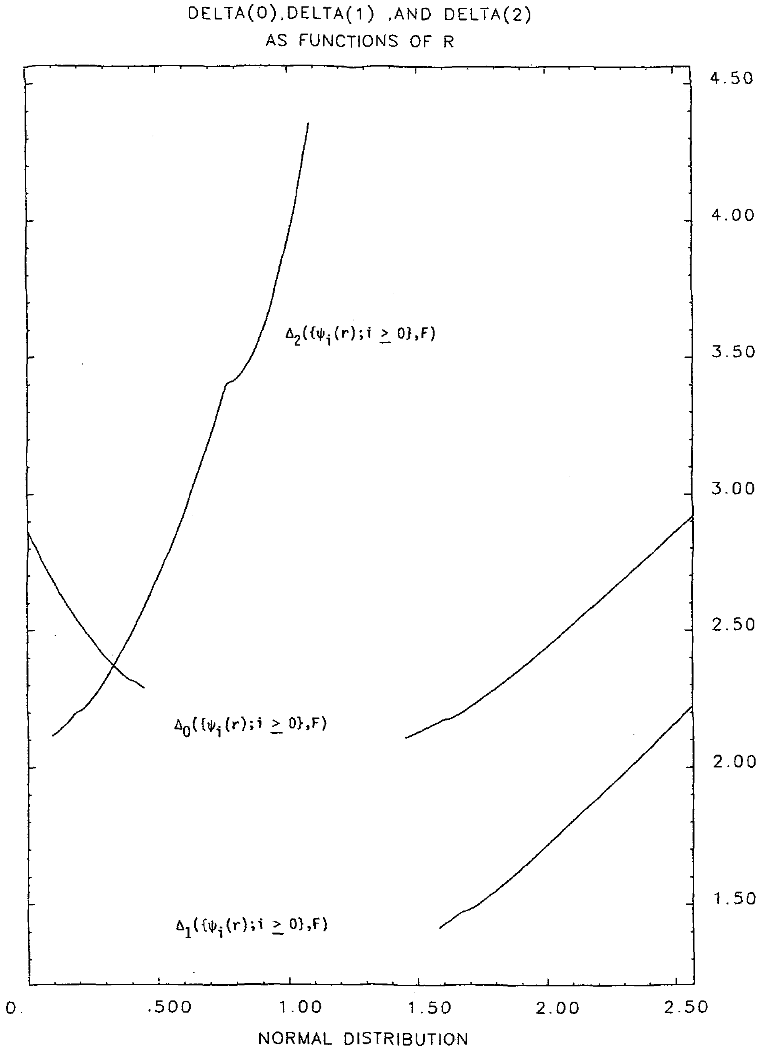

Figure 2.

[(4.31) and (4.32) hold also for $k=0$ with summations starting from $i=1$]. Let $D_i = da_i/dr$, then for $k=1$, 2, 3 and 4 we clearly have $D_0=1$, and then from (3.6), (3.7) and (3.8) we obtain

$$(4.32) \qquad D_1 = \frac{da_1}{dr} = \frac{(-1)^{k+1} \operatorname{sign}(r) f'(r)}{2f^2(r)} + 1; \qquad k=1, 2, 3 \text{ and } 4$$

whereas $D_1 = 1$ for $k=0$.

Since for $i \geqq 1$ each a_{i+1} is a function of a_i and a_{i-1}, we have

$$(4.33)\quad D_{i+1}=\frac{da_{i+1}}{dr}=\frac{\partial a_{i+1}}{\partial a_i}\cdot\frac{da_i}{dr}+\frac{\partial a_{i+1}}{\partial a_{i-1}}\cdot\frac{da_{i-1}}{dr}$$

$$=\frac{\partial a_{i+1}}{\partial a_i}D_i+\frac{\partial a_{i+1}}{\partial a_{i-1}}D_{i-1};\qquad i\geqq 1.$$

With the convention that $D_0=0$ for $k=0$, then simple calculations on (3.4) and (3.5) yield the following recursive formula.

$$(4.34)\quad \begin{aligned} D_{i+1}&=\left[2+\operatorname{sign}(a_i)\frac{f'(a_i)}{f(a_i)}(|a_i|+|a_{i+1}|)\right]D_i-\frac{f(a_{i-1})}{f(a_i)}D_{i-1}\\ &\quad i \text{ and } k \text{ are given by (3.2).}\end{aligned}$$

For the other values of i and k we may obtain D_{i+1} from (3.9) through (3.11). Now from (4.32), (4.33) and (4.34), D_i is a function of r for all $i \geqq 0$. And both $d\Delta_k/dr$, $d^2\Delta_k/dr^2$ could be expressed as functions of r. These results may, in fact, facilitate the task of studying the convexity and concavity of the functions $\Delta_k(\{\psi_i(r);\ i\geqq 0\}, F)$; $k=0$, 1, 2, 3 and 4. Moreover, the values of D_i's as functions of r will provide us with good indications of how the changes in r will affect the entries $a_i=\psi_i(r)$ as will be seen in the next example.

Example 4.1 (continued): Let us return to the distribution (4.8) and consider the case $k=1$. The optimal value of $\Delta_1(\{\psi_i(r);\ i\geqq 0\}, F)$ has been found to occur at an extreme point for which we have obtained Computer Results (1) below concerning the optimal search path $\{x(i);\ i\geqq 0\}$ and the optimal value of r with the corresponding optimal value of $\Delta_1(\{\psi_i(r);\ i\geqq 0\}, F)$ together with the derivatives of $x(i)$'s with respect to r.

Making an infinitesimal change in r gives Computer Results (2) with more of the entries $x(i)$'s.

The infinitesimal changes in r have been continued to be made upon reaching 29 decimal digits giving us Computer Results (3) with about 20 of the entries $x(i)$'s, upon reaching the system capacity.

One can easily see, from these results, how the changes in r affect significantly the entries of a (C.S.P.) especially the last ones.

```
   MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
 NORMAL DISTRIBUTION START RO= .2500000000000000000000000000000D+01

 .27352071072249251 7D+01   .37258834920974171 1D+01   .46045703267604123 6D+01   .54039860843968608 8D+01
 .61439577809973735 0D+01   .68697015195151083 8D+01   .11121041100057334 1D+02   .36728734999999915 1D+02
 .140993038941158524+319
--R MIN= .15711834694999996582892114346D+01---MIN_DELTA(1)= .141309704879697850D+01
  THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS= .25884804769151 51D+00 IF=    0

   DERIVATIVES OF X(I)'S (W.R.T) R

 .57661296873607736 8D+01   .78108864844398874 6D+02   .21266308042364865 2D+04   .90717532578815929 8D+05
 .53636179753495171 8D+07   .41161784123208590 8D+09   .49446030577630652 5D+11  -.16708904253144809 9D+26
```

Computer Results (1)

```
   MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
 NORMAL DISTRIBUTION START RO= .15711834694173698250000000000D+01

 .27352071067484690 6D+01   .37258834856434658 6D+01   .46045701510423170 2D+01   .54039785886100233 5D+01
 .61435146129236449 1D+01   .68357402201768637 1D+01   .74893648768177995 8D+01   .81160295881360174 3D+01
 .87139104192022933 0D+01   .93366010604816698 7D+01   .21092405266004051 9D+02   .36728734999999915 1D+02
 .757436937546393391+318
--R MIN= .15711834694173698249999999970D+01---MIN_DELTA(1)= .141309704876551590D+01
  THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS= .25884804767956 03D+00 IF=    0

   DERIVATIVES OF X(I)'S (W.R.T) R

 .57661296861265089 5D+01   .78108864712387060 0D+02   .21266307464359811 2D+04   .90717453557479033 6D+05
 .53633890329112718 5D+07   .41045286275751611 9D+09   .38892658554635483 0D+11   .44234681455307926 8D+13
 .59019547443609733 1D+15   .90974501738760323 1D+17   .25501405422597648 5D+20  -.43278773024656110 5D+95
```

Computer Results (2)

```
   MINIMUM VALUE OF DELTA(1) AS A FUNCTION OF R AT
 NORMAL DISTRIBUTION START RO= .15711834694173698246531916566 5D+01

 .27352071067484690 6D+01   .37258834856434658 2D+01   .46045701510423160 2D+01   .54039785886099782 0D+01
 .61435146129208724 4D+01   .68357402199585647 1D+01   .74893648556596714 1D+01   .81160271364488377 2D+01
 .87135830448846768 1D+01   .92861875492449770 4D+01   .98370480801186219 5D+01   .10368712132519706 0D+02
 .10883275985318550 6D+02   .11389103612938191 6D+02   .13700026696530064 8D+02   .36728734999999915 1D+02
 .115488061836672268+319
--R MIN= .15711834694173698246531916563 5D+01---MIN_DELTA(1)= .141309704876551589D+01
  THE INTEGRAL OF ABS(X)*DF(X) FROM 0 TO X0 IS ANS= .25884804767956 03D+00 IF=    0

   DERIVATIVES OF X(I)'S (W.R.T) R

 .57661296861265089 4D+01   .78108864712387059 3D+02   .21266307464359808 0D+04   .90717453557478559 7D+05
 .53633890329098424 6D+07   .41045286275003663 5D+09   .38892658493364466 4D+11   .44234674123125900 6D+13
 .59018345419858002 0D+15   .90710122785634559 0D+17   .15824950240065111 3D+20   .30962121420956991 3D+22
 .67267096797265553 6D+24   .16097413889856406 5D+27   .45487227862075578 1D+29  -.62624880728098307 1D+39
```

Computer Results (3)

5. ALGORITHM AND ILLUSTRATION

(a) Computational algorithm

We have pointed out that there are bounds on the main variable r. And that the functions $|\psi_i(r)|$, $\psi_i(r)$ may have some kinds of fixed points causing a gap in the graph of $\Delta_k(\{\psi_i(r); i \geqq 0\}, F)$ (recall, Figure 2). In the case of no fixed points (*i.e.* the side conditions (3.16) or (3.17) are satisfied) then the function $\Delta_k(\{\psi_i(r); i \geqq 0\}, F)$ would be of continuous type. For example the side conditions are always fulfilled for any of the special search paths (4.9). Note that, for these special search paths, the plot of $\Delta_1(r, F)$ at the F given by (4.8), as shown in [1] figure 26, is of continuous type.

If $\Delta_k(\{\psi_i(r); i \geqq 0\}, F)$ is continuous and of convex or concave type, we may use the following algorithm for finding the optimal value r^* of r and the corresponding $\Delta_k^*(r^*, F)$ as defined by (3.15).

The algorithm

r_1 = left bound of r, r_2 = right bound of r.

ε is an infinitely small positive quantity say $\varepsilon = 0.1 \times 10^{-10}$.

$I = 1$, and N is the number of suitable iterations, say $N = 100$.

Step (1):

$$r_{11} = r_1 + (1/2)(r_2 - r_1 - \varepsilon)$$
$$r_{21} = r_{11} + \varepsilon$$
$$Q_1 = \Delta_k(\{\psi_i(r_{11}); i \geqq 0\}, F)$$
$$Q_2 = \Delta_k(\{\psi_i(r_{21}); i \geqq 0\}, F)$$
$$\delta = Q_1 - Q_2.$$

If δ is greater than zero go to step (2).

If δ equals to zero go to step (5).

If δ is less than zero go to step (3).

Step (2): $r_1 = r_{11}$ go to step (4).

Step (3): $r_2 = r_{21}$.

Step (4): If I is greater than N go to step (8) otherwise $I = I + 1$.

If $(r_2 - r_1)$ is greater then 2ε go to step (1) otherwise go to step (5).

Step (5): If Q_1 is less than Q_2 go to step (6) otherwise go to step (7).

Step (6): The optimal values are $r^* = r_{11}$, $\Delta_k^*(r^*, F) = Q_1$ go to step (8).

Step (7): The optimal values are $r^* = r_{21}$, $\Delta_k^*(r^*, F) = Q_2$.

Step (8): Stop.

If, however, $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ is piecewise continuous so that its curve is constituted of several parts each of which is either convex or concave. Then we minimize on each part and take the overall minimum value of the minimums of those parts. In cases with gaps like figure 2 we have first to find the extreme points of these gaps (the points after which or before which the side conditions (3.16) or (3.17) start to be violated). Then we consider the left or right bounds of r starting from these points. It is then to be noted that if $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ is concave or convex on the parts that result from the extreme points as it is the case in figure 2. Then one of the extreme points would be a strong candidate to represent the optimal solution as will be seen in the next example.

(*b*) **Example (5.1)**

By the result of Theorem 4.1, a family of distributions called the Bimodal Normal is to be considered. This family is characterized by the positive parameters μ and σ so that their densities are given by

$$(5.1) \qquad f(x) = \frac{1}{2\sigma\sqrt{2\pi}}\left[e^{-(1/2)((x+\mu)/\sigma)^2} + e^{-(1/2)((x-\mu)/\sigma)^2}\right]; \qquad x \in \mathbb{R}$$

Each member of these densities is symmetric and have two modals occuring at $-\mu$, μ. The results presented here are concerned with $\sigma = 1$ (recall remark 2.1). The following formula for the F's has been used for computations.

$$(5.2) \qquad F_\mu(x) = \frac{1}{2} + \frac{1}{4}\left[\mathrm{ERF}((x+\mu)/\sqrt{2}) + \mathrm{ERF}((x-\mu)/\sqrt{2})\right]; \qquad x \in \mathbb{R}$$

because the error function $\mathrm{ERF}(t) = 2/\sqrt{\pi}\int_0^t e^{-x^2}\,dx$ does exist in the computer library. Table I contains the extremal values of $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ for $k = 0$, 1, 2 and 4, at different values of μ ($\mu = 1$, 2, 3, 5, 7 and 10). We note that, for $\mu \geqq 2$, $k = 0$, 1 and 2 there are two extreme values of r, between which there is a gap. The optimal value of $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$; $k = 0$, 1 and 2, occurs at one of these values. When $\mu = 1$, however, there is only one (right) extreme point for each of $k = 0$, 1 and 2 at which $\Delta_k(\{\psi_i(r);\ i \geqq 0\}, F)$ attains its optimal value. On the other hand, it has been found that $|a_2| < 0$ for $k = 4$, $\mu = 1$, which means that, for $\mu = 1$, the set of critical search

TABLE I

Extremal and Optimal values of $\Delta_k(\{\psi_i(r);\ i \geqq 0\})$; $K=0, 1, 2$ and 4 at the Bimodal Normal distribution with $\mu=1, 2, 3, 5, 7$ and 10 (1).

	$\mu=1$	$\mu=2$		$\mu=3$		$\mu=5$		$\mu=7$		$\mu=10$	
	R.E	L.E	R.E	L.E.	R.E	L.E	R.E	L.E	R.E	L.E	R.E
r	3.21465597	0.70859354	4.09254095	1.56163869	5.11721282	3.29084407	7.25217051	5.11881371	9.35505798	7.94681641	12.4676503
$\Delta_0(\psi_i(r), F)$ . . .	3.32090576	5.21343476	4.27029067	7.95437801	5.32016456	13.57922604	7.44846807	19.3539984	9.54370455	28.1375843	12.6486717
r	2.87127273	0.24252615	4.01255460	1.20034313	5.11436289	2.95163349	7.25777990	4.79792487	9.35911335	7.94681641	12.4676503
$\Delta_1(\psi_i(r), F)$ . . .	1.34282437	4.35438795	0.99108155	6.27893356	0.28200713	10.14593723	−1.53007101	14.0816070	−3.4595767	20.0308165	−6.3883916
r	1.29895724	0.47224569	2.36211261	1.56658200	3.44041114	3.53053031	5.53787975	5.50727624	7.59972249	8.48347245	10.6623089
$\Delta_2(\psi_i(r), F)$ . . .	2.43603227	3.92741316	1.96960308	3.92740145	0.64565836	3.84644137	−2.0697918	3.53862390	−4.4330439	1.68345247	−7.9111008
r	EMPTY	0.84507105	1.56944457	1.74669804	2.967171472	3.64904043	5.22680713	5.60277842	7.34980658	8.56197713	10.4579105
$\Delta_4(\psi_i(r), F)$ . . .	EMPTY	8.30154518	7.69095107	63.4327457	63.0653466	174601.077	179110.674	28102975788	28634304062	0.3309 D+22	0.3353 D+22
r	EMPTY	1.56944457		2.18565712		4.37132823		6.13634973		9.10982432	
$\Delta_4^*(r^*, F)$	EMPTY	7.69095107		9.94929608		4.99109095		−1.60934651		−7.23919203	
$M(F)$	1.16663094	2.01698141		3.00076431		5.00000011		7.00000000		10.0000000	
β	1.09054628	1.66761124		2.33520672		3.91069803		5.65344406		8.40782712	
$\alpha(K=0)$	1.46679755	2.53350005		3.53362158		5.33621581		7.533621581		10.53362158	
$\alpha(K=1,3)$	1.26866546	2.37199753		3.37223389		5.37223389		7.37223898		10.3722389	
$\alpha 1(K=2,4)$	DOES NOT	0.03614831		0.00110797		0.00000124		0		0	
$\alpha 2(K\neq 2,4)$	EXIST	1.62193035		2.62776036		4.62776110		6.62776110		9.62776110	

(1) Except for α, β and $M(F)$ all other computations has been done in double precision on the CDC system in Brussels University. This system has the range of -10^{322} to 10^{322} and the zero value for 10^{-319}. A double precision constants in this system are accurate up to 29 decimal digits.

TABLE II

Optimal search paths with the corresponding optimal searching time $T_k^*(r^*, F)$, K=0, 1, 2 and 4, at the Bimodal Normal distribution with μ=1, 2, 3, 5, 7 and 10.

	μ=1	μ=2	μ=3	μ=5	μ=7	μ=10
r=X(1)	3.214655969691	4.092540945083	5.117212816163	7.252170504691	9.355057978548	12.467650287755
X(2)	4.5781116365823	5.542111348044	6.604853980157	8.770081640992	10.89500687797	14.03427712881
X(3)	5.677195267114	6.680801794243	7.761684042148	9.944188494140	12.08291794395	15.23963224067
X(4)	6.631499284435	7.659004497700	8.751095368336	10.94535129501	13.09404268582	16.26373306377
X(5)	7.490148322959	8.533987361095	9.633861788931	11.83630675114	13.99280362461	17.17288295498
X(6)	8.278497845543	9.334426654661	10.44014018388	12.56430916946	14.72871123983	17.91991558497
X(7)	8.930619034936	9.996607893149	11.10793844002	13.24475116216	15.41583297176	18.61515810652
X(8)	9.549306848264	10.62353149938	11.73949180679	13.88685420154	16.06367221874	19.27072686671
X(9)	10.13981546776	11.22092243456	12.34076759983	14.49710062605	16.67891222231	19.89278273603
X(10)	10.70609224498	11.79304538644	12.91619826807	15.08027508699	17.26648375693	20.48642986134
X(11)	11.25120529003	12.34319138234	13.46919495106	15.64004061475	17.83015821062	21.05566249509
X(12)	11.77760354238	12.87397281395	14.00247359020	16.17973375750	18.37325938459	21.64774194701
X(13)	12.28755940641	13.38827944804	14.52379231777	16.86789645114	19.04962863467	51.56963744008
X(14)	12.86082112568	14.12748603736	16.98009384417	225.6843442466	217.4519872384	0.1241 D+263
X(15)	53.30179803774	496.051340949	3398934225457	0.9162 D+261	0.1028 D+261	....
$T_0^*(r^*, F)$	4.487536696651	6.287272071844	8.320928873315	12.44846814427	16.543704549231	22.64867174062
r=X(0)	2.871272727294	4.012554601867	5.114362890815	7.257779901117	9.369113348241	12.47046847882
X(1)	4.323974474067	5.484476729814	6.602822133626	8.774054468106	10.89786304705	14.03625998502
X(2)	5.463738198691	6.632838785268	7.760000273145	9.947469049003	12.08527642621	15.24126254647
X(3)	6.442801274872	7.616805176464	8.749617389325	10.94822432661	13.09610507902	16.26515648280
X(4)	7.318642493803	8.495743006204	9.632523797171	11.83890262061	13.99466524501	17.17416624613
X(5)	8.119851693623	9.299106903994	10.43884989179	12.56672129910	14.73043915647	17.92034550092
X(6)	8.780782374215	9.963303974223	11.10672301079	13.24701782293	15.41745513688	18.61627376487
X(7)	9.406785317904	10.59189412294	11.73833813338	13.88900154828	16.06520775475	19.27178198567
X(8)	10.00351410267	11.19069690378	12.33966614088	14.49914754403	16.68037495727	19.89378704225
X(9)	10.57517574104	11.76403829538	12.91514179239	15.08223588603	17.26788441210	20.48739071895

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=5$	$\mu=7$	$\mu=10$
X (12).......	12.16910965223	13.36276325907	14.52096200129	16.77310019688	20.32142370956	59.81754328176
X (13).......	12.68263630811	14.29711892548	16.3099387317	61.00679178693	2286439126.48	0.1102 D+263
X (14).......	18.24658809394	5409.840101868	375230458.92	0.2831 D+262	0.9191 D+253	....
X (15).......	0.7492 D+34	0.4954 D+259	0.1071 D+254	....	....	....
$T_1^*(r^*, F)$	2.509455308716	2.998062954314	3.282771438385	3.469929095727	3.540423289462	3.61160837871
$r=$X (0)	1.298957237647	2.362112607942	3.440411139417	5.537879749456	7.5997224853423	10.66230892911
X (1).	2.518761061530	3.700245548679	4.821362063684	6.986264057825	9.099962165863	12.22298313543
X (2).	4.076125359810	5.266377857421	6.398856447007	8.585791411980	10.71858525841	13.86539513699
X (3).	5.258850040400	6.452993643917	7.592145216363	9.792974586089	11.93843031567	15.10156467532
X (4).	6.263057390019	7.459254901504	8.602758570657	10.81331788044	12.96805343702	16.14351149787
X (5).	7.156006210906	8.353317546974	9.499880409506	11.71720660648	13.87927697074	17.0646793321
X (6).	7.969837056064	9.167852310270	10.31674908506	12.45375966346	14.62344746399	17.81894612630
X (7).	8.639369907420	9.839672534178	10.99179867520	13.14094824480	15.31708368053	18.52124635524
X (8).	9.272469436418	10.47454267750	11.62931837567	13.78857145535	15.97024738096	19.18195715248
X (9).	9.875203813347	11.07865181001	12.23562903744	14.40345454981	16.58995443491	19.80832209783
X (10).......	10.45204546328	11.65656340985	12.81539085842	14.99060190374	17.18134917086	20.40565315397
X (11).......	11.00642148569	12.21177214511	13.37217006180	15.55382500234	17.74834979043	20.97812356632
X (12).......	11.54103609885	12.74703276688	13.90877747618	16.09641814051	18.29515419030	21.59029757083
X (13).......	12.05808827521	13.26462507892	14.42826304467	16.73154849965	19.29810621527	67.23559025930
X (14).......	12.56357577384	13.78400941274	15.19573110789	106.9534597356	11857.56389053	0.2431 D+263
X (15).......	14.32227530839	20.73148301570	723.7402716630	0.4629 D+262	0.4894 D+259	....
$T_2^*(r^*, F)$	3.602663214331	3.98658448861	3.646422684033	2.930208268070	2.566956060858	2.088899182088
$r=$X (0)		1.569444567300	2.185657118867	4.371328233919	6.136349731960	9.109824318074
X (1).		2.941904273447	3.93172528129	5.898481584703	7.956169095549	10.97246969880
X (2).		0.0000000000424	1.255419404521	3.178922849853	5.257144503031	8.253386690598
X (3).		5.432025432230	8.233790907466	11.19771600260	12.18345435590	15.19983865320
X (4).		447.3859103095	1067366.451212	265395999.0592	321055.6533022	894255.236693
X (5).		0.1603 D+290	0.4445 D+286	0.1533 D+284	0.5830 D+286	0.5339 D+286 (*)
$T_4^*(r^*, F)$		8.857582009614	12.49372507934	9.991091059647	5.390653488692	2.760807975461

(*) All such values in this table means that we have reached the computer ranges, therefore, such values should not be taken seriously.

paths from the class Q_4 is empty. The calculations showed that $\Delta_k(\{\psi_i(r);\ i \geqq 0\},\ F)$; $k=0$, 1 and 2, decrease when r varies from zero to the left extreme (l. e.) point, and increase when r takes on values greater than the right extreme (r. e.) point, so that the optimal value of Δ_k's, $k=0$, 1 and 2 occurs at the right extreme points. However, the situation for $k=4$ is quite different from those of $k=0$, 1 and 2. For $k=4$, $\mu \geqq 2$ there are two extreme points with a gap before the first, and a gap after the second so that the side conditions (3.17) is fulfilled between these two extremes. It happens that the optimal value of $\Delta_4(\{\psi_i(r);\ i \geqq 0\},\ F)$ for $\mu=2$ occurs at the second extreme, whereas for $\mu>2$ this optimal value occurs at a point that lies between the indicated two extreme points. In all cases ($k=0$, 1, 2 and 4), the existence of a gap before or after an extreme point means that the side condition (3.17) is violated, hence $\Delta_k(\{\psi_i(r);\ i \geqq 0\},\ F)$ is not defined. Some other significant values in Table I has the following meaning: The value (values) of α indicates the bounds on r that can be obtained from (4.4) for $k=0$, (4.5) for $k=1$, 3, and (4.6) for $k=2$, 4. The value of β indicates the fixed points that have been obtained from (4.30). Some other kinds of fixed points may be obtained from (4.26). The $M(F)$ indicates the first absolute moment of F. Table II contains some of the entries $x(i)=|a_i|$; $i \geqq 0$ ($i \geqq 1$ for $k=0$) of the optimal search path for each value of μ given in Table I and for each of the cases $k=0$, 1, 2 and 4. These entries has been calculated as far as the system capacity. The fact that the optimal value of $\Delta_k(\{\psi_i(r);\ i \geqq 0\},\ F)$; $k=0$, 1 and 2, occurs at an extreme point was a very helpful tool in studying the strong relations between r and the entries $x(i)$; $i \geqq 1$ as these relations are given by (4.32) and (4.34) (recall the last example in the previous section). The entries $x(0)$, $x(1)$, $x(2)$, $x(3)$, $x(4)$, ..., in Table II should be understood, for $k=1$ for instance, as follows; $a_0=x(0)$, $a_1=-x(1)$, $a_2=x(2)$, $a_3=-x(3)$, $a_4=x(4)$, ... with similar understanding for $k=0$, 2 and 4. Table II contains also the optimal searching time denoted by $T_k^*(r^*,\ F)$ for $k=0$, 1, 2 and 4. One can easily verify that $T^*(r_0^*,\ F)/M(F)$ satisfies Theorem 2.5.

We would finally like to mention that the results in Table I and Table II are only roughly correct due to many difficulties in the calculational system such as accumulation errors, the bounds on the system ranges, the system capacity, etc. Thus the large values of $x(i)$; $i \geqq 0$ in Table II would, in fact, result in less precision than the small ones. Nevertheless, this will cause a very slight change in the resulting $\Delta_k(\{\psi_i(r);\ i \geqq 0\},\ F)$. Because the term $2-[F(x_i)-F(x_{i-1})]$, for those large values of x_i, equals zero in the computer digits. [Recall the values of $\Delta_1^*(r^*,\ F)$ in the three computer results concerning the distribution (4.8), in the last example of Section 4.]

6. CONCLUSION

In this paper we have introduced analytical methods for constructing and studying some important properties of optimal search paths for the (GLSP) at the absolutely continuous class of target distributions that have strictly positive densities. We have shown, then, that the (GLSP) can be characterized by only a single variable instead of infinitely many. The techniques used in this study are those of standard calculus so that an optimal search path would, in general, be a critical one. It has also been shown that for three of the only five possible cases of search, and for the distributions of unimodal type with the mode occuring at zero, then these (C.S.P.)'s are not minimal (maximum, saddle, or extreme). We would finally, note that the results of Table I indicate that for the distributions (5.1), the class Q_1 is better than the class Q_0, and for most of the values of μ the classes Q_2, and Q_4 are better than the class Q_0 in the sense that they give less expected cost. Note also that some of the classes Q_k is better than some others justifying, thus, the generalization of the linear search problem that has been introduced in [2]. Some other results concerning the search for a target located in the plane or on a line may be found in [14], and [15].

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