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FIXED COST MINIMIZATION OVER A LEONTIEF SUBSTITUTION SYSTEM (1)

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Résumé. — *Using Bender's decomposition method and properties of a Leontief substitution system, a formulation for solving fixed cost minimizations over Leontief substitution systems is presented.*

It has been pointed out in the literature [2] that Bender's decomposition method appears useful as a solution strategy. Using Bender's method and the properties of a Leontief substitution system, a solution procedure is specified for solving fixed cost minimization problems over a Leontief substitution system. Preliminary results indicate that for some large problems, this procedure is a viable alternative to procedures specified elsewhere [4].

A Leontief substitution system is characterized by a constraint matrix having exactly one positive element per column and non-trivial rows [3]. Substitute activities are identified by columns whose positive elements fall in the same row. A salient feature of Leontief substitution systems is that one and only one activity is optimal for each set of substitute activities.

Consider the problem

$$\text{Min } f(x)$$

subject to

$$Ax = b$$
$$x \geqslant 0$$
$$b \geqslant 0$$

where A is m by n and Leontief, x is n by 1, b is m by 1, and $f(\cdot)$ is a quasi-concave function of the form :

$$f(x) = \sum_{i=1}^{n} f_i(x_i)$$

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where

$$f_i(x_i) = \begin{cases} c_i x_i + d_i & x_i > 0 \quad d_i \geqslant 0 \\ 0 & x_i = 0 \end{cases}$$

The above problem is termed a fixed cost minimization over a Leontief substitution system.

Let the Set J_i contain the column identifications for all columns whose positive coefficients fall in row i. Associate with each continuous variable x_i a binary variable y_i and reformulate the fixed cost problem as :

(1)
$$\text{Min } c'x + d'y$$

subject to

$$\begin{aligned} Ax &= b \\ x - ly &\leqslant 0 \\ x &\geqslant 0 \\ b &\geqslant 0 \\ y &\in Y \end{aligned}$$

where

$$Y = \{ y : y \text{ is binary and } \sum_{j \in J_i} y_j = 1,\ i = 1, 2, \dots m \}$$

and

$$l > x_i$$

for all possible x_i. The requirement that

$$\sum_{j \in J_i} y_j = 1 \qquad i = 1, 2, \dots m$$

is a result of the substitution property of Leontief substitution systems.

The problem given in equation (1) may be reformulated as :

$$\text{Min } d'y + \begin{bmatrix} \underset{x}{\text{Min}}\ c'x \\ \text{subject to} \\ Ax = b \\ x \leqslant ly \\ x \geqslant 0 \\ b \geqslant 0 \end{bmatrix}$$

Using Bender's decomposition method [1], an approximation to the outer problem is now given. Let N denote the number of times the inner problem has been solved. Also, let π_1^i be the dual multipliers to the constraint set

$$Ax = b$$

at the solution of the inner problem at the i^{th} iteration. Likewise, let π_2^i be the dual multiplier to the constraint set

$$x \leqslant ly$$

at the i^{th} iteration. The approximation to the outer problem is

$$\text{Min } z + d'y \tag{2}$$

subject to

$$z \geqslant b'\pi_1^i + ly'\pi_2^i \qquad i = 1,2,\ldots N$$

$$y \in Y$$

Given a $y \in Y$, the inner problem is easily solved. Let y^i be the fixed y vector supplied to the inner problem at the beginning of iteration i. Then the solution to the inner problem is :

$$\pi_1^i = (A'_K)^{-1} c_K \tag{3}$$

where

$$K = \{ j : y_j^i = 1 \}$$

and A'_K is the transpose of the submatrix of A consisting of the columns in the set K. Also

$$\pi_2^i = [c - A'\pi_1^i]^+ \tag{4}$$

where $[a]^+$ is a vector of $[a_i]^+$ $i = 1, 2, \ldots, n$, and $[a_i]^+ = \min [a_i, 0]$.

For computational purposes, it is beneficial to note that if A_k is recursive [4], as is usually the case, then $[A'_K]^{-1}$ is easily computed.

As an example, consider an arborescence-structured production and inventory system. Let x_i^K, y_i^K, and r_i^K be, respectively, the production, end-of-period inventory, and known requirements at facility K in period i. For illustrative purposes let the number of facilities be 2 and the number of periods be 3. The arborescence model is

$$\text{Min} f(x, y)$$

subject to

$$y_{i-1}^1 + x_i^1 - y_i^1 - x_i^2 = r_i^1 \qquad i = 1, 2, 3$$

$$y_{i-1}^2 + x_i^2 - y_i^2 = r_i^2 \qquad i = 1, 2, 3$$

$$y_0^1 = y_0^2 = y_3^1 = y_3^2 = 0$$

$$x_i^K, y_i^K \geqslant 0$$

where

$$f(x, y) = \sum_{K=1}^{2} \sum_{i=1}^{3} \{ f_i^K(x_i^K) + f_i^K(y_i^K) \}$$

and

$$f_i^K(x_i^K) = \begin{cases} c_i^K x_i^K + d_i^K & x_i^K > 0 \quad d_i^K \geqslant 0 \\ 0 & x_i^K = 0 \end{cases}$$

and $f_i^K(y_i^K)$ is similarly defined. Let

$$r^1 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} C_x^2 = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} C_y^2 = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} d_x^1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} d_y^1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$C_x^1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} C_y^1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} r^2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} d_x^2 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} d_y^2 = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}$$

By the structure of the problem, the set of all feasible basis is

$$K = \bigtimes_{i=1}^{6} J_i$$

where

$$J_1 = \{ x_1^1 \} \qquad J_4 = \{ x_1^2 \}$$

$$J_2 = \{ y_1^1, x_2^1 \} \qquad J_5 = \{ y_1^2, x_2^2 \}$$

$$J_3 = \{ y_2^1, x_3^1 \} \qquad J_6 = \{ y_2^2, x_3^2 \}$$

To start the solution procedure, an arbitrary initial feasible solution is specified. Let the optimal solution to the continuous portion of the problem be the initial starting basis. Then

$$K_1 = \{ x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, y_2^2 \}$$

and the outer problem can be written as follows. Let the continuous variable identifications, x_i^K and y_i^K represent their binary counterparts.
Then equation (2) becomes :

$$\text{Min } z + d_x'x + d_y'y$$

subject to

$$z \geqslant 54$$

Then

$$K_2 = \{ x_1^1, y_1^1, y_2^1, x_1^2, y_1^2, x_3^2 \}$$

or

$$K_2 = \{ x_1^1, y_1^1, x_3^1, x_1^2, y_1^2, x_3^2 \}$$

and the outer problem becomes (using the first K_2) :

$$\text{Min } z + d_x'x + d_y'y$$

subject to

$$z \geqslant 54$$

$$z \geqslant 87 - 200x_2^1 - 300x_3^1 - 500x_2^2$$

where $l = 100$.

Continuing in like manner and using all alternative optimal points to form constraints, the following was found.

Iteration		
3	$x_1^1, y_1^1, x_3^1, x_1^2, y_1^2, x_3^2$	$z \geqslant 78 - 200x_2^1 - 500x_2^2$
4	$x_1^1, x_2^1, x_3^1, x_1^2, y_1^2, x_3^2$	$z \geqslant 70 - 700x_2^2$
	$x_1^1, x_2^1, y_2^1, x_1^2, y_1^2, x_3^2$	$z \geqslant 73 - 100x_3^1 - 700x_2^2$
5	$x_1^1, y_1^1, y_2^1, x_1^2, x_2^2, x_3^2$	$z \geqslant 77 - 200x_2^1 - 300x_3^1 - 300y_2^2$
	$x_1^1, y_1^1, x_3^1, x_1^2, x_2^2, x_3^2$	$z \geqslant 68 - 200x_2^1$
6	$x_1^1\ x_2^1, y_2^1, x_1^2, x_2^2, x_3^2$	$z \geqslant 59 - 100x_3^1 - 300y_2^2$
	$x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2$	$z \geqslant 56 - 200y_2^2$
7	$x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^3$	

STOP

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