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CONSTRUCTION OF INTERACTING QUANTUM FIELDS :

A SURVEY

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In this exposition, I attempt to summarize the main results, and the most interesting methods, of an approach to the study of specific interactions in quantum field theory. This subject is largely historically motivated, so I begin with a few historical remarks. When people first began to study quantum field theory, they had in mind theories with specific interactions (the electromagnetic field interacting with various things), which were to be treated by essentially the same methods are were used in non-relativistic quantum mechanics. This procedure led, after some very complicated manipulations, to infinite series (the renormalized perturbation series) which were supposed to represent physical quantities, and the first few terms of the series gave remarkably good agreement with experiment in quantum electrodynamics. This agreement is probably the best justification for thinking that field theory has something to do with nature. Unfortunately, the series are very complicated, so that it is feasible to compute only a very few terms; besides, not much is known about their convergence. Hence, for strong-interaction physics, the series are not of much use, and, in the early 1950's, research took a different direction - the investigation of the general properties that a satisfactory theory, if it exists, should have. This led to such things as the Wightman axioms and the study of analyticity properties of scattering amplitudes. The work I am going to describe returns to the original direction of investigation : One writes down specific interactions between fields and tries to treat them in analogy with ordinary quantum mechanics. Instead of manipulating formally with power series expansions, however, one uses

Hilbert space methods. The central problem is to construct the Hamiltonian as a self-adjoint operator on a Hilbert space. In the end, one hopes to arrive at a theory which fits into one or another of the general theoretical frame works for relativistic quantum mechanics which have been devellopped in the past twenty years (Wightman fields, rings of local observables, etc.) and thus to obtain a non-trivial model for these systems of axioms. However that may come out, the subject has considerable interest in its own right, both from a physical point of view because it is closely tied to renormalized perturbation theory, and from a mathematical point of view because it leads into an area of concrete operator theory with as much structure as the theory of differentiation operators.

With these remarks to serve as an introduction, I want next to explain the formal procedure one would like to use to construct interacting fields with a specific interaction, and to show why the construction procedure doesn't work. For purposes of illustration, we will consider a self-interacting boson field; this theory is not the most interesting one physically, but it has the advantage of giving rise to the simplest formulas. Although one is evidently most interested in a theory in four-dimensional space-time, theories tend to become more tractable as the number of space dimensions is reduced. (Divergent integrals become convergent.) Hence, one frequently studies theories in 2 or 3 dimensional space-time. We will use ν to denote the number of dimensions of space.

We start with a free scalar boson field at time zero :

$$\phi(x) = \frac{1}{\sqrt{2}\,(2\pi)^{\nu/2}} \int \frac{dk}{\sqrt{\mu(k)}} \; e^{ik\cdot x} \; [a(k) + a^*(-k)]$$

Here, x, k denote space variables only, since we are considering the field at time zero. The creation and annihilation operators have the non-relativistic normalization

$$[a(k),\; a^*(\ell)] = \delta(k-\ell) \quad ,$$

and

$$\mu(k) = \sqrt{\mu_o^2 + k^2} \quad , \quad \mu_o \text{ the mass of the boson.}$$

There is a corresponding free Hamiltonian :

$$H_o = \int dk \quad \mu(k) \quad a^*(k)\; a(k) \quad ,$$

and we want to consider a total Hamiltonian :

$$H = H_o + V$$

$$V = \lambda \int dx \; : \; \phi^4 \; : (x) \quad ,$$

where λ is the "coupling constant" and $: \;\; :$ means Wick ordering. (Wick ordering is the operation on formal expressions for operators in terms of creation and annihilation operators which puts all the annihilation operators to the right of the creation operators; if there are fermion operators present, the resulting expression is also multiplied by (-1) to the number of interchanges of pairs of fermion creation and

annihilation operators necessary to carry out this re-arrangement.)
Written out in terms of creation and annihilation operators :

$$V = \frac{\lambda}{4\,(2\pi)^{\nu}} \int \frac{dk_1 \ldots dk_4}{[\mu(k_1)\ldots\mu(k_4)]^{1/2}} \delta(k_1+\ldots+k_4)[a^*(k_1)\ldots a^*(k_4) + 4a^*(k_1)a^*(k_2)a^*(k_3)a(-k_4) + 6a^*(k_1)a^*(k_2)a(-k_3)a(-k_4)+\ldots]$$

Interacting fields are to be constructed by propagating the free fields at time zero with the total Hamiltonian :

$$\phi^H(x,t) = e^{i\,Ht}\,\phi(x)\;e^{-i\,Ht}$$

and the "physical vacuum" Ψ_o should be the lowest eigenstate of H .
The Wightman functions for the interacting fields are then given by

$$(\Psi_o,\ \phi^H(x_1,\ t_1)\ldots\ \phi^H(x_n,\ t_n)\ \Psi_o)\quad ,$$

and from these Wightman functions one should be able to compute such physical quantities as scattering amplitudes, vacuum polarization, etc.

In deriving the perturbation series for vacuum expectation values, one uses the above formal procedure and treats V as a perturbation on H_o. In point of fact, however, V is not only not small in any reasonable sense, but is so large that it is not an operator at all. To see how this comes about, we remark as a rule of thumb that a formal expression

$$\int f(k_1,\ldots,k_n)\, a^*(k_1)\ldots a^*(k_n)\, dk_1\ldots dk_n,$$

with a symmetric kernel f, cannot define an operator on Fock space unless f is square-integrable. This is true because Fock space is a space of symmetric tensors over the one-particle space $L^2(dk)$, and $\int f(k_1,\ldots,k_n)\, a^*(k_1)\ldots a^*(k_n)\, dk_1\ldots dk_n$ acts by tensoring with $f(k_1,\ldots,k_n)$ and symmetrizing; it is very hard to see how this action can give anything square-integrable unless f is square-integrable itself. (Conversely, if f is square-integrable, it is well-known that $\int f(k_1,\ldots,k_n)\, a^*(k_1)\ldots a^*(k_n)\, dk_1\ldots dk_n$ gives a densely defined operator.)

Now the kernel expressing V in terms of a's and a^*'s contains a δ-function and therefore cannot be square-integrable. It is worth knowing that this particular difficulty, unlike some others we shall see later, is very persistent and cannot be eluded by operator-theoretic tricks. To see this, observe that the δ-function in the kernel is a reflection of the fact that V is defined to be translation-invariant. Thus, a self-adjoint operator on Fock space which is constructed by any reasonable interpretation of the formal expression for $H_o + V$ should commute with translations, and so should the one-parameter group of unitary operators which it generates. This one-parameter group should therefore map the subspace of translation invariant vectors onto itself; since this subspace is one-dimensional and is spanned by the no-particle state, the no-particle state must be an eigenvector for $H_o + V$. But this contradicts the formal expression for

$H_o + V$, which is a sum of terms annihilating the no-particle state plus a term carrying the no-particle state to a four-particle state.

Conclusion : It is impossible to give a reasonable definition for $H_o + V$ as a self-adjoint operator on Fock space.

Some changes must therefore be made in the formal expression for V . We can either :

a. Put the whole theory "in a box with periodic boundary conditions" , i.e., replace physical space $\mathbb{R}^\nu$ by the torus T^ν .

or

b. Put a space cut-off in V , i.e., write

$$V = \lambda \int dx \;\; h(x) \;\; :\phi^4: (x)$$

where h is non-negative and goes to zero at infinity.

Both methods have their merits; for definiteness, we will consider the second.

Then

$$V = \frac{\lambda}{4\,(2\pi)^\nu} \int \frac{dk_1 \ldots dk_4}{[\mu(k_1)\ldots\mu(k_4)]^{1/2}} \;\; \tilde{h}\,(k_1+\ldots+k_4)\; [a^*(k_1)\ldots\; a^*(k_4)+\ldots\;]$$

We have thus to ask whether the kernel :

$$\frac{\tilde{h}\,(k_1+\ldots+k_4)}{[\mu(k_1)\ldots\mu(k_4)]^{1/2}}$$

is square-integrable. Here we get a first glimpse of the advantages of considering space-time of dimension 2 : The kernel is square-integrable if

$\nu = 1$ (provided $\tilde{h}$ decreases reasonably rapidly at infinity) but not in higher dimensions. ($\mu(k) \approx |k|$ for large $|k|$.) Thus, as it turns out, V needs only a space cut-off to make sense in two-dimensional space-time. In more dimensions, we have to do something about the contributions from large values of k , i.e., from high energies. What we will do is simply to remove them by introducing an "ultraviolet cut-off" ; let

$$V(\sigma) = \frac{\lambda}{4\,(2\pi)^{\nu/2}} \int_{|k_i| \leq \sigma} \frac{dk_1 \ldots dk_4}{[\mu(k_1)\ldots\mu(k_4)]^{1/2}} \tilde{h}(k_1+\ldots+k_4)[a^*(k_1)\ldots a^*(k_4)+$$

Then $V(\sigma)$ and $H_o + V(\sigma)$ are easily interpreted as densely defined symmetric operators on Fock space, and we are in a position to begin doing operator theory.

At this point a straightforward, if ambitious, long-range program suggests itself : Use $H_o + V(\sigma)$ to construct interacting fields and the physical vacuum, and hence construct the Wightman functions for the theory with cut-offs. Then study these Wightman functions as the cut-offs are removed and, hopefully, prove that they have a limit. The choice of Wightman functions as the right quantities to study is motivated mostly by the fact that their construction is in principle straightforward and that there is no evidence from perturbation theory that they <u>don't</u> have limits as the cut-offs are removed. The study of the removal of the cut-offs in this context has unfortunately not progressed very far, but there exist at least fairly complete investigations of the theories with cut-offs, contained in the theses of Jaffe and myself [1], [2] .

We consider two specific interactions :

a. Boson self-interaction (Jaffe). Let $P(\xi)$ be a polynomial in one variable which is non-negative on the real axis, and take (formally) :

$$V = \lambda \int dx \ :P(\phi):(x)$$

The analysis requires both a box or space cut-off, and an ultra-violet cut-off, even in two-dimensional space-time where there are no ultra-violet divergences. In fact, the cut-off must be strong enough so that V can be expressed as a polynomial in finitely many a's and a^*'s . The problem then reduces to studying the differential operator

$$-\Delta + f \quad ,$$

f a non-negative polynomial, in a large but finite number of variables, and the method of attack is to use the theory of partial differential equations.

b. Yukawa interaction (Lanford). Here, there are two fields interacting with each other, a Dirac field ψ and a scalar field ϕ . The interaction is given formally by :

$$V = \lambda \int dx \ :\psi^+(x)\,\psi(x):\ \phi(x)$$

We need both a space cut-off or box and an ultra-violet cut-off. (In this theory, there are ultra-violet divergences even in two-dimensional space-time.) With these cut-offs, V becomes a small operator with respect to H_o , and the investigation of the theory with cut-offs is based on perturbation techniques.

Although attention has been directed primarily at these two interactions, it is possible to combine the techniques used to give fairly complete results for any cut-off interaction between fields provided that

1. The total Hamiltonian is formally semi-bounded
2. There are no zero-mass particles.

The problem splits into three parts :

a. The Hamiltonian. In both theories the Hamiltonian, defined on a natural domain, is a semi-bounded essentially self-adjoint operator.

b. Interacting fields. We want to define, for appropriate test-functions $f(x, t)$,

$$\int f(x,t)\, \phi^H(x,t)\, dx\, dt = \int dt \quad e^{i\, Ht} \left[\int dx\, f(x,t)\, \phi(x) \right] e^{-i\, Ht}$$

For this definition to make sense, we have to be sure that $e^{-i\, Ht}$ does not disturb the domain of unbounded operators of the form :

$$\int dx \quad g(x) \quad \phi(x)$$

too much. In both theories, this problem has been controlled; any polynomial in operators of the form $\int f(x,t)\, \phi^H(x,t)\, dx\, dt$, f continuous and rapidly decreasing at infinity, is densely defined.

c. The vacuum. We want there to be an eigenvector of the total Hamiltonian (the vacuum) with eigenvalue at the bottom of the spectrum of H . Moreover, we want the corresponding eigenvalue to have multiplicity one (uniqueness of the vacuum), and we want the eigenvector to belong to the domain of any polynomial in the smeared interacting fields. All these things are true for the boson self-interaction theory; they become true for the Yukawa interaction after a finite mass renormalization, i.e. after a finite change in the masses of the particles.

These results combine to permit the construction of vacuum expectation values of the interacting fields as tempered numerical distributions.

So much for the theory with cut-offs. I now turn to the more interesting question of the existence of limits as the cut-offs are removed. Here, one adopts the pragmatic position of seeking the simplest context to study any given limit. For the limit as the volume goes to infinity, or as the space cut-off goes to a constant, Guenin [3] proposed to study the time-evolution of bounded local observables. This investigation is simpler in at least two respects than the study of Wightman functions :

a. Because one deals with bounded observables, rather than with unbounded smeared fields, domain difficulties are not present.

b. The difficult problem of the existence of a vacuum state is completely separated from other considerations.

The formal idea is the following : If A is a bounded operator which is a function of the fields and their canonical conjugates at time zero smeared with test functions having support in some fixed bounded region $\mathcal{O}$, and if

$$H_h = H_o + \int dx \; h(x) \, \mathcal{H}_I(x)$$

where $\mathcal{H}_I(x)$, the interaction density, is a local quantity, i.e., a function of the fields at the point x , then

$$e^{i H_h t} \, A \, e^{-i H_h t}$$

is independent of h provided h is one on the set of all points from which light signals can be sent into $\mathcal{O}$ in time $|t|$. Hence, trivially,

$$\lim_{h \to 1} . \, e^{i H_h t} \, A \, e^{-i H_h t}$$

exists. If we let $\mathfrak{A}(\mathcal{O})$ denote the von Neumann algebra of all operators A and $\mathfrak{A}$ the norm closure of the union of the $\mathfrak{A}(\mathcal{O})$'s , then a one-parameter group of time-evolution automorphisms τ_t of $\mathfrak{A}$ may be defined by

$$\tau_t(A) = \lim_{h \to 1} e^{i H_h t} \, A \, e^{-i H_h t}$$

The key point in all this is the fact that $e^{iH_h t} A e^{-iH_h t}$ becomes independent of h as soon as h is equal to one on a large enough set. This assertion may be supported by a formal perturbation theory argument (see [3]), but more recently an essentially rigorous argument has been given for boson self-interactions in two-dimensional space-time. It is due to Segal [4] and goes as follows : Let

$$V_h = \int h(x) :P(\phi): (x)\, dx ,$$

where P is a non-negative polynomial. It is known that H_o and V_h are self-adjoint operators and that their sum $H_o + V_h = H_h$ is densely defined. At this point, we come to the only place where the argument is not complete : We have to <u>assume</u> that H_h is essentially self-adjoint for each h *). Then the Trotter product formula (see [5] and the references given there) gives :

$$e^{iH_h t} = \underset{n \longrightarrow \infty}{\text{strong limit}} \left(e^{iH_o t/n} e^{iV_h t/n}\right)^n$$

For any bounded operator A , we have therefore :

$$e^{iH_h t} A e^{-iH_h t} = \lim_{n \to \infty} \left(e^{iH_o t/n} e^{iV_h t/n}\right)^n A \left(e^{-iV_h t/n} e^{-iH_o t/n}\right)^n$$

*) It has very recently been shown by Glimm and Jaffe that this is true, and even that H_h is self-adjoint. See [14] .

We now make two elementary remarks :

1. If $B \in \mathfrak{A}((\alpha, \beta))$, then $e^{iH_o\tau} B e^{-iH_o\tau} \in \mathfrak{A}((\alpha-|\tau|, \beta+|\tau|))$.

2. If $B \in \mathfrak{A}((\alpha, \beta))$, and if $\alpha' < \alpha$, $\beta' > \beta$, then

$$e^{iV_h t} B e^{-iV_h t}$$ belongs to $\mathfrak{A}((\alpha', \beta'))$ and depends only on the values of h on (α', β') .

Applying each of these remarks n times, then taking the limit $n \to \infty$, shows that, if $A \in \mathfrak{A}((a, b))$,

$$e^{iH_h t} A e^{-iH_h t}$$

depends only on the values of h on a neighborhood to $[a - |t|, b + |t|]$ and belongs to $\mathfrak{A}((\alpha, \beta))$ for any $\alpha < a - |t|$, $\beta > b + |t|$.

Besides the existence of the infinite-volume limit for automorphisms, there is a result, due to Jaffe and Powers [6], on the infinite-volume limit of the vacuum state. The idea is as follows : Let f, g be two smooth functions of compact support. For any cubical region Λ whose interior contains the supports of f and g , construct the cut-off ϕ^4 Hamiltonian in the box Λ with periodic boundary conditions; let Ω_Λ be the corresponding vacuum state, and let

$$\omega_\Lambda (f, g) = (\Omega_\Lambda, e^{i(\phi(f)+\pi(g))}\Omega_\Lambda)$$

(where π denotes the field canonically conjugate to ϕ).

If we take a sequence Λ_n of cubes which eventually contains any bounded set, elementary compactness arguments show that there exists a subnet n_α such that

$$\omega(f, g) = \lim_{\alpha} \omega_{\Lambda_{n_\alpha}}(f, g)$$

exists for all f, g . Then ω defines a translation-invariant state of the Weyl algebra for the infinite-volume fields ϕ and π at time zero and is a reasonable candidate for the physical vacuum state. What Jaffe and Powers show is that $\omega(f, g)$ is continuous in (f, g) on finite-dimensional subspaces and that therefore the state defined on the Weyl algebra is <u>regular</u> , i.e., gives rise to a representation of the canonical commutation relations. Although the proof in [6] applies only to the ϕ^4 interaction, the result may be extended to almost any physically reasonable interaction with ultraviolet cut-off. It could also be extended to boson self-interactions in two-dimensional space-time <u>without</u> an ultraviolet cut-off, if it could be shown for these theories that, for all values of the coupling constant, the vacuum energy in a box of volume V decreases at most linearly with V as V goes to infinity.

I come now to the most substantial contribution which has been made to the solution of the problem of the removal of cut-offs : Glimm's work on the definition of the total Hamiltonian without ultraviolet cut-offs. Glimm starts with a formal expression for the total Hamiltonian, containing infinite counterterms which are supposed to cancel the worst-behaved parts of the interaction. The spirit of the investigation is to make these cancellations explicit and thus to construct (on an appropriate concrete Hilbert space) a self-adjoint operator which can reasonably be interpreted as the total Hamiltonian without ultraviolet cut-offs. A space cut-off is always present in the interaction; moreover, problems concerning interacting fields and the vacuum state are at present untouched.

Glimm has studied two specific interactions : The Yukawa interaction in two-dimensional space-time and the ϕ^4 interaction in three dimensions. Although the same underlying formal ideas are used in the two cases, the technical details are quite different. The Yukawa interaction is by far the simpler, and I shall not discuss the methods of proof for the ϕ^4 interaction. However, to begin, I give a summary of the results that have been obtained for both interactions.

First, the Yukawa theory. The problem is to define.

$$H_{ren} = H_o + \int h(x) :\psi^+(x)\psi(x): \phi(x)\, dx + \frac{\delta m^2}{2}\int h^2(x) :\phi^2:(x)dx + c\, \mathbb{1} ,$$

where δm^2 and c are infinite, i.e., are given as divergent integrals. (The term $\frac{\delta m^2}{2}\int h^2(x) :\phi^2:(x)dx$ is a mass renormalization counterterm, and the constant c is to be thought of as adjusting the energy of the ground state.) The procedure followed is first to introduce a cut-off σ in the interaction and the counterterms; this gives a well-defined operator

$$H_{ren}(\sigma) = H_o + \int h(x):\psi_\sigma^+(x)\psi_\sigma(x):\ \phi_\sigma(x)\,dx + \frac{\delta m^2(\sigma)}{2}\int h^2(x):\phi_\sigma^2:(x)\,dx + c(\sigma)\,\mathbb{1}$$

where the quantities $\delta m^2(\sigma)$ and $c(\sigma)$ are finite numbers obtained by putting a corresponding cut-off in the divergent integrals defining δm^2 and c . Next, one constructs a family of unbounded operators $T(\sigma)$ on a dense domain $\mathfrak{D}(T)$, and a limiting operator T such that

$$\lim_{\sigma\to\infty} T(\sigma)\,\Psi = T\,\Psi$$

for all $\Psi \in \mathfrak{D}(T)$, and such that $T\,\mathfrak{D}(T)$ is dense in Fock space. The operator T is called a "dressing transformation" ; its function is to take analytically well-behaved vectors into vectors which have a chance of being in the domain of the singular operator H_{ren} . The first major result is the following : There is a <u>symmetric bilinear form</u> H_{ren} on $T\,\mathfrak{D}(T)$ such that

$$\lim_{\sigma\to\infty} (H_{ren}(\sigma)\,T(\sigma)\Psi,\ T(\sigma)\Phi) = (H_{ren}\,T\Psi,\ T\Phi)$$

for all $\Psi, \Phi \in \mathfrak{D}(T)$. This is essentially the content of [7] .

The second step is to pass from the bilinear form to an operator.

In [8], Glimm shows that, if an appropriate finite change is made in the mass renormalization (i.e., if a fixed finite constant is added to $\delta m^2(\sigma)$ for all σ), then the bilinear form H_{ren} is semi-bounded and closeable and therefore corresponds to a self-adjoint operator by Friedrichs extension techniques. The finite change that must be made in the mass renormalization is annoying, especially since it seems to go to infinity as the space cut-off goes to one. Fortunately, in [9] it is shown that this finite change was not really necessary; H_{ren} is semi-bounded and closeable whatever finite change has been made in the mass renormalization. (In the same reference it is shown that, if $P(\xi)$ is an even non-negative polynomial and if h is non-negative, then in two-dimensional space-time the total boson self-interaction Hamiltonian

$$H_o + \int dx\ h(x)\ :P(\emptyset):\ (x)$$

is a semi-bounded operator on Fock space. This generalizes an earlier result of Nelson [10])

So much for the Yukawa interaction. In [11], Glimm makes a similar attack on the $\emptyset^4$ interaction in three-dimensional space-time. Here again, one wants to define :

$$H_{ren} = H_o + \lambda \int dx\ :\emptyset^4:\ (x)\ h(x) + \frac{\delta m^2}{2} \int :\emptyset^2:\ (x)\ h^2(x)\ dx + c\ \mathbb{1}\ ,$$

where δm^2 and c are infinite. Again, one defines a cut-off Hamiltonian and a family $T(\sigma)$ of cut-off dressing transformations on a fixed domain $\mathfrak{D}(T)$ in Fock space. This time, because the theory is "more divergent" than the Yukawa interaction in two dimensions, $\lim\limits_{\sigma \to \infty} T(\sigma)\Psi$ does not exist in Fock space.

However, if Φ, Ψ belong to $\mathfrak{D}(T)$,

$$\lim_{\sigma \to \infty} \frac{(T(\sigma)\Phi,\ T(\sigma)\Psi)}{\|T(\sigma)\ \Phi_o\|^2}$$

exists (Φ_o is the Fock vacuum). This limit can be used to define a new Hilbert space $\mathcal{H}_{ren}$, and a dressing transformation T mapping $\mathfrak{D}(T)$ to a dense subset of $\mathcal{H}_{ren}$ can be defined by

$$(T\Phi,\ T\Psi)_{ren} = \lim_{\sigma \to \infty} \frac{(T(\sigma)\Phi,\ T(\sigma)\Psi)}{\|T(\sigma)\ \Phi_o\|^2}$$

The main result is the existence of a symmetric operator H_{ren} on $T\,\mathfrak{D}(T) \subset \mathcal{H}_{ren}$ such that

$$(H_{ren}\ T\Phi,\ T\Psi)_{ren} = \lim_{\sigma \to \infty} \frac{(H_{ren}(\sigma)\ T(\sigma)\Phi,\ T(\sigma)\Psi)}{\|T(\sigma)\ \Phi_o\|^2}$$

for all Φ, Ψ in $\mathfrak{D}(T)$.

The appearance of a new Hilbert space on which the renormalized Hamiltonian acts is a phenomenon of considerable physical interest and deserves further investigation. Glimm constructs the Hilbert space $\mathcal{H}_{ren}$ in a fairly concrete, if extremely complicated, way. It would be useful to have a simpler realization of it as a function space, to see whether the creation and annihilation operators act on this function space (i.e., whether the corresponding formal operations give densely defined operators satisfying the canonical commutation relations in Weyl form) and, if so, to study the properties of the representation of the canonical commutation relations so obtained. It would also be useful to know to what extent the space $\mathcal{H}_{ren}$ is uniquely determined by the fact that H_{ren} gives a densely defined operator on it.

It is out of the question, in an exposition of reasonable length, to give detailed proofs for any of the main results. Instead, it seems more useful to try to give an idea of how the proofs work by illustrating the main formal and technical ideas used in the construction of the Hamiltonian for the Yukawa interaction. To start, we will look at an analytically transparent example to show how the dressing-transformation technique can be used to define an operator which, on first sight, looks too singular to make any sense. The operator we want to define is :

$$- i\frac{d}{dx} + M_\delta$$

on $L^2(dx)$, where M_δ means the operator of multiplication by the δ function.

To define this operator, we start by "introducing a cut-off", i.e., by approximating the δ-function by a continuous positive function f with integral one. Let

$$h_f(x) = \exp(-i \int_{-\infty}^{x} f(t)dt)$$

and let $T_f = M_{h_f}$, the operator of multiplication by h_f .
Then a simple calculation gives :

$$[- i \frac{d}{dx} + M_f] T_f = i M_{h'_f} + T_f[- i \frac{d}{dx} + M_f]$$

$$= T_f [- i \frac{d}{dx} - M_f + M_f] = T_f[- i \frac{d}{dx}] .$$

If we now let $f \longrightarrow \delta$ in some reasonable sense, then the operators T_f converge strongly to an operator which can call T_δ , and we can define

$$[- i \frac{d}{dx} + M_\delta] T_\delta \Psi = \lim_{f \to \delta} [- i \frac{d}{dx} + M_f] T_f \Psi = T_\delta[- i \frac{d}{dx}] \Psi$$

for Ψ in $\mathcal{D}(- i \frac{d}{dx})$.

Note that this procedure does <u>not</u> give a definition of M_δ by itself. Instead, commuting the "free Hamiltonian" $- i \frac{d}{dx}$ past the dressing transformation T_δ gives something which cancels the singular "interaction Hamiltonian" M_δ .

We next turn to a more realistic example. The interaction Hamiltonian for the Yukawa theory splits into a sum of eight terms : A term which creates a fermion, an antifermion, and a boson ; a term which creates a fermion and an antifermion and annihilates a boson ; etc.

Let Q_1 be the pure-creation term :

$$Q_1 = \int dp\, dp'\, dk\, \tilde{q}_1(p,p',k)\, a^*(k)\, b^*(p)\, b'^*(p')$$

(b^* denotes a fermion creation operator, b'^* an antifermion creation operator). This expression is only formal, i.e., does not define an operator in any straight-forward way ; the kernel $\tilde{q}_1$ is not square-integrable. We will show how to define

$$H_o + Q_1 + Q_1^* + \Delta + c\,\mathbb{1},$$

where Δ is an (infinite) mass-renormalization counterterm and c an (infinite) constant. We first look just at $H_o + Q_1$, and we proceed formally. Let

$$\Gamma Q_1 = \int dp\, dp'\, dk\ \frac{\tilde{q}_1(p,p',k)}{\omega(p) + \omega(p')+\mu(k)}\ a^*(k)\ b^*(p)\ b'^*(p')$$

($\omega(p) = \sqrt{\omega_o^2 + p^2}$, ω_o the fermion mass).

The kernel $\dfrac{\tilde{q}_1(p,p',k)}{\omega(p)+\omega(p')+\mu(k)}$ is square-integrable, so ΓQ_1 , unlike Q_1 itself, defines an operator on Fock space. Note, however, that

$$[H_o\ ,\Gamma Q_1] = Q_1$$

Now

$$(H_o + Q_1)\, e^{-\Gamma Q_1} = e^{-\Gamma Q_1} \{H_o + Q_1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [\ldots[H_o, \Gamma Q_1], \ldots \Gamma Q_1]\}$$

$$= e^{-\Gamma Q_1} \{H_o + Q_1 - Q_1\} = e^{-\Gamma Q_1} H_o$$

(We are making use of the fact that Q_1 commutes with ΓQ_1 since both are made up out of creation operators alone and each contains an even number of fermion operators.)

Formally, then, we should be able to define

$$(H_o + Q_1)\, e^{-\Gamma Q_1} = e^{-\Gamma Q_1} H_o$$

wherever the right-hand side makes sense. There remains the problem of constructing $e^{-\Gamma Q_1}$; furthermore, to justify the above definition, we should check that

$$(H_o + Q_1(\sigma))\, e^{-\Gamma Q_1(\sigma)} = e^{-\Gamma Q_1(\sigma)} H_o$$

for all values of the ultraviolet cut-off σ. The latter identity follows easily from the argument we just gave if $e^{-\Gamma Q_1(\sigma)}$ can be defined by the power-series expansion for the exponential, i.e., if there is a sufficiently large set of vectors Ψ such that

$$\sum_n \frac{1}{n!} \|(\Gamma Q_1(\sigma))^n \Psi\| < \infty$$

Here, we need a technical lemma :

Lemma. There exists a constant K such that, for any square-integrable kernel $\tilde{r}(p,p',k)$, the corresponding operator

$$R = \int dp\, dp'\, dk\ \ \tilde{r}(p,p',k)\ a^*(k)\ b^*(p)\ b'^*(p')$$

satisfies :

$$\|R\,\Psi\| \le K \cdot \|\tilde{r}\|_2 \cdot \|(N+1)\Psi\|$$

for all $\Psi \in \mathfrak{D}(N)$, where N is the total particle number operator.

From this lemma, and the fact that R increases the particle number by 3, it follows easily that, for $\|\tilde{r}\|_2$ small enough,

$$\sum_{n=1}^{\infty} \frac{1}{n!}\|R^n\Psi\| < \infty$$

for all Ψ in $\mathfrak{D}_o$, the set of vectors with bounded total particle number. Unfortunately,

$$\int \left| \frac{q_1(p,p',k)}{\omega(p)+\omega(p')+\mu(k)} \right|^2 dp\, dp'\, dk$$

need not be small. To get around this difficulty, we use a technical device apparently first used by Nelson in [12] ; we introduce a <u>lower</u> cut-off on the momenta. If, instead of ΓQ_1 , we consider $\Gamma Q_1 - \Gamma Q_1(\rho)$ with ρ sufficiently large, we get an operator whose kernel has L^2 norm which is as small as we like. Thus, by making ρ

large enough, we can guarantee that

$$\sum_n \frac{1}{n!} \|(\Gamma Q_1 - \Gamma Q_1(\rho))^n \Psi\| < \infty \quad \text{for all } \Psi \text{ in } \mathfrak{D}_o$$

and also that :

$$e^{-(\Gamma Q_1 - \Gamma Q_1(\rho))}\{\mathfrak{D}_o \cap \mathfrak{D}(H_o^2)\}$$

is dense in Fock space. Then, formally,

$$(H_o + Q_1)\; e^{-(\Gamma Q_1 - \Gamma Q_1(\rho))} = e^{-(\Gamma Q_1 - \Gamma Q_1(\rho))}(H_o + Q_1(\rho)) .$$

The right-hand side is well-defined on $\mathfrak{D}(H_o) \cap \mathfrak{D}_o$, so $H_o + Q_1$ may be rigorously defined on the dense domain

$$e^{-(\Gamma Q_1 - \Gamma Q_1(\sigma))}\{\mathfrak{D}_o \cap \mathfrak{D}(H_o)\}$$

by this formula. Similarly, for $\sigma > \rho$,

$$(H_o + Q_1(\sigma))\; e^{-(\Gamma Q_1(\sigma) - \Gamma Q_1(\rho))} = e^{-(\Gamma Q_1(\sigma) - \Gamma Q_1(\rho))}(H_o + Q_1(\rho))$$

so, letting

$$T(\sigma) = e^{-(\Gamma Q_1(\sigma) - \Gamma Q_1(\rho))} \quad , \; T = e^{-(\Gamma Q_1 - \Gamma Q_1(\rho))}$$

we get

$$\lim_{\sigma\to\infty} T(\sigma)\Psi = T\Psi \qquad (\Psi \in \mathfrak{D}_o)$$

and

$$\lim_{\sigma\to\infty} (H_o + Q_1(\sigma))\, T(\sigma)\Psi = (H_o + Q_1)\, T\Psi \qquad (\Psi \in \mathfrak{D} \cap \mathfrak{D}(H_o)) .$$

Thus, $H_o + Q_1$ has been constructed as a densely-defined operator which is the limit, in a reasonable sense, of $H_o + Q_1(\sigma)$ as σ goes to infinity. Note that no renormalization counterterms have been needed in this construction.

It remains to deal with Q_1^* + mass renormalization. To simplify the formulas, we will assume that we don't need the lower momentum cut-off ρ . Because we have left out some terms in the interaction, we can't use the full mass renormalization counterterm, but only the number-conserving part, i.e.,

$$\Delta(\sigma) = \text{const}(\sigma) \cdot \int\limits_{\substack{|k|\leq\sigma \\ |\ell|\leq\sigma}} \frac{dk\, d\ell}{[\mu(k)\mu(\ell)]^{1/2}} \quad \tilde{h}^2\,(k-\ell)\; a^*(k)a(\ell)$$

(The constant will be determined later and will go to infinity as σ does.)

What we have to do is to study :

$$(Q_1^*(\sigma) + \Delta(\sigma) + c(\sigma)\,\mathbb{1})\; e^{-\Gamma Q_1(\sigma)}$$

as $\sigma \longrightarrow \infty$. The technique used is to commute the operator on the left through the exponential and to write the result in Wick order, i.e., with the annihilation operators on the right. This gives a polynomial of bounded degree in the creation and annihilation operators, multiplied on the left by $e^{-\Gamma Q_1(\sigma)}$. If the operator defined by the polynomial has a limit as σ goes to infinity, so does

$$(Q_1^*(\sigma) + \Delta(\sigma) + c(\sigma)\,\mathbb{1})\; e^{-\Gamma Q_1(\sigma)}$$

Thus, the problem reduces to studying the finite number of kernels defining the polynomial, i.e., to questions of computation. The computations are forbiddingly complicated if approached in a straightforward way ; fortunately, there is a formal device, due to Friedrichs [13], which greatly simplifies the grouping of terms.

To see how this formal device works, we have to recall how the operation of Wick-ordering a product of two polynomials in creation and annihilation operators goes. Let P, R be two such polynomials ; we will suppose R to be made up out of creation operators only and P to be Wick-ordered. To get P.R expressed as a Wick-ordered polynomial, the annihilation operators in P must be commuted through R , using the commutation relations. Each time an annihilation operator is commuted past a creation operator, one obtains a new term with a δ-function in the corresponding variables. Such a term we will refer to as a <u>contraction</u>. Each contracted term must itself be written in Wick order ; this gives new terms with more variables contracted. The net result is that P.R = R.P + the sum of all possible contractions between P and R , Wick ordered.

The operator that we actually want to analyze is of the form :

$$P\, e^{-R} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!}\, P.\, R^n$$

For each n, we define the <u>connected product</u> $P \multimap R^n$ to be the sum of all contractions between P and R^n in which at least one variable in each factor R is contracted. Note that $P \multimap R^n = 0$ if n is greater than the number of annihilation operators in P. The formula of Friedrichs now says :

$$P \cdot e^{-R} = e^{-R}\left[P + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} : P \multimap R^n :\right]$$

(For a proof of this formula, see [7], §3. 3.)

Thus we get :

$$\left.\begin{aligned} (Q_1^*(\sigma) + \Delta(\sigma) + c(\sigma)\mathbb{1})\, e^{-\Gamma Q_1(\sigma)} &= e^{-\Gamma Q_1(\sigma)}\{Q_1^*(\sigma) + \Delta(\sigma) + c(\sigma)\mathbb{1} \\ &- : Q_1^*(\sigma) \multimap \Gamma Q_1(\sigma) : - : \Delta(\sigma) \multimap \Gamma Q_1(\sigma) : \\ &+ \frac{1}{2} : Q_1^*(\sigma) \multimap (\Gamma Q_1(\sigma))^2 : - \frac{1}{6} : Q_1^*(\sigma) \multimap (\Gamma Q_1(\sigma))^3 :\} \end{aligned}\right] \quad (*)$$

When written out in detail, i.e., indicating the different ways the contractions may be made, the expression in braces becomes even more complicated. We will not discuss all the terms, but will look at a few representative ones to show what tricks are used for handling them.

Before doing this, it will be useful to make a few remarks about the problem of finding dense domains for formal polynomials in the creation and annihilation operators. We have already discussed what happens for pure creation operators : An expression like

$$R = \int \tilde{r}(k_1,\ldots,k_n)\ a^*(k_1)\ldots\ a^*(k_n)\ dk_1\ldots dk_n$$

makes sense as an operator if and only if $\tilde{r}$ is square-integrable. Moreover, if $\tilde{r}$ is square-integrable, then the domain of R contains $\mathfrak{D}_0$ and, for $\Psi \in \mathfrak{D}_0$, $R\Psi$ varies continuously with $\tilde{r}$. The latter features persist if some of the creation operators are replaced by annihilation operators ; as long as we have to do with a square-integrable kernel, everything is easily controlled. If there are annihilation operators present, however, the condition that the kernel be square - integrable can be weakened. For example, if $\tilde{r}$ is <u>any</u> Lebesgue-measurable complex-valued function, then

$$\int \tilde{r}(k_1,\ldots,k_n)\ a(k_1)\ \ldots\ a(k_n)\ dk_1\ldots dk_n$$

is in a natural way a densely-defined operator. (It is easily defined on those vectors Ψ which have bounded free energy and which are such that $\tilde{r}(k_1,\ldots,k_n)$ is essentially bounded on

$$\{(k_1,\ldots,k_n)\ :\ a(k_1)\ldots\ a(k_n)\ \Psi \neq 0\}\ .$$

One can also easily make more precise statements : If $\dfrac{\tilde{r}(k_1,\ldots,k_n)}{\mu(k_{n-1})\mu(k_n)}$

is square-integrable, then

$$R = \int \tilde{r}(k_1,\ldots,k_n)\ a(k_1)\ldots\ a(k_n)\ dk_1\ldots dk_n$$

is defined on any Ψ in $\mathfrak{D}_o \cap \mathfrak{D}(H_o^2)$; and $R\,\Psi$ varies continuously with the kernel $\tilde{r}$ in the obvious sense. This remains true if some or all of the annihilation operators $a(k_1) \cdots a(k_{n-2})$ are replaced by creation operators. Finally, although we have discussed only boson operators, the same remarks hold for fermion operators or for mixed expressions.

Returning to the consideration of the expression in braces in $(*)$, we apply first the remark just made to show that, if $\Psi \in \mathfrak{D}_o \cap \mathfrak{D}(H_o^2)$,

$$\lim_{\sigma \to \infty} \quad Q_1{}^*(\sigma)\Psi$$

exists since the kernel $\dfrac{\overline{\tilde{q}_1(p,p',k)}}{\omega(p)\omega(p')}$ is square-integrable.

Second, we write out $:\ Q_1{}^*(\sigma) \angle\, Q_1(\sigma)\ :$ in terms of the various ways the contractions can be made :

$$: Q_1{}^*(\sigma) \angle \ulcorner Q_1(\sigma) : \; = \; : Q_1{}^*(\sigma) \underset{1,0}{-\!\circ\!-} \ulcorner Q_1(\sigma) : + : Q_1{}^*(\sigma) \underset{0,1}{-\!\circ\!-} \ulcorner Q_1(\sigma) :$$

$$+ \; : Q_1{}^*(\sigma) \underset{2,0}{-\!\circ\!-} \ulcorner Q_1(\sigma) : + : Q_1{}^*(\sigma) \underset{2,1}{-\!\circ\!-} \ulcorner Q_1(\sigma) :$$

(Here $\underset{j,j}{-\!\circ\!-}$ means the sum of all terms with exactly i fermion contractions and j boson contractions.) The terms $: Q_1{}^*(\sigma) \underset{1,0}{-\!\circ\!-} \ulcorner Q_1(\sigma) :$ and $: Q_1{}^*(\sigma) \underset{0,1}{-\!\circ\!-} \ulcorner Q_1(\sigma) :$ are all right for the same sort of reasons as $Q_1{}^*(\sigma)$; they have enough fermion annihilation variables free to take advantage of the fact that they are being applied to vectors in $\mathfrak{D}(H_0^2)$. The term $Q_1^*(\sigma) \underset{2,1}{-\!\circ\!-} \Gamma Q_1(\sigma)$ is just a number, which goes to infinity as σ does ; we adjust $c(\sigma)$ to cancel it.

The term $Q_1^*(\sigma) \underset{2,0}{-\!\circ\!-} \Gamma Q_1(\sigma)$ is more interesting ; it has to be cancelled by the infinite mass renormalization.

$$:Q_1^*(\sigma) \underset{2,0}{-\!\circ\!-} \Gamma Q_1(\sigma): = \int_{\substack{|p|\le\sigma \quad |k|\le\sigma \\ |p|\le\sigma \quad |\ell|\le\sigma}} dp\, dp'\, dk\, d\ell \; \frac{\tilde{q}_1(p,p',k)\overline{\tilde{q}_1(p,p',\ell)}}{\omega(p) + \omega(p') + \mu(k)}$$

$$a^*(k)\; a(\ell) \; .$$

We now need an explicit formula for $\tilde{q}_1$:

$$\tilde{q}_1(p,p',k) = \tilde{h}(p+p'+k) \; \frac{S(p,p')}{\mu(k)^{1/2}} \qquad ,$$

where $S(p,p')$ is real and bounded. Hence :

$$:Q_1^*(\sigma) \underset{2,0}{-\!\circ\!-} \Gamma Q_1(\sigma): = \int_{\substack{|k|\le\sigma \\ |\ell|\le\sigma}} dk\, d\ell \; \frac{a^*(k)\; a(\ell)}{\mu(k)^{1/2}\; \mu(\ell)^{1/2}} \int_{\substack{|p|\le\sigma \\ |p'|\le\sigma}}$$

$$dp\, dp'\; \tilde{h}(p+p'+k)\overline{\tilde{h}}(-p-p'-\ell)\, \frac{(S(p,p'))^2}{\omega(p)+\omega(p')+\mu(k)}$$

$$= \int_{\substack{|k|\le\sigma \\ |\ell|\le\sigma}} dk\, d\ell\, \frac{a^*(k)a(\ell)}{\mu(k)^{1/2}\mu(\ell)^{1/2}} \left\{ \frac{1}{2} \int_{I_\sigma} ds\, dt\, \frac{\tilde{h}(s+k)\tilde{h}(-s-\ell)(S(\frac{s+t}{2}, \frac{s-t}{2}))^2}{\omega(\frac{s+t}{2}) + \omega(\frac{s-t}{2}) + \mu(k)} \right\}$$

(We have changed variables from (p,p') to $s = p+p'$; $t = p-p'$; I_σ denotes the region of integration in the new variables.) For any fixed s,k, the integral over t diverges logarithmically as $\sigma \longrightarrow \infty$. The above expression is to be subtracted from :

$$\Delta(\sigma) = \int_{\substack{|k|\le\sigma \\ |\ell|\le\sigma}} dk\, d\ell\, \frac{a^*(k)a(\ell)}{\mu(k)^{1/2}\mu(\ell)^{1/2}} \left\{ \text{const}(\sigma) \int ds\, \tilde{h}(s+k)\, \tilde{h}(-s-\ell) \right\}$$

If we take

$$\text{const}(\sigma) = \int_{|t|\le\sigma} dt\, \frac{[S(t,-t)]^2}{2\omega(t) + \mu_o}$$

we get exact cancellation between these two expressions for $s = 0$, $k = 0$. It turns out that, with this choice for const(σ) , the kernel of

$$- : Q_1^*(\sigma) \underset{2,0}{\multimap} \Gamma Q_1(\sigma) : + \Delta(\sigma)$$

converges in L^2 as σ goes to infinity, so

$$\lim_{\sigma\longrightarrow\infty} \{- : Q_1^* \underset{2,0}{\multimap} \Gamma Q_1(\sigma) : + \Delta(\sigma)\} \Psi$$

exists for every Ψ in $\mathfrak{D}_o$.

It will be left to the reader to investigate the behavior of the remaining terms in (*) . To conclude, we summarize Glimm's treatment of the total Yukawa Hamiltonian. First, the interaction $\mathcal{V}$ is split into a pair creation and annihilation part $\mathcal{V}_1$, and the remainder $\mathcal{V}_2$ which is made up of terms corresponding to the emission and absorption of bosons by fermions. We have shown how to deal with half of $\mathcal{V}_1$. The whole of $\mathcal{V}_1$ can be handled by similar techniques, using a more complicated dressing transformation and the full mass renormalization counterterm. This gives :

$$H_o + \mathcal{V}_1 + \text{counterterms}$$

as a symmetric operator (not just as a bilinear form) on the (dense) range of the dressing transformation. The remainder $\mathcal{V}_2$ of the interaction, without counterterms, is then shown to define a bilinear form on the range of the dressing transformation.

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