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A Non-Central Functional Limit Theorem for Quadratic Forms in Martingale Difference Sequences

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Abstract

For quadratic forms with nulls on the diagonal the partial-sum process is both a martingale and a stochastic integral. Using corresponding tools we derive a result on the convergence to the double Wiener-Ito integral.

Let $\{\mathbf{A}^n = (a_{i,j}^n)\}_{n \in \mathbf{N}}$ be a sequence of infinite matrices with nulls on the diagonals:

$$a_{i,i}^n = 0, \quad i = 1, 2, \dots, n \in \mathbf{N}. \quad (1)$$

Let $\{X_j\}_{j \in \mathbf{N}}$ be a martingale difference sequence with respect to some filtration $\{\mathcal{F}_j\}_{j \in \mathbf{N} \cup \{0\}}$ such that

$$E(X_j^2 | \mathcal{F}_{j-1}) = 1, \quad j = 1, 2, \dots \quad (2)$$

Then for each $n \in \mathbf{N}$ the process

$$S_{n,k} = \sum_{1 \leq i, j \leq k} a_{j,i}^n X_i X_j = \sum_{j \leq k} \left(\sum_{i < j} (a_{j,i}^n + a_{i,j}^n) X_i \right) X_j \quad k = 1, 2, \dots, \quad (3)$$

is both a *square integrable* martingale and a stochastic integral with respect to $\{\mathcal{F}_j\}$.

Using the martingale structure, a Functional Central Limit Theorem for suitably scaled $\{S_{n,k}\}_{k \in \mathbf{N}}$ was proved in [JaMé90]. In the present paper we state a non-central functional limit theorem based on limit theorems for stochastic integrals given in [JMP89]. The limit is identified with a double Wiener-Ito stochastic integral (see e.g. [Ito51] or [Maj81]). In some sense it is not surprising: such limits arise, for example, in limit theory for quadratic forms in stationary gaussian sequences exhibiting long-range dependence (see [Ros79], also [SuHo86]).

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For each $n \in \mathbf{N}$, let $\overline{\mathbf{A}}^n$ be the symmetrization of \mathbf{A}^n , i.e. the matrix with entries $(a_{i,j}^n + a_{j,i}^n)/2$. For \mathbf{A}^n , define its representation in $L^2([0, T]^2)$ by the formula

$$\widetilde{\mathbf{A}}^n(u, v) = a_{[nu], [nv]}^n \quad \text{for } T \geq u, v \geq 0. \quad (4)$$

Finally, let

$$X^n(t) = n^{-1/2} \sum_{1 \leq j \leq [nt]} X_j \quad (5)$$

$$Y^n(t) = n^{-1} \sum_{1 \leq i, j \leq [nt]} a_{i,j}^n X_i X_j. \quad (6)$$

Theorem Suppose \mathbf{A}^n 's satisfy (1) and $\{X_j\}$ is a martingale difference sequence such that (2) holds.

If $\widetilde{\mathbf{A}}^n$ converges in $L^2([0, T]^2)$ to some function A , then

$$Y^n \xrightarrow{\mathcal{D}} Y, \quad (7)$$

where

$$Y(t) = \int \int A(u, v) I_{[0,t]^2}(u, v) dW_u dW_v \quad (8)$$

is the classical Wiener-Ito integral.

PROOF. For each $\varepsilon > 0$ there are continuous functions ϕ_l, ψ_l , $l = 1, 2, \dots, m$ and numbers $\alpha_1, \dots, \alpha_m$ such that

$$\Phi(s, t) = \sum_{l=1}^m \alpha_l \cdot \frac{\phi_l(s)\psi(t) + \psi_l(s)\phi_l(t)}{2}$$

satisfies

$$\|A - \Phi\|_2 < \varepsilon. \quad (9)$$

Hence for $n \geq n_1$, $\|\widetilde{\mathbf{A}}^n - \Phi\|_2 < 2\varepsilon$ and by continuity of Φ $\|\widetilde{\mathbf{A}}^n - \widetilde{\Phi}_n\|_2 < 3\varepsilon$ for $n \geq n_2$, where $\widetilde{\Phi}_n(s, t) = \Phi([ns]/n, [nt]/n)$.

If

$$Z^n(t) = n^{-1} \sum_{1 \leq i \neq j \leq [nt]} \Phi(i/n, j/n) X_i X_j, \quad (10)$$

then for $n \geq n_2$

$$E \sup_{1 \leq t \leq T} |Y^n(t) - Z^n(t)|^2 \leq 4E|Y^n(T) - Z^n(T)|^2 \leq 4\|\widetilde{\mathbf{A}}^n - \widetilde{\Phi}_n\|_2^2 \leq 4 \cdot 9\varepsilon^2. \quad (11)$$

Let

$$Z(t) = \int \int \Phi(u, v) I_{[0,t]^2}(u, v) dW_u dW_v. \quad (12)$$

By the isometry property for Wiener-Ito integrals

$$E \sup_{1 \leq t \leq T} |Y(t) - Z(t)|^2 \leq 4E|Y(T) - Z(T)|^2 = 4\|A - \Phi\|_2^2 \leq 4\varepsilon^2. \quad (13)$$

Hence it is enough to prove that

$$Z^n \xrightarrow{\mathcal{D}} Z \quad (14)$$

on the space $\mathcal{D}([0, T])$. But

$$\begin{aligned} Z_n(t) &= 2n^{-1} \sum_{1 \leq i < j \leq [nt]} \Phi(i/n, j/n) X_i X_j \\ &= \sum_{l=1}^m \alpha_l \left(\sum_{1 \leq i < j \leq [nt]} \phi_l(i/n) \psi_l(j/n) \frac{X_i}{\sqrt{n}} \frac{X_j}{\sqrt{n}} + \right. \\ &\quad \left. + \sum_{1 \leq i < j \leq [nt]} \psi_l(i/n) \phi_l(j/n) \frac{X_i}{\sqrt{n}} \frac{X_j}{\sqrt{n}} \right) \\ &= \sum_{l=1}^m \alpha_l \left(\int_0^t \psi_l(v) \left(\int_0^{v-} \phi_l(u) dX_u^n \right) dX_v^n + \right. \\ &\quad \left. + \int_0^t \phi_l(v) \left(\int_0^{v-} \psi_l(u) dX_u^n \right) dX_v^n \right) \\ &\xrightarrow{\mathcal{D}} \sum_{l=1}^m \alpha_l \left(\int_0^t \psi_l(v) \left(\int_0^{v-} \phi_l(u) dW_u \right) dW_v + \right. \\ &\quad \left. + \int_0^t \phi_l(v) \left(\int_0^{v-} \psi_l(u) dW_u \right) dW_v \right) \\ &= Z(t), \end{aligned}$$

where the convergence in distribution holds by [JMP89, Theorem 2.6] \square

Example 1. Let $f : [0, 1]^2 \rightarrow \mathbf{R}^1$. Define quadratic forms

$$a_{i,j}^n = f(i/n, j/n).$$

If $g_n(u, v) = f([nu], [nv]) \xrightarrow{L^2} f(u, v)$, then

$$n^{-1} \sum_{1 \leq i, j \leq [nt]} a_{i,j}^n X_i X_j \xrightarrow{\mathcal{D}} \int_0^1 \int_0^t \bar{f}(u, v) dW_u dW_v,$$

where \bar{f} is the symmetrization of f .

Example 2. Let

$$c_0 = 0, c_1 = b_1, c_2 = b_2, \dots, c_d = b_d, c_{d+1} = b_1, c_{d+2} = b_2, \dots$$

Define a matrix \mathbf{A} by

$$a_{i,j} = c_{|i-j|}.$$

In this case $\widetilde{\mathbf{A}}^n$ does not converge, so we cannot apply directly our theorem.

Take $n = N \cdot d$ and define a rearranging of coordinates:

$$\begin{aligned}
e_1 &\mapsto e_1 \\
e_2 &\mapsto e_{N+1} \\
e_3 &\mapsto e_{2N+1} \\
&\vdots \\
e_d &\mapsto e_{(d-1)N+1} \\
e_{d+1} &\mapsto e_2 \\
e_{d+2} &\mapsto e_{N+2} \\
&\vdots
\end{aligned}$$

(The general formula is of the form $e_{k \cdot d + i} \mapsto e_{(i-1)N + k + 1}$ if $0 \leq k \leq N - 1$, $1 \leq i \leq d$).

Under this rearranging, the distribution of X^n remains unchanged, while \mathbf{A}^n transforms to the form $\mathbf{D}^n = (d_{p,q}^n)$, where for $p = (i - 1)N + k + 1$ and $q = (j - 1)N + l + 1$

$$d_{p,q}^n = \begin{cases} 0 & \text{if } j = i, l = k \\ b_d & \text{if } j = i, l \neq k \\ b_{|j-i|} & \text{if } j > i, l \geq k \text{ or } j < i, l \leq k \\ b_{d-|j-i|} & \text{if } j > i, l < k \text{ or } j < i, l > k. \end{cases}$$

It is easy to see that now $\widetilde{\mathbf{D}}^n \rightarrow D$ in $L^2([0, 1]^2)$, where

$$D(u, v) = \begin{cases} 0 & \text{if } u = v \\ b_d & \text{if } [d \cdot u] = [d \cdot v], u \neq v \\ b_{|[d \cdot u] - [d \cdot v]|} & \text{if } [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} \geq \{d \cdot v\} \\ & \text{or} \\ & [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} \leq \{d \cdot v\} \\ b_{d - |[d \cdot u] - [d \cdot v]|} & \text{if } [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} < \{d \cdot v\} \\ & \text{or} \\ & [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} > \{d \cdot v\} \end{cases}$$

Here $[x]$ is the integer part of x and $\{x\} = x - [x]$.

The convergence is easily extendible from subsequence $N \cdot d$ to the whole \mathbf{N} . But we **do not have the functional convergence**, since

$$n^{-1} \sum_{1 \leq i, j \leq [nt]} a_{i,j}^n X_i X_j$$

and

$$n^{-1} \sum_{1 \leq i, j \leq [nt]} d_{i,j}^n X_i X_j$$

have nothing common except for $t = 1$, where they coincide.

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