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# ERGODIC THEORY FOR

# INNER FUNCTIONS OF THE UPPER HALF PLANE

Jon Aaronson

Abstract :

The real restriction of an inner function of the upper half plane leaves Lebesgue measure quasi-invariant. It may have a finite or infinite invariant measure. We give conditions for the rational ergodicity and exactness of such restrictions.

Abstrait :

La restriction à la droite réelle d'une fonction intérieure du demi-plan supérieur laisse la mesure de Lebesgue quasi-invariante, et peut avoir une mesure invariante finie ou infinie. Nous donnons les conditions pour l'ergodicité rationnelle et l'exactitude de telles transformations.

Rennes,

December 1977

# ERGODIC THEORY FOR

# INNER FUNCTIONS OF THE UPPER HALF PLANE

Jon Aaronson

## §0 - Introduction

In this paper, we consider the ergodic properties of the real restrictions of inner functions on the open upper half plane : $\mathbb{R}^{2+} = \{x+y : x,y \in \mathbb{R},\ y > 0\}$ .

Let $f:\mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an analytic function. We say that $f$ is an inner function on $\mathbb{R}^{2+}$ if for $\lambda$-a.e. $x \in \mathbb{R}$ the limit $\lim_{y\downarrow 0} f(x+iy)$ exists, and is real. (Here, and throughout the paper, $\lambda$ denotes Lebesgue measure on $\mathbb{R}$). Consider the limit $\lim_{y\downarrow 0} f(x+iy) = Tx$ . This is defined $\lambda$-a.e. on $\mathbb{R}$ . We call this limit the (real) restriction of $f$ , and will sometimes write this as $T = T(f)$ . We will denote the class of inner functions on $\mathbb{R}^{2+}$ by $I(\mathbb{R}^{2+}) = I$ , and their real restrictions by $M(\mathbb{R})$. We note that $f \in I(\mathbb{R}^{2+})$ iff $\phi^{-1} f \phi(z)$ is an inner function of the unit disc, according to the definition on p. 370 of [8] (where $\phi(z) = i(\frac{1+z}{1-z})$) .

The following characterisation of $I(\mathbb{R}^{2+})$ appears in [6] and [17].

$f \in I(\mathbb{R}^{2+})$ iff

(0-1) $\quad f(\omega) = \alpha\omega + \beta + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t)$ where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\mu$ is a bounded, positive Borel measure, singular w.r.t. $\lambda$ . Since we shall be refering to (0-1) rather a lot, we shall denote the class of bounded, positive, singular measures on $\mathbb{R}$ by $S(\mathbb{R})$ .

G. Letac ([6]) has shown that a measurable transformation $T$ of $\mathbb{R}$ preserves the class of Cauchy distributions iff either $T \in M(\mathbb{R})$ or $-T \in M(\mathbb{R})$. In particular, if $dP_{a+ib}(x) = \frac{b}{\pi} \frac{dx}{(x-a)^2+b^2}$ for $a+ib \in \mathbb{R}^{2+}$ and $T = T(f) \in M(\mathbb{R})$, then :

$$(0.2) \qquad P_\omega \circ T^{-1} = P_{f(\omega)} \quad \text{for} \quad \omega \in \mathbb{R}^{2+}$$

This equation shows that $M(\mathbb{R})$ is a class of non-singular transformations of the measure space $(\mathbb{R},\mathcal{B},\lambda)$, and is therefore an object of ergodic theory.

Let $f \in I(\mathbb{R}^{2+})$ have a fixed point $\omega_0 \in \mathbb{R}^{2+}$. By (0.2), $T(f)$ preserves the Cauchy distribution $P_{\omega_0}$. It was shown in [16], that if $f$ is $1-1$, then $T(f)$ is conjugate to a rotation of the circle, and shown in [15] that otherwise, $T(f)$ is mixing. We show in §1 that if $f$ is not $1-1$ then $T(f)$ is exact.

In §2 we recall some well known facts about inner functions of $\mathbb{R}^{2+}$. The Denjoy-Wolff theorem (see [13], [14] and [18]) adapted to $\mathbb{R}^{2+}$ shows that when studying the ergodic properties of $T(f)$, for $f \in I(\mathbb{R}^{2+})$ with no fixed points in $\mathbb{R}^{2+}$, we may assume that $\alpha(f) \geq 1$. In case $\alpha(f) > 1$, $T(f)$ is dissipative, and when $\alpha(f) = 1$, $T(f)$ préserves Lebesgue measure.

In §3, we consider the case $\alpha(f) = 1$. Here, the conservativity of a restriction $T(f)$ is sufficient for its rational ergodicity ([1]) (ergodicity was established in [15]). We also give sufficient conditions for exactness, and discuss the similarity classes ([1]) of restricitions.

The ergodic theory of certain restrictions has been considered in [2], [5], [7], [10], [11], [15] and [16]

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## § 1 - Mixing restrictions preserving finite measures

Theorem 1.1

Let $f \in I(\mathbb{R}^{2+})$ and assume that $f$ is not $1-1$. If $f$ has a fixed point $\omega_0 \in \mathbb{R}^{2+}$, then $(R,\mathcal{B},P_{\omega_0},T(f))$ is an exact measure preserving transformation.

i.e. $\bigcap_{n\geq 1} T^{-1}\mathcal{B} = \{\emptyset,R\} \bmod \lambda$.

Before proving theorem 1.1., we shall need some auxiliary results. The first of these is Lin's criterion for exactness of Markov operators (theorem 4.4. in [7]) as applied to our case. To state this, we shall need some extra notation :

Let $T \in M(\mathbb{R})$, then $(\mathbb{R},\mathcal{B},\lambda,T)$ is a non-singular transformation, and so $g \in L^\infty(\mathbb{R},\mathcal{B},\lambda)$ iff $g\circ T \in L^\infty(\mathbb{R},\mathcal{B},\lambda)$. We define the dual operator of $T$, $\hat{T} : L^1(\mathbb{R},\mathcal{B},\lambda) \to L^1(\mathbb{R},\mathcal{B},\lambda)$ by

$$\int_R \hat{T}h.g\,d\lambda = \int_R h.g\circ T\,d\lambda \qquad \text{for} \quad h \in L^1 \quad \text{and} \quad g \in L^\infty$$

If we write, for $\omega = a+ib \in \mathbb{R}^{2+}$

$$\frac{dP_\omega}{d\lambda}(x) = \phi_\omega(x) = \frac{b}{\pi} \cdot \frac{1}{(x-a)^2+b^2}$$

then equation (0.2) translates to :

(1.1) $\qquad \hat{T}\phi_\omega = \phi_{f(\omega)}$ for $T = T(f) \in M(\mathbb{R})$

Clearly, $\hat{T}$ is a positive linear operator, $\int_R \hat{T}h\,d\lambda = \int_R h\,d\lambda$ for $h \in L^1$.

Lin's Criterion (for restrictions) Let $T = T(f) \in M(\mathbb{R})$.

$T$ is exact iff

(1.2) $\qquad ||\hat{T}^n u||_1 \to 0$ for every $u \in L^1$, $\int_R u\,d\lambda = 0$. (Here, and throughout, $||u||_1 = \int_R |u|\,d\lambda$).

We will need the following elementary lemma:

Lemma 1.2 If $\omega_n \in \mathbb{R}^{2+}$ and $\omega_n \to \omega \in \mathbb{R}^{2+}$ then :

$$||\phi_{\omega_n} - \phi_\omega||_1 \to 0$$

Proof of the theorem 1.1

We first show that $f^n(\omega) \to \omega_0 \quad \forall \omega \in \mathbb{R}^{2+}$, where $f^1(\omega) = f(\omega)$ and $f^{n+1}(\omega) = f(f^n(\omega))$.

Let $\emptyset : U = [|Z|<1] \to \mathbb{R}^{2+}$ be a conformal map. Then $g = \emptyset^{-1} f \emptyset : U \to U$ is analytic, and $g(\emptyset(\omega_0)) = \emptyset(\omega_0)$. By the Schwartz lemma ([9]) : $|g'(\emptyset(\omega_0))| < 1$ as $g$ is not 1 - 1 . It is now not hard to see that $g^n(Z) \to \emptyset(\omega_0) \quad \forall z \in U$, and hence that $f^n(\omega) \to \omega_0 \quad \forall \omega \in \mathbb{R}^{2+}$ .

Hence, by lemma 1.2

$$||\hat{T}^n \phi_\omega - \phi_{\omega_0}||_1 = ||\phi_{f^n(\omega)} - \phi_{\omega_0}||_1 \to 0 \quad \text{for} \quad \omega \in \mathbb{R}^{2+} .$$

We will now establish that

$$||\hat{T}^n u||_1 \to 0 \quad \text{for} \quad u \in L^1 \quad \text{with} \quad \int_{\mathbb{R}} u d\lambda = 0$$

which, by Lin's criterion, will ensure the exactness of $T$ .

Let $u \in L^1$ with $\int_{\mathbb{R}} u d\lambda = 0$ and let $\varepsilon > 0$ . By Wiener's Tauberian theorem (see [12] p.357), there exist $\alpha_1 \ldots \alpha_N$ , $a_1 \ldots a_N \in \mathbb{Q}$ such that

$$||u - \sum_{j=1}^{N} \alpha_j \phi_{a_j + i}||_1 < \varepsilon/2$$

Clearly, this implies that $|\sum_{j=1}^{N} \alpha_j| < \varepsilon/2$ and so :

$$||\hat{T}^{n} u||_1 \leq$$

$$\leq ||\hat{T}^{n} (u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i})||_1 + ||\hat{T}^{n} (\sum_{j=1}^{N} \alpha_j (\phi_{a_j+i} - \phi_{\omega_0}))||_1 + ||\sum_{j=1}^{N} \alpha_j \phi_{\omega_0}||_1 \leq$$

$$\leq ||u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i}||_1 + \sum_{j=1}^{N} |\alpha_j| \, ||\hat{T}^{n} \phi_{a_j+i} - \phi_{\omega_0}||_1 + |\sum_{j=1}^{N} \alpha_j| <$$

$$< \varepsilon + o(1) \quad \text{as} \quad k \to \infty \quad \square$$

Since $\varepsilon > 0$ was arbitrary : $||\hat{T}^{n} u||_1 \to 0$ . $\square$

## §2 - Basic Classification

**Proposition 2.1** ([17] p.151-2)

Let $f \in I(\mathbb{R}^{2+})$

Then $\dfrac{f(ib)}{ib} \to \begin{cases} \alpha(f) = \alpha \in [0,\infty) & \text{as } b \to \infty \quad (\alpha \text{ as in 0.1}) \\ \gamma(f) \in [\alpha,\infty] & \text{as } b \downarrow 0 \ . \end{cases}$

Moreover $\alpha = \gamma$ iff $f(\omega) = \alpha\omega$

*Proof.* From the representation 0.1 , we immediatly calculate that :

$$\frac{f(ib)}{ib} = \alpha + \frac{\beta}{ib} + \frac{1-b^2}{ib} \int_{-\infty}^{\infty} \frac{t\,d\mu(t)}{t^2+b^2} + \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2}\, d\mu(t) \qquad (2.1)$$

It follows from elementary integration theory that

$$\frac{f(ib)}{ib} \to \alpha = \alpha(f) \quad \text{as} \quad b \to \infty \ .$$

To check the limit as $b \to 0$ , we "flip" $f$ to get :

$$\tilde{f}(\omega) = -1/f(-1/\omega)$$

Since $\tilde{f} \in I(\mathbb{R}^{2+})$, we have that

$$\frac{\tilde{f}(ib)}{ib} \to \alpha(\tilde{f}) \in [0,\infty) \quad \text{as} \quad b \to \infty$$

but this decodes to :

$$\frac{f(ib)}{ib} \to \gamma(f) = \frac{1}{\alpha(f)} \in (0,\infty] \quad \text{as} \quad b \downarrow 0 \ .$$

Now, if $\gamma(f) < \infty$ then, by 2.1 :

$$\gamma(f) = \alpha + \int_{-\infty}^{\infty} \frac{1+t^2}{t^2}\, d\mu(t)$$

Hence $\gamma(f) \geq \alpha(f)$ with equality iff $\mu \equiv 0$ . □

**Proposition 2.2**

Let $f \in I(\mathbb{R}^{2+})$ and $T = T(f)$ .

If $\alpha(f) > 1$ then $T$ is dissipative .

Proof. Write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$ .

From the representation (0.1), we have :

$$v_{n+1}(\omega) = \alpha v_n(\omega) + v_n(\omega) \int_{-\infty}^{\infty} \frac{1+t^2}{(t-u_n)^2+v_n^2} \geq \alpha v_n$$

Hence $v_n(i) \geq \alpha^n$ for $n \geq 1$ , and

$$\hat{T}^n \phi_i(t) = \frac{v_n(i)}{\pi((t-u_n)^2+v_n^2)} \leq \frac{1}{\pi\alpha^n}$$

Clearly $\sum_{n=1}^{\infty} \hat{T}^n \phi_i(t) \leq \frac{1}{(\alpha-1)} \quad \forall t \in \mathbb{R}$

and so $\sum_{n=1}^{\infty} 1_A \circ T^n < \infty$ a.e. $\forall A \in \mathcal{B}$ ; $\lambda(A) < \infty$ □

Proposition 2.3 (Letac [6])

Let $f \in I(\mathbb{R}^{2+})$ , $T = T(f)$ .

If $\alpha(f) = 1$ then $\lambda \circ T^{-1} = \lambda$ .

Proof. Let $f(ib) = u(b) + iv(b)$

we have : $\frac{u(b)}{b} \to 0$ and $\frac{v(b)}{b} \to 1$ as $b \to \infty$ .

Hence, for $A \in \mathcal{B}$ :

$$\pi b P_{ib}(A) \to \lambda(A)$$

and $\pi b P_{f(ib)}(A) \to \lambda(A)$ as $b \to \infty$ .

Since $P_{ib}(T^{-1}A) = P_{f(ib)}(A)$ , we have that

$\lambda(T^{-1}A) = \lambda(A)$ for $A \in \mathcal{B}$ □

The next result is the Denjoy-Wolff theorem stated on $\mathbb{R}^{2+}$, which shows shows that if $f \in I(\mathbb{R}^{2+})$ has no fixed point in $\mathbb{R}^{2+}$, then $\exists\ \tilde{f} \in I(\mathbb{R}^{2+})$ with $\alpha(\tilde{f}) = 1$, and such that $(\mathbb{R}, \mathcal{B}, \lambda, T(f))$ and $(R, B, \lambda, T(\tilde{f}))$ are conjugate, (and therefore have the same ergodic properties).

Theorem 2.4

Let $f \in I(\mathbb{R}^{2+})$ have no fixed points in $\mathbb{R}^{2+}$, and assume that $\alpha(f) < 1$; then

$\exists!\ t \in R$ such that $\alpha(\phi_t\ f\phi_t^{-1}) \geq 1$ where $\phi_t(\omega) = \frac{1+t\omega}{t-\omega}$. (Note that $\alpha(\phi_0^{-1}\ f\phi_0) = 1/\gamma(f)$).

Proof.

Let $\phi(z) = i(\frac{1+Z}{1-Z})$. Then $g = \phi^{-1}\ f\phi : U \to U$ is analytic, and has no fixed points in $U$. The Denjoy-Wolff theorem on $U$ (see [13] or [14]) shows that $\exists!\ \rho \in T$ such that

$$(*) \qquad \mathrm{Re}(\frac{\rho+g(Z)}{\rho-g(Z)}) \geq \mathrm{Re}(\frac{\rho+Z}{\rho-Z}) \qquad \forall Z \in U$$

Now let $t = \phi(\rho)$, $\psi = i(\frac{\rho+Z}{\rho-Z})$ and $\tilde{f} = \psi g\ \psi^{-1} \in I(\mathbb{R}^{2+})$. It follows that $\phi\psi^{-1} = \phi_t^{-1}$ and hence that $\tilde{f} = \phi_t\ f\phi_t^{-1}$. Also, (*) means that $\mathrm{Im}\ \psi g(Z) \geq \mathrm{Im}\ \psi(Z)$ for $Z \in U$, and hence $\mathrm{Im}\ \tilde{f}(\omega) \geq \mathrm{Im}\omega$ for $\omega \in \mathbb{R}^{2+}$, which implies $\alpha(\tilde{f}) \geq 1$. □

If $\alpha(\phi_t\ f\ \phi_t^{-1}) > 1$ for some $t$, then by proposition 2.2, $T(f)$ is dissipative. If $\alpha(\phi_t\ f\ \phi_t^{-1}) = 1$, then, by proposition 2.3, $T(\phi_t\ f\ \phi_t^{-1}) = \phi_t\ T(f)\ \phi_t^{-1}$ preserves Lebesgue measure. Hence $T(f)$ preserves the measure $\nu_t$, where $d\nu_t(x) = dx/(x-t)^2$.

The rest of this section is devoted to odd restrictions.

(We say that a restriciton $T$ is <u>odd</u> if $T(-x) = -T(x)$).

<u>Lemma 2.5</u>

Let $f \in I(\mathbb{R}^{2+})$ and let $T = T(f)$. The following are equivalent :

(i) $T$ is odd (ii) $\mathrm{Re}\, f(ib) = 0$ for $b > 0$

(iii) $f(-\bar{\omega}) = -\overline{f(\omega)}$ for $\omega \in \mathbb{R}^{2+}$

(iv) $f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t)$ where $\mu \in S(\mathbb{R})$ is symetric.

<u>Proof</u>. The implications (iv) => (iii) => (i) and (iii) => (ii) are elementary. That (ii) => (iii) is because of the Schartz reflection principle (see [9]). The fact that for $t \geq 0$ :

$$e^{itf(\omega)} = \int_{-\infty}^{\infty} e^{itT(x)}\, \phi_{\omega}(t)\, dt$$

gives the implication (i) => (iii) .

We show that (iii) => (iv). Assume (iii). It is evident that

$\beta = 0$ in the representation 0.1 , so we have

$$f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t) \quad \text{where} \quad \alpha \geq 0 \quad \text{and} \quad \mu \in S(\mathbb{R}) .$$

We must show that $\mu$ is symetric. To see this, we first rewrite the equation $v(-a+ib) = v(a+ib)$ (implied by (iii)) as :

$$(2.2) \qquad \int_{\infty}^{\infty} \phi_b(t-a)(1+t^2)\, d\mu(t) = \int_{\infty}^{\infty} \phi_b(t+a)(1+t^2)\, d\mu(t)$$

Next, we take $g(t)$ a continuous function of compact support and let $g_b(t) = \phi_{ib} * g$ . for $b > 0$ . It follows from (2.2) that

$$\int_{\infty}^{\infty} g_b(-t)(1+t^2)\, d\mu(t) = \int_{\infty}^{\infty} g_b(t)(1+t^2)\, d\mu(t) .$$

The symetry of $\mu$ is established by the (elementary) facts that

$$g_b(t) \to g(t) \quad \text{as} \quad b \to 0$$

$$\sup_{\substack{t\in\mathbb{R} \\ b>0}} (1+t^2)|g_b(t)| < \infty \qquad \square$$

We denote the collection of those inner functions on $\mathbb{R}^{2+}$ satisfying the conditions of the above lemma by $I_0(\mathbb{R}^{2+})$, and remark that $f \in I_0(\mathbb{R}^{2+})$ iff $\phi^{-1} f \phi$ is an essentially real inner function of $U$ . (Here $\phi(z) = i(\frac{1+z}{1-z})$).

Theorem 2.6

Let $f \in I_0(\mathbb{R}^{2+})$ and $T = T(f)$ .

If $\alpha(f) < 1 < \gamma(f)$ then $T$ preserves a Cauchy distribution. Moreover, if $\omega f(\omega)$ is not constant, then $T$ is exact.

Proof. If $f \in I_0(\mathbb{R}^{2+})$ then it follows from the lemma

$$\gamma(f) = \alpha(f) + \int_{\infty}^{\infty} \frac{1+t^2}{t^2}\, d\mu(t) .$$

Now since $\alpha(f) < 1 < \gamma(f)$, we have that

$$\int_{-\infty}^{\infty} \frac{1+t^2}{t^2}\, d\mu(t) > 1 - \alpha > 0 .$$

But $\int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2}\, d\mu(t) \downarrow 0$ as $b \to \infty$ so there is a $b_0 > 0$ such that $\int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b_0^2}\, d\mu(t) = 1 - \alpha$, i.e. $f(ib_0) = ib_0$, hence

$$P_{ib_0} \circ T^{-1} = P_{ib_0} .$$

The result now follows from theorem 1.1 ▨

To illustrate the results of this section, we consider $T_x = \alpha x + \beta \tan x$ where $\alpha, \beta > 0$.

If either $\alpha > 1$, or $\alpha + \beta < 1$, $T$ is dissipative.

If $\alpha < 1 < \alpha+\beta$, then $T$ preserves a Cauchy distribution and is exact. (This was established in [5] for $\alpha = 0$, $\beta > 1$).

The remaining cases ($\alpha = 1$ and $\alpha+\beta = 1$) are contained in the discussion of :

## §3 - Restrictions Preserving Infinite Measures

In this section, we consider those restrictions preserving infinite measures with $\alpha = 1$, and $\gamma = 1$.

We will see that for these transformations, conservativity is sufficient for ergodicity and rational ergodicity ([1]) - a stronger property (example 1-2 in [1]). We then give sufficient conditions for exactness.

Firstly, we recall the definition of rational ergodicity. Let $(X,\mathcal{B},m,\tau)$ be a conservative, ergodic, measure preserving transformation of a non-atomic, $\sigma$-finite measure space. We say that $\tau$ is rationally ergodic if there is a set $A$, of positive finite measure and $K < \infty$ such that

$$\text{(B)} \qquad \int_A \Big(\sum_{k=0}^{n-1} 1_A \circ \tau^k\Big)^2 dm \le K \Big(\sum_{k=0}^{n-1} m(A \cap \tau^{-k}A)\Big)^2 \quad \text{for} \quad n \ge 1$$

For a rationally ergodic transformation $\tau$, we let $B(\tau)$ denote the collection of sets with the property (B). It was shown in [1] that there is a sequence $\{a_n(\tau)\}$ such that

$$\frac{1}{a_n(\tau)} \sum_{k=0}^{n-1} m(A \cap T^{-k}A) \to m(A)^2 \quad \text{for every} \quad A \in B(\tau)$$

The sequence $\{a_n(\tau)\}_n$ is known as a <u>return sequence for $\tau$</u> and the collection of all sequences asymptotically proportional to $a_n(\tau)$ (i.e. $\frac{a_n}{a_n(\tau)} \to c \in (0,\infty)$) is known as the <u>asymptotic type of $\tau$</u> and denoted by $\mathcal{Q}(\tau)$. It was shown in [1] (theorem 2.4) that if $\tau_1$ and $\tau_2$ are rationally ergodic transformations which are both factors of the same measure preserving transformation, then

$$\mathcal{Q}(\tau_1) = \mathcal{Q}(\tau_2) \quad (\text{i.e. } \exists \lim_{n\to\infty} \frac{a_n(\tau_1)}{a_n(\tau_2)} \in (0,\infty)) .$$

We commence with the case $\alpha(f) = 1$.

Lemma 3.1

Let $f \in I(\mathbb{R}^{2+})$ be non-linear and let $T = T(f)$,

$f^n(\omega) = u_n(\omega) + iv_n(\omega)$ for $n \ge 1$ $\omega \in \mathbb{R}^{2+}$.

If $\alpha = 1$ then $T$ is conservative

$$\text{iff} \quad \sum_{n=1}^{\infty} \frac{V_n(\omega)}{|f^n(\omega)|^2} = \infty \quad \forall\, \omega \in \mathbb{R}^{2+} .$$

Proof. It will be more comfortable to work on the unit disc $U$. Accordingly, we let $M(z) = \phi^{-1} f \phi(z)$ *. Then $M$ is an inner function on $U$. Let $M(re^{i\theta}) \to \tau e^{i\theta}$ as $r \to 1$ a.e. . Denoting $Im(\frac{e^{i\theta}+z}{e^{i\theta}+z})$ by $q_z(\theta)$ and $q_z(\theta)\, d\theta$ by $d\pi_z(\theta)$, we see that $\pi_z \circ \phi^{-1} = \pi_\phi P_{\phi(z)}$ and this combined with the fact that $\phi^{-1} T \phi = \tau$ gives us that :

$$\pi_z \circ \tau^{-1} = \pi_{M(z)} \quad .$$

So $\tau$ is a non-singular transformation of $(\mathbf{T}, \lambda)$, and is conservative iff $T$ is conservative.

Let $\hat{\tau}$ be the operator dual to $\tau$, acting on $L'$. Then $\hat{\tau}\, q_z(t) = q_{M(z)}(t)$ and $\tau$ is conservative iff

$$(3.1) \qquad \sum_{n=1}^{\infty} q_{M^n(z)}(t) = \infty \quad \text{a.e. } \forall z \in U \quad .$$

We next show that $M^n(z) \to 1$ as $n \to \infty$ $\forall z \in U$. This will follow from the fact that $f^n(\omega) \to \infty$ as $n \to \infty$ $\forall \omega \in R^{2+}$ which we now demonstrate. From 0.1 :

$$v_{n+1}(\omega) = v_n(\omega) + v_n(\omega) \int_{-\infty}^{\infty} \frac{(1+t^2)\, d\mu(t)}{(t-U_n)^2+v_n^2} \geq v_n(\omega) \quad .$$

Hence $v_n \uparrow v_\infty$. It is not hard to see that if $v_\infty < \infty$, we must have $|U_n| \to \infty$. Hence $M^n(z) \to 1$.

Now choose $z \in U$ and let $M^n(z) = r_n e^{i\theta_n}$. We have $r_n \to 1$ and $\theta_n \to 0$. Also :

$$q_{M^n(z)}(t) = \frac{1-r_n^2}{1-2r_n\cos(\theta_n - t)+r_n^2} \sim \frac{1-r_n}{1-\cos t} \quad \text{as } n \to \infty \text{. For } t \neq 0 \text{.}$$

Thus :

$$(3.2) \qquad T \text{ is conservative iff } \sum_{n=1}^{\infty} 1-|M^n(z)| = \infty \quad \forall z \in U \text{.}$$

Since $M^n(z) \to 1$, the second condition is the same as

* where $\phi(z) = i\left(\frac{1+z}{1-z}\right)$

$$\sum_{n=1}^{\infty} 1 - |M^n(z)|^2 = \infty \quad \forall\, z \in U .$$

Now if $\omega = a + ib \in \mathbb{R}^{2+}$, then

$$1 - \left|\frac{\omega-i}{\omega+i}\right|^2 = \frac{4b}{a^2+(b+1)^2}$$

From the definition of $M$, we have

$$1 - |M^n(\tfrac{\omega-i}{\omega+i})|^2 = \frac{4v_n(\omega)}{U_n(\omega)+(v_n+1)^2} \sim \frac{4v_n(\omega)}{|f^n(\omega)|^2} \quad \text{as} \quad n \to \infty$$

□

Theorem 3.2

Let $f \in I(\mathbb{R}^{2+})$ be non-linear, $T = T(f)$ and $\alpha(f) = 1$. If $T$ is conservative then $T$ is rationally ergodic, and $\mathcal{Q}(T) = \{\sum_{k=1}^{n} \frac{v_k(\omega)}{|f^k(\omega)|^2}\}$ for every $\omega \in \mathbb{R}^{2+}$.

Proof. We first prove ergodicity, and here again, it is more comfortable to work on $U$. We prove the ergodicity of $\tau$. If $T$ is conservative then by (3.2) $\sum_{n=1}^{\infty} 1-|M^n(z)| = \infty \quad \forall z \in U$. Since $M^n(z) \to 1$, we must have that the points $\{M^n(z)\}_{n\geq 1}$ are distinct. Now, let $h \in N(U)$ (defined on p. 303 of [9]). If $h(M(z)) = h(z)$ for some $z \in U$ then by theorem 15-23 of [9], $h$ must be constant. The ergodicity of $\tau$ is deduced from this as follows :

Let $A \subseteq \mathbf{T}$ be an $\tau$-invariant measurable set.
The function $u(z) = \int_T q_z(\theta)\, 1_A(\theta)\, d\theta$ is a bounded harmonic function on $U$, and $u(g(z)) = u(z)$ on $U$. By theorem 17-26 of [8], $u$ is the imaginary part of an analytic function $F(z) \in H^p(u)$ for $1 \leq p < \infty$ ($H^p \subseteq N$).

Clearly $F(g(z)) = F(z) + c$ where $c \in \mathbb{R}$.

Let $F^*(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta})$, then $F^*(\tau e^{i\theta}) = F^*(e^{i\theta})_{+c}$. The

conservativity of $\tau$ yields that $c = 0$ (since the set $[|F^*| \leq M]$ has positive measure for some $M$, and so every point of this set returns infinitely often to it under iterations of $\tau$ — an impossibility if $c \neq 0$). Thus, by step 3, $F$ is constant and hence $u$ is constant, hence $1_A(\omega)$.

We now turn to rational ergodicity.

Let $b_n(\omega) = \frac{|f^n(\omega)|^2}{v_n(\omega)}$

Since $f^n(\omega) \to \infty$, it is clear that :

(3.3) $\pi b_n(\omega)\, \hat{T}^n \phi_\omega(t) \to 1$ uniformly on compact subsets of $\mathbb{R}$.

Let $a_n(\omega) = \sum_{k=1}^{n} \frac{1}{\pi b_k(\omega)}$. From (3.3) we have that

(3.4) $\frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k \phi_\omega \to 1$ uniformly on compact subset of $\mathbb{R}$.

Now, since $T$ is a conservative ergodic transformation, it follows that $\hat{T}$ is a conservative ergodic Markov operator, and we have from (3.4), by the Chacon-Ornstein theorem (see [3]) that :

(3.5) $\frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k f \to \int_{\mathbb{R}} f d\lambda$ a.e. $\forall f \in L^1$.

Hence $\exists a_n \to \infty$ s.t. $\frac{a_n(\omega)}{a_n} \to 1$ for every $\omega \in \mathbb{R}^{2+}$.

We will prove rational ergodicity of $T$ by showing that bounded intervals are in $B(T)$

Let $A = [a,b]$ where $-\infty < a < b < \infty$

Then $1_A \leq c\phi_i$

Hence, by (3.4), there is a $C_1 < \infty$ s.t.

(3.6) $\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A(x) \leq C_1$ for $n \geq 1$, $x \in A$.

This, combined with (3.5), gives (by dominated convergence)

$$(3.7) \qquad \frac{1}{a_n} \sum_{k=0}^{n-1} \lambda(A \cap T^{-k}A) \to \lambda(A)^2$$

To complete the proof that $T$ is rationally ergodic, we show that :

$$(3.8) \qquad \int_A \Big(\sum_{k=0}^{n-1} 1_A \circ T^k\Big)^2 \, d\mu \le 2\, C_1\, a_n^2 \quad \text{for} \quad n \ge 1 \ .$$

$$\int_A \Big(\sum_{k=0}^{n-1} 1_A \circ T^k\Big)^2 \, d\mu \le 2 \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} \lambda(A \cap T^{-k}(A \cap T^{-\ell}A))$$

$$= 2 \sum_{\ell=0}^{n-1} \int_{A \cap T^{-\ell}A} \sum_{k=0}^{n-1} \hat{T}^k 1_A \, d\lambda$$

$$\le 2\, C_1\, a_n^2 \qquad \square$$

Remark : If, in addtion, we assume that $f \in I_0(\mathbb{R}^{2+})$, we have that $b_n(i) = v_n(i)$, and that (3.6) holds for every $x \in \mathbb{R}$ . In this situation, we have that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} p(T^{-k}A) \to \lambda(A) \quad \text{for} \quad p \ \text{a} \ \lambda\text{-absolutely continuous probability measure, and} \ A \ \text{a bounded measurable set. (see [4] §4).}$$

We now turn to exactness. The following elementary lemma plays a similar role to that of lemma 1.2.

Lemma 3.3

If $b_n \to \infty$ , $B_n \sim b_n$ and $\frac{a_n}{b_n} \to 0$ as $n \to \infty$ then

$$\| \phi_{a_n + ib_n} - \phi_{iB_n} \|_1 \to 0 \quad \text{as} \quad n \to \infty \ .$$

Theorem 3.4

Let $f \in I(\mathbb{R}^{2+})$ , $T = T(f)$ and assume

$$f(\omega) = \omega + \int_{-K}^{K} \frac{d\nu(t)}{t-\omega}$$

then : T is exact, rationally ergodic and $\mathcal{Q}(T) = \{\sqrt{n}\}$ .

Proof. Let $L = \max\{\nu(\mathbb{R}), \nu(\mathbb{R})^2\}$ and assume that $K \geq \frac{1}{4}$ . We write $f^n(\omega) = u_n(\omega) + iv_n(\omega)$. The assumption of the theorem means that

$$u_{n+1} = u_n + \int_{-K}^{K} \frac{t-u_n}{(t-u_n)^2+v_n^2} \, d\nu(t) \tag{3.9}$$

$$v_{n+1} = v_n + v_n \int_{-K}^{k} \frac{d\nu(t)}{(t-u_n)^2+v_n^2}$$

The first part of the proof of this result consists of deducing the asymptotic behaviour of $u_n$ and $v_n$ . For this, we assume that $\omega = a + iL$ where $a \in \mathbb{R}$ . The recurrence relations (3.9) show us that

$$v_n(\omega) \geq L \quad \text{for every} \quad n \geq 1 .$$

and this enables us to deduce the boundless of $|u_n(\omega)|$ as follows :

Not-ing that :

$$\left| \int_{-K}^{K} \frac{t-u_n}{(t-u_n)^2+v_n^2} \, d\nu(t) \right| \leq \frac{\nu(\mathbb{R})}{2 v_n} \leq \frac{1}{2}$$

we see that :

If $u_n \geq K$ then $-K \leq K - \frac{1}{2} \leq u_{n+1} \leq u_n$ .

If $u_n \leq -K$ then $u_n \leq u_{n+1} \leq -K + \frac{1}{2} \leq K$

If $u_n \leq K$ then

$$u_{n+1} \le u_n + (K-u_n) \int_{-K}^{K} \frac{d\nu}{(t-u_n)^2+v_n^2} \le u_n + \frac{(K-u_n)}{v_n^2}\,\nu(R) \le K$$

If $u_n \ge -K$ then $u_{n+1} \ge -K$ .

Hence $|u_n(a+iL)| \le |a| \vee K$ for $n \ge 1$

The recurrence relations (3.9) now imply that $v_n \to \infty$ as $n \to \infty$ and hence

$$v_{n+1}^2 - v_n^2 = 2\,v_n^2 \int_{-K}^{K} \frac{d\nu(t)}{(t-u_n)^2+v_n^2} + v_n^2 \left(\int_{-K}^{K} \frac{d\nu(t)}{(t-u_n)^2+v_n^2}\right)^2$$

$$\to 2\nu(\mathbb{R}) \quad \text{as} \quad n \to \infty$$

Hence $v_n(a+iL) \sim \sqrt{2\nu n}$ as $n \to \infty$ .

Lemma 3.3 now shows us that for every $a \in \mathbb{R}$ :

$$(3.10) \qquad \|\hat{T}^n \phi_{a+iL} - \phi_{i\sqrt{2\nu n}}\| \to 0 \quad \text{as} \quad n \to \infty \ .$$

We now obtain exactness by Lin's criterion by an argument similar to that of theorem 1.1. (The rational ergodicity of T has already been established, and its asymptotic type characterised by theorem 3.2).

Let $u \in L^1$ , $\int_R u\,d\lambda = 0$ , and $\varepsilon > 0$ :

By Wiener's Tauberian theorem, there are $\alpha_1 \dots \alpha_N$ , $a_1 \dots a_N \in R$ such that

$$\|u - \sum_{k=1}^{N} \alpha_k \phi_{a_k+iL}\|_1 < \varepsilon/2$$

Whence :

$$\|\hat{T}^n u\|_1 \le \|\hat{T}^n(u - \sum_{k=1}^{N} \alpha_k \phi_{a_k+iL})\|_1 + \|\hat{T}^n \sum_{k=1}^{N} \alpha_k \phi_{a_k+iL} - \sum_{k=1}^{N} \alpha_k \phi_{i\sqrt{2\nu n}}\|_1$$

$$+ \|\sum_{k=1}^{N} \alpha_k \phi_{i\sqrt{2\nu n}}\|_1$$

$$||\hat{T}^n u||_1 \le ||u - \sum_{k=1}^{N} \alpha_k \phi_{a_k+iL}||_1 + \sum_{k=1}^{N} \alpha_k \, ||\hat{T}^n \phi_{a_k+iL} - \phi_{i\sqrt{2}vn}||_1 + |\sum_{k=1}^{N} \alpha_k|$$

$$< \varepsilon + o(1) \qquad \square$$

We note that the "generalised Boole transformation" (proven ergodic in [7]) falls within the scope of this last theorem.

If we added $\beta \neq 0$ to $f$ in theorem 3.4, we would obtain that for $Im\omega$ large enough $|u_n(\omega)| \ge c_1 n$ and $v_n(\omega) \le c_2 \log n$ (where $f^n(\omega) = u_n(\omega) + iv_n(\omega)$). The methods of lemma 3.1 would yield that $T(f)$ is dissipative.

The following corollary follows immediately from lemma 3.1 and theorem 3.2 .

Corollary 3.5.

Let $f \in I_0(R^{2+})$ and let $T = T(f)$, $f^n(i) = iv_n(i)$ . If $\alpha(f) = 1$ then :

$T$ is conservative iff $\sum_{n=1}^{\infty} \frac{1}{v_n(i)} = \infty$

and in this case, $T$ is rationally ergodic with

$$\mathcal{C}(T) = \{ \sum_{k=1}^{n} \frac{1}{\pi v_k(i)} \} \quad .$$

Moreover, in case $f \in I_0$ and $\alpha(f) = 1$ : we have that $v_n \to \infty$ and so :

$$v_{n+1}^2 - v_n^2 = 2v_n^2 \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+v_n^2} \, d\mu(t) + v_n^2 \left(\int_{-\infty}^{\infty} \frac{1+t^2}{t^2+v_n^2} \, d\mu(t)\right)^2$$

$$\to 2 \int_{-\infty}^{\infty} (1+t^2) \, d\mu(t) \le \infty$$

Hence :

$$\frac{v_n(i)}{\sqrt{n}} \to \sqrt{2\int_{-\infty}^{\infty}(1+t^2) \, d\mu(t)} \le \infty$$

which means :

(a) $T \times T \times T$ is dissipative

(b) $\frac{a_n(T)}{\sqrt{n}} \to c \in [0,\infty)$ as $n \to \infty$ . (in case $T$ is r.e.) .

These last two properties are held in common with the restrictions of theorem 3.4, and with the Markov shifts of random walks on $\mathbb{Z}$ .

The following example does not fall within the scope of theorem 3.4, (though theorem 3.2 does apply).

Example 3.6 $Tx = x + \alpha \tan x$ is exact, rationally ergodic with $a_n(T) \sim \frac{\text{Log } n}{\alpha}$ for $\alpha > 0$ .

Proof. Let $f(\omega) = \omega + \alpha \tan \omega$ and $f^n(\omega) = u_n(\omega) + iv_n(\omega)$

Then :

$$u_{n+1} = u_n + \frac{2\alpha \sin 2u_n e^{2v_n}}{e^{4v_n} - 2\cos 2u_n e^{2v_n} + 1}$$

and

$$v_{n+1} = v_n + \alpha \frac{e^{4v_n}}{e^{4v_n} - 2\cos 2u_n e^{v_n} + 1}$$

Whence : $v_{n+1} - v_n \geq \alpha \tanh v_n \geq \alpha \tanh v_0 > 0$

so $v_n \sim \alpha n$ as $n \to \infty$.

On the other hand :

$$|u_{n+1} - u_n| \leq \frac{2\alpha e^{2v_n}}{(e^{2v_n}-1)^2} \leq 4\alpha e^{-2v_n} \leq 4\alpha e^{-\alpha n} \quad \text{for } n \text{ large} .$$

Hence $u_n \to u_\infty$ , and the argument that $T$ is exact now proceeds identically to the last argument of theorem 3.4.

□

The following lemma will give examples of $f \in I_0(R^{2+})$ with $\alpha(f) = 1$ and $T = T(f)$ dissipative, and also uncountably many dissimilar (see [1]) rationally ergodic restrictions $T(f)$ with $f \in I_0(\mathbb{R}^{2+})$ , $\alpha(f) = 1$ .

Lemma 3.7

Let $\mu \in S(\mathbb{R})$ be symetric with

$c(x) = \mu(|t| \geq x) \sim \frac{1}{x^\alpha}$ where $0 < \alpha < 2$ .

Let $f_\alpha(\omega) = \omega + \int_{-\infty}^{\infty} \frac{1+t^2}{t-\omega}\, d\mu(t)$ and $f^n(i) = iv_n$ .

Then : $v_n \sim cn^{1/\alpha}$ where $c$ depends only on $\alpha$ .

Proof. We have

$$v_{n+1} = v_n(1+F(v_n))$$

where $F(b) = \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2}\, d\mu(t)$ .

It is not difficult to see that

$$F(b) = \frac{\mu(\mathbb{R})}{b^2} + 2(b^2-1) \int_0^\infty \frac{xc(x)}{(x^2+b^2)^2}\, dx$$

We first show that $F(b) \sim \frac{c_1}{b^\alpha}$ as $\alpha \to \infty$

Let $\varepsilon > 0$, and $M$ be such that

$$\frac{1-\varepsilon}{x^\alpha} \leq c(x) \leq \frac{1+\varepsilon}{x^\alpha} \quad \forall\, x \geq M$$

Writing

$$L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2+b^2)^2}\, dx$$

we have that :

$$(1-\varepsilon)\, L_M(b) = \int_M^\infty \frac{xc(x)dx}{(x^2+b^2)^2} \leq (1+\varepsilon)\, L_M(b) .$$

Now $L_M(b) = \int_M^\infty \frac{x^{1-\alpha}}{(x^2+b^2)^2}\, dx = \frac{1}{b^{2+\alpha}} \int_{M/b}^\infty \frac{x^{1-\alpha}dx}{(x^2+1)^2} \sim \frac{c}{b^{2+\alpha}}$ as $b \to \infty$

where $c = \int_0^\infty \frac{x^{1-\alpha}dx}{(x^2+1)^2}$

Since $\varepsilon > 0$ was arbitrary and $\alpha < 2$, we have that

$$F(b) \sim \frac{c}{b^{\alpha}} \quad \text{as} \quad b \to \infty \ .$$

Clearly, $v_n \to \infty$ , hence :

$$v_{n+1}^{\alpha} - v_n^{\alpha} = v_n^{\alpha}\left[(1+F(v_n))^{\alpha} - 1\right]$$

$$\sim \alpha v_n^{\alpha} F(v_n) \quad \text{as} \quad n \to \infty$$

$$\to \alpha c \quad \text{as} \quad n \to \infty$$

Thus $v_n \sim (\alpha cn)^{1/\alpha}$ as $n \to \infty$ □

We now let $T_\alpha = T(f_\alpha)$ .

By corollary 3.5 :

If $0 < \alpha < 1$ then $T_\alpha$ is dissipative .

If $1 \leq \alpha < 2$ then $T_\alpha$ is rationally ergodic and

$$\mathcal{Q}(T_\alpha) = \begin{cases} \{\log n\} & \text{if } \alpha = 1 \\ \{n^{1-1/\alpha}\} & \text{if } 1 < \alpha < 2 \end{cases} .$$

If follows from theorem 2.4. of [1] that if $1 \leq \alpha_1 < \alpha_2 < 2$ then $T_{\alpha_1}$ and $T_{\alpha_2}$ are not factors of the same measure preserving transformation.

Theorem 3.8.

Let $f \in I(\mathbb{R}^{2+})$ and $T = T(f)$

Suppose $x_0 \in \mathbb{R}$ and $f$ is analytic in a neighbourhood around $x_0$.

If $Tx_0 = x_0$, $T'(x_0) = 1$ and $T''(x_0) = 0$ then $T$ preserves the measure $\nu_{x_0}$ where $d\nu_{x_0}(x) = \frac{dx}{(x-x_0)^2}$, and is exact, rationally ergodic with asymptotic type $\{\sqrt{n}\}$

Remarks : The conditions $Tx_0 = x_0$ and $T'(x_0) = 1$ correspond to : $\alpha(\phi_{x_0} f \phi_{x_0}^{-1}) = 1$. If, in this situation, $T''(x_0) \neq 0$ : then $T$ is dissipative. By possibly considering $g(\omega) = f(\omega+x_0)-x_0$, we may (and do) assume $x_0 = 0$.

Proof. Let $f(\omega) = \omega + \sum_{n=3}^{\infty} a_n \omega^n$ for $|\omega|$ small.

Then $\frac{1}{f(\omega)} - \frac{1}{\omega} = \frac{\omega - f(\omega)}{f(\omega)} = \frac{\omega}{f(\omega)} \sum_{n=3}^{\infty} a_n \omega^n$

$\to 0$ as $\omega \to 0$.

Hence $\frac{1}{f(\omega)} = \frac{1}{\omega} + \sum_{n=1}^{\infty} b_n \omega^n$ for $|\omega|$ small.

Let $\tilde{f}(\omega) = -1/f(-\frac{1}{\omega})$.

Then :

(3.11) $\tilde{f}(\omega) = \omega + \sum_{n=1}^{\infty} b_n \omega^{-n}$ for $|\omega|$ large, say $|\omega| \geq K$ and, since $\tilde{f} \in I(\mathbb{R}^{2+})$, $\alpha(\tilde{f}) = 1$ :

(3.12.) $\tilde{f}(\omega) = \omega + \beta + \int_{-\infty}^{\infty} \frac{1+t\omega}{t-\omega}\, d\mu(t)$ where $\mu \in S(\mathbb{R})$ , $\beta \in \mathbb{R}$

In order to prove the theorem by applying theorem 3.4, we will show that

(3.13.) $\tilde{f}(\omega) = \omega + \int_{-K}^{K} \frac{d\nu(t)}{t-\omega}$ where $\nu \in S(\mathbb{R})$ .

Firstly, let $g(\omega) = \tilde{f}(\omega) - \omega$ . By (3.11.) :

$- ibg(ib) \to b_1$ as $b \to \infty$

But by (3.12.) :

$$- ibg(ib) = -ib(\beta - b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2}) + ib \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2}$$

$$+ b^2 \int_{-\infty}^{\infty} \frac{1+t^2}{t^2+b^2}\, d\mu(t) .$$

Hence, we obtain, from the convergence of the real part, that

$$\int_{-\infty}^{\infty}(1+t^2)\, d\mu(t) < \infty$$

and from the convergnece of the imaginary part that :

$$b^2 \int_{-\infty}^{\infty} \frac{td\mu(t)}{t^2+b^2} \to \beta \quad \text{as} \quad b \to \infty .$$

which convergence, when combined with the previous one, gives

$$\int_{-\infty}^{\infty} td\mu(t) = \beta .$$

Now, let $d\nu(t) = (1+t^2)\, d\mu(t)$ , then $\nu \in S(\mathbb{R})$ and it follows easily that

(3.14.) $\tilde{f}(\omega) = \omega + \int_{-\infty}^{\infty} \frac{d\nu(t)}{t-\omega}$

Now, let $h_b(a) = \text{Im } g(a+ib) = b\int_{-\infty}^{\infty} \frac{d\nu(t)}{(t+a)^2+b^2}$ . By (3.11.) $g$ is uniformly continuous on compact subsets of $[|\omega| \geq K]$, and so $h_b(a) \to 0$ as $b \to 0$ uniformly on compact subsets of $[|a| > K]$ .

Let $dQ_b(x) = h_b(x)dx$ , then $Q_b = P_{ib} * \nu$ , and so $Q_b(A) \to \nu(A)$ for $A$ a compact set. If $A$ is a compact subset of $[|x| > K]$, then

$$\nu(A) = \lim_{b \downarrow 0} Q_b(A) = \lim_{b \downarrow 0} \int_A h_b(x)dx = 0 .$$

Thus $\nu$ is concentrated on $[-K,K]$ and (3.13.) is established.

□

The transformations $T_\alpha x = \alpha x + (1-\alpha) \tan x$ for $0 \leq \alpha < 1$ fall within the scope of theorem 3.9. (It was shown in [11] that $T_0$ is ergodic). It follows from asymptotic type considerations that the above transformations are dissimilar to $Tx = x + \alpha \tan x$ .

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