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Action on a Compact Space**

Publications des séminaires de mathématiques et informatique de Rennes, 1975, fascicule S4

« International Conference on Dynamical Systems in Mathematical Physics », , p. 1-10

<http://www.numdam.org/item?id=PSMIR_1975__S4_A18_0>

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A SHORT PROOF OF THE VARIATIONAL PRINCIPLE FOR A $\mathbb{Z}_+^N$ ACTION ON A COMPACT SPACE

by

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0. Introduction. Ruelle in [12] introduced the notion of pressure for an action of the group $\mathbb{Z}^N$ on a compact metric space. It is a generalization of the notion of topological entropy. The variational principle (proved in [12] under some strong conditions) is a generalization of the Dinaburg's theorem ([5,8,7]) on a connection between the topological and measure entropies. A general proof of the variational principle was given by Walters in [13] (see also Denker [3]) for an action of $\mathbb{Z}_+$ and by Elsanousi in [6] for an action of $\mathbb{Z}_+^N$.

The first part of the proof given below (an inequality $h_\mu(T) + \int f \leq P(T, f)$) is a natural generalization of the proof from [10]. The second part ($\sup_\mu (h_\mu(T) + \int f) \geq P(T, f)$) is quite new (although the idea is close to the Ruelle's one).

1. Notations.

$\mathbb{Z}_+$ denotes

the set of all non-negative integers. Let us

fix a positive integer N .

$G = \mathbb{Z}_+^N$ is a commutative semigroup with respect to addition. For $n \in G$ we denote by n_i the i -th coordinate

of all $\mathcal{U}^n$, $n \in \mathbb{G}$).

$\mathcal{W}$ is the set of all neighbourhoods of the diagonal in $X \times X$, directed by the inclusion. It is a uniform structure (uniformity) for X (see [9]).

2. Definitions of pressure and entropy.

We may in the natural way extend Ruelle's definition of pressure of $\mathbb{Z}^N$ -action ([12]).

Let $n \in \mathbb{G}$, $\delta \in \mathcal{W}$, $f \in C(X)$. We define successively:

$$\Lambda(n) = \{m \in \mathbb{G} : m_i < n_i \text{ for } i = 1, \dots, N\},$$

$$\lambda(n) = \text{Card } \Lambda(n) = n_1 \cdot \dots \cdot n_N,$$

$$\delta_n = \bigcap_{k \in \Lambda(n)} (T^k \times T^k)^{-1} \delta,$$

$$f_n = \sum_{k \in \Lambda(n)} T^{nk} f$$

Of course $\delta_n \in \mathcal{W}$, $f_n \in C(X)$.

A finite subset e of X is called:

(n, δ) -separated, if for any $x, y \in e$, $x \neq y$, we have $(x, y) \notin \delta_n$,

(n, δ) -spanning, if for any $x \in X$ there exists $y \in e$ such that $(x, y) \in \delta_n$.

We denote $p(f, e) = \log \sum_{x \in e} \exp f(x)$. We define further

$$(1) \quad P_{n, \delta}(T, f) = \sup \{p(f_n, e) : e \text{ is } (n, \delta)\text{-separated}\}$$

$$(2) \quad P_\delta(T, f) = \limsup_{n \in \mathbb{G}} \frac{1}{\lambda(n)} P_{n, \delta}(T, f)$$

Of course, $P_\delta(T, f) \geq P_\varepsilon(T, f)$ for $\delta \subset \varepsilon$. Therefore

it is possible to define the pressure

$$(3) \quad P(T, f) = \lim_{\delta \in \mathcal{W}} P_\delta(T, f) = \sup_{\delta \in \mathcal{W}} P_\delta(T, f)$$

In the sequel we shall use the measure entropy function

of n ($i = 1, \dots, N$). For $n, m \in G$ let
 $nm = (n_1 m_1, \dots, n_N m_N)$. The relation $n \geq m$ iff
 $n_i \geq m_i$ for $i = 1, \dots, N$ directs G .

X is a non-empty compact Hausdorff space.

$C(X)$ is the space of all continuous real functions
on X with the norm $\|f\| = \sup_{x \in X} |f(x)|$.

$\mathcal{M}(X)$ is the space of all positive Borel regular
normed measures on X . It can be identified with the
space of all positive linear functionals on $C(X)$
having norm 1 (therefore we shall write μf instead of
 $\int_X f d\mu$ for $f \in C(X)$). We consider the weak- * topology
on $\mathcal{M}(X)$ (then $\mathcal{M}(X)$ is compact).

T ($n \mapsto T^n$) is an action of G on X , i.e.
a homomorphism of G into the semigroup of all continuous
transformations of X into itself (i.e. $T^n: X \rightarrow X$,
 $T^{n+m} = T^n \circ T^m$).

$T^{*n}: C(X) \rightarrow C(X)$ (for $n \in G$) is the operator
induced by T^n (i.e. $T^{*n}f = f \circ T^n$).

$T^{*n}: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ (for $n \in G$) is a restriction
of the operator induced by T^n (i.e. $T^{*n}\mu = \mu \circ T^n$)
to $\mathcal{M}(X)$. It is easy to check that indeed $T^{*n}(\mathcal{M}(X)) \subset$
 $\mathcal{M}(X)$ and that T^{*n} is continuous.

Of course T^* and T^{**} are actions of G on
 $C(X)$ and $\mathcal{M}(X)$, respectively.

$\mathcal{M}(X, T)$ is a space of all T -invariant measures
(i.e. these elements of $\mathcal{M}(X)$ which are fixed points

$H_\mu(\cdot)$ (for the definition and properties see e.g. [11]).

For $\mu \in \mathcal{M}(X, T)$ the entropy $h_\mu(T)$ may be defined in the same way as in [2] for the action of $\mathbb{Z}^N$. For a Borel finite partition A of the space X we define

$$A^n = \bigvee_{k \in \Lambda(n)} (T^k)^{-1} A \quad \text{for } n \in G,$$

$h_\mu(T, A) = \lim_{n \in G} \frac{1}{\lambda(n)} H_\mu(A^n)$. (it is easy to show that the limit exists, cf. [2]). Finally,

$$h_\mu(T) = \sup \{ h_\mu(T, A) : A - \text{Borel finite partition} \}.$$

3. The variational principle.

We shall prove the following variational principle:

$P(T, f) = \sup_{\mu \in \mathcal{M}(X, T)} (h_\mu(T) + \int \mu f)$. The proof consists of two parts.

Part I. $h_\mu(T) + \int \mu f \leq P(T, f)$ for $\mu \in \mathcal{M}(X, T)$.

Proof. Let $\mu \in \mathcal{M}(X, T)$. Let us fix $\xi > 0$ and a Borel finite partition A . Take $m \in G$ such that

$$(4) \quad \log 2 \leq \xi \cdot \lambda(m)$$

Let A^m consist of the sets $a_1, \dots, a_s$. For any of them there exists a compact set $b_i \subset a_i$ such that

$$\mu(a_i \setminus b_i) \leq \frac{\xi}{s \log s}. \quad \text{Let } b_0 = X \setminus \bigcup_{i=1}^s b_i. \quad \text{For the}$$

partition $B = \{b_0, b_1, \dots, b_s\}$ we have

$$(5) \quad H_\mu(A^m | B) \leq \mu(b_0) \cdot \log s \leq \xi$$

We take $\varepsilon = (X \times X) \setminus \bigcup_{\substack{i, j=1 \\ i \neq j}}^s (b_i \times b_j) \in \mathcal{H}$ and next $\delta \in \mathcal{H}$

such that $\delta \circ \delta \subset \varepsilon$ (i.e. if $(x, y), (y, z) \in \delta$, then

$(x, z) \in \varepsilon$) and $|f(x) - f(y)| \leq \xi$ if $(x, y) \in \delta$.

Let us fix $n \in G$. There exists a maximal (nm, δ) -separated (i.e. being also (nm, δ) -spanning) set $e \subset X$.

Denote by $C = \bigvee_{k \in \Lambda(n)} (T^{km})^{-1} B$. Further, denote

$$\alpha(b) = \sup_{x \in b} f_{nm}(x) \quad \text{for } b \in C, \quad \beta = \sum_{b \in C} \exp \alpha(b).$$

We have $\int_b f_{nm} d\mu \leq \alpha(b) \cdot \mu(b)$, therefore

$$\begin{aligned} H_\mu(C) + \mu f_{nm} &\leq \sum_{b \in C} \mu(b) (\alpha(b) - \log \mu(b)) = \\ &= \beta \cdot \sum_{b \in C} \frac{\exp \alpha(b)}{\beta} \cdot \eta\left(\frac{\mu(b)}{\exp \alpha(b)}\right), \quad \text{where } \eta(x) = -x \log x. \end{aligned}$$

The function η is concave, therefore

$$(6) \quad H_\mu(C) + \mu f_{nm} \leq \beta \cdot \eta\left(\sum_{b \in C} \frac{\exp \alpha(b)}{\beta} \cdot \frac{\mu(b)}{\exp \alpha(b)}\right) = \log \beta$$

e is a (nm, δ) -spanning set, hence for every $b \in C$ there exists a point $z(b) \in e$ such that

$\alpha(b) = \sup \{f_{nm}(x) : x \in b, (x, z(b)) \in \delta_{nm}\}$. But if $(x, z(b)) \in \delta_{nm}$, then for $k \in \Lambda(nm)$ $(T^k x, T^k z(b)) \in \delta$, thus, in view of the definition of δ ,

$$|(T^{nk} f)(x) - (T^{nk} f)(z(b))| \leq \xi. \quad \text{Hence}$$

$$(7) \quad f_{nm}(z(b)) \geq \alpha(b) - \xi \cdot \lambda(nm)$$

From the definitions of δ and ξ we obtain for $y \in e$, $k \in \Lambda(n)$ $\text{Card} \{a \in B : \exists_{x \in a} (T^{km} x, T^{km} y) \in \delta\} \leq 2$,

thus for $y \in e$ $\text{Card} \{b \in C : \exists_{x \in b} (x, y) \in \delta_{nm}\} \leq 2^{\lambda(n)}$,

because $\delta_{nm} \subset \bigcap_{k \in \Lambda(n)} (T^{km} \times T^{km})^{-1} \delta$. Hence

$$(8) \quad \text{Card} \{b \in C : z(b) = y\} \leq 2^{\lambda(n)}$$

Hence, from (7) and (8) we get:

$2^{\lambda(n)} \cdot \sum_{y \in e} \exp f_{nm}(y) \geq \sum_{b \in C} \exp \alpha(b) \cdot \exp (-\xi \cdot \lambda(nm))$, thus

$$(9) \quad \lambda(n) \cdot \log 2 + p(f_{nm}, e) \geq \log \beta - \xi \cdot \lambda(nm)$$

But $\mu f_{nm} = \lambda(nm) \cdot \mu f$, so from this, from (1), (4), (6) and (9) we obtain (notice that $\lambda(nm) = \lambda(n) \cdot \lambda(m)$) :

$$(10) \quad \frac{1}{\lambda(nm)} H_{\mu}(C) + \mu f \leq \frac{1}{\lambda(nm)} P_{nm, \delta}(T, f) + 2\xi$$

In view of (5), for $k \in \Lambda(n)$ we have

$$H_{\mu}((T^{km})^{-1}A^m | (T^{km})^{-1}B) \leq \xi \quad , \text{ therefore}$$

$$H_{\mu}(A^{nm} | C) = H_{\mu}\left(\bigvee_{k \in \Lambda(n)} (T^{km})^{-1}A^m \mid \bigvee_{k \in \Lambda(n)} (T^{km})^{-1}B\right) \leq \xi \cdot \lambda(n) \quad .$$

$$\text{Hence } H_{\mu}(A^{nm}) \leq H_{\mu}(C) + H_{\mu}(A^{nm} | C) \leq H_{\mu}(C) + \xi \cdot \lambda(n) \quad ,$$

$$\text{thus we obtain from (10): } \frac{1}{\lambda(nm)} H_{\mu}(A^{nm}) + \mu f \leq$$

$$\leq \frac{1}{\lambda(nm)} P_{nm, \delta}(T, f) + 3\xi \quad . \text{ Taking } \limsup \text{ with respect}$$

$$\text{to } n \text{ we get } h_{\mu}(T, A) + \mu f \leq P_{\delta}(T, f) + 3\xi \leq P(T, f) + 3\xi \quad .$$

But ξ and A were arbitrary , hence $h_{\mu}(T) + \mu f \leq P(T, f)$. □

Part II. $\sup_{\mu \in \mathcal{M}(X, T)} (h_{\mu}(T) + \mu f) \geq P(T, f) \quad .$

Proof. Let us fix $\delta \in \mathcal{W}$. For every $n \in G$, we choose such an (n, δ) -separated set e_n that

$$(11) \quad p(f_n, e_n) \geq P_{n, \delta}(T, f) - 1$$

Let us define a measure σ_n , concentrated on e_n , by a formula $\sigma_n(\{y\}) = \exp(f_n(y) - p(f_n, e_n))$ for $y \in e_n$.

We have $\sum_{y \in e_n} \sigma_n(\{y\}) = 1$, therefore $\sigma_n \in \mathcal{M}(X)$.

$$\text{Let } \mu_n = \frac{1}{\lambda(n)} \sum_{k \in \Lambda(n)} T^{*k} \sigma_n \quad .$$

For some sequence $(n_i)_{i=1}^{\infty}$ cofinal with G we have

$$(12) \quad \lim_{i \rightarrow \infty} \frac{1}{\lambda(n_i)} P_{n_i, \delta}(T, f) = P_{\delta}(T, f)$$

We choose some cluster point of the sequence $(\mu_{n_i})_{i=1}^{\infty}$ and denote it by μ . Of course, μ is also a cluster point of the net $(\mu_n)_{n \in G}$. For $g \in C(X)$ and $k \in G$ fixed, the function $\bar{\Phi}: \mathcal{M}(X) \longrightarrow \mathbb{R}$, given by the formula $\bar{\Phi}\nu = \nu g - \nu(T^{*k}g)$, is continuous, therefore $\bar{\Phi}\mu$ is a cluster point of the net $(\bar{\Phi}\mu_n)_{n \in G}$. We have

$$|\bar{\Phi}\mu_n| \leq \frac{1}{\lambda(n)} \cdot 2 \cdot (\lambda(n) - \lambda(n-k)) \cdot \|g\| \quad (\text{for } n \geq k),$$

because $\text{Card}(\Lambda(n) \setminus (k + \Lambda(n))) = \text{Card}((k + \Lambda(n)) \setminus \Lambda(n)) = \lambda(n) - \lambda(n-k)$. But $\lim_n \frac{\lambda(n-k)}{\lambda(n)} = 1$, thus $\bar{\Phi}\mu = 0$. Hence $\mu g = (T^{*k}\mu)g$. But g and k were arbitrary, therefore $\mu \in \mathcal{M}(X, T)$.

There exists a Borel finite partition A of X such that $a \times a \subset \delta$ for $a \in A$. Then for $a \in A^n$ $a \times a \subset \delta_n$, therefore $\text{Card}(e_n \cap a) \leq 1$. Hence

$$H_{\sigma_n}(A^n) + \sigma_n f_n = \sum_{y \in e_n} \sigma_n(\{y\})(f_n(y) - \log \sigma_n(\{y\})) = p(f_n, e_n).$$

Let us fix $m, n \in G$, $n \geq 2m$. For given $j \in \Lambda(m)$ let $s(j) = (E(\frac{n_1 - j_1}{m_1}), \dots, E(\frac{n_N - j_N}{m_N}))$. We have:

$$A^n = \bigvee_{r \in \Lambda(s(j))} (T^{rm+j})^{-1} A^m \vee \bigvee_{k \in \Xi} (T^k)^{-1} A, \text{ where}$$

$$\Xi = \Lambda(n) \setminus (j + \Lambda(ms(j))). \text{ But } \text{Card } \Xi = \lambda(n) - \lambda(ms(j)) \leq \lambda(n) - \lambda(n-2m), \text{ thus } p(f_n, e_n) = H_{\sigma_n}(A^n) + \sigma_n f_n \leq$$

$$\leq \sum_{r \in \Lambda(s(j))} H_{\sigma_n}((T^{rm+j})^{-1} A^m) + \sigma_n f_n + (\lambda(n) - \lambda(n-2m)) \log \text{Card } A.$$

Summing the inequalities obtained for $j \in \Lambda(m)$ we get
(notice that for $k \in \Lambda(n)$ there exists a unique $j \in \Lambda(m)$
and a unique $r \in \Lambda(s(j))$ such that $k = rm + j$) :

$$(13) \quad \sum_{k \in \Lambda(n)} H_{\sigma_n}((T^k)^{-1} A^m) + \lambda(m) \cdot \sigma_n f_n \geq \\ \geq \lambda(m) \cdot (p(f_n, e_n) - (\lambda(n) - \lambda(n-2m)) \cdot \log \text{Card } A)$$

We have also

$$(14) \quad \sigma_n f_n = \sigma_n \left(\sum_{k \in \Lambda(n)} T^{*k} f \right) = \left(\sum_{k \in \Lambda(n)} T^{*k} \sigma_n \right) f = \lambda(n) \cdot \mu_n f$$

From the definition of entropy it follows that

$$(15) \quad H_{\sigma_n}((T^k)^{-1} A^m) = H_{T^{*k} \sigma_n}(A^m)$$

From the definition of entropy and from the concavity of
the function $-x \log x$ it follows that

$$(16) \quad H_{\mu_n}(A^m) \geq \frac{1}{\lambda(n)} \sum_{k \in \Lambda(n)} H_{T^{*k} \sigma_n}(A^m)$$

The formulas (11) and (13) - (16) give us

$$(17) \quad \frac{1}{\lambda(m)} H_{\mu_n}(A^m) + \mu_n f \geq \frac{1}{\lambda(n)} P_{n,f}(T, f) - \frac{1}{\lambda(n)} \cdot \\ \cdot ((\lambda(n) - \lambda(n-2m)) \log \text{Card } A + 1)$$

The partition A can be chosen in such a way that
the boundaries of elements of A have measure μ zero.
(see [1], Chapt. IV, §5, exerc. 13 d; see also [10]).

Then A^n has the same property. But for a set a with
the boundary of measure μ zero, the function $\mathcal{M}(X) \rightarrow \mathbb{R}$,
given by $\nu \mapsto \nu(a)$, is continuous in the point μ .

Hence the function $\mathcal{M}(X) \rightarrow \mathbb{R}$, given by $\nu \mapsto H_\nu(A^m)$ is also continuous at the point μ , therefore in view of (12), (17) and the definition of μ , we have

$$\frac{1}{\lambda(m)} H_\mu(A^m) + \mu f \geq P_\delta(T, f).$$

Taking the limit with respect to m and using the inequality $h_\mu(T, A) \leq h_\mu(T)$, we obtain $h_\mu(T) + \mu f \geq P_\delta(T, f)$. But δ was arbitrary, hence $\sup_{\mu \in \mathcal{M}(X, T)} (h_\mu(T) + \mu f) \geq P(T, f)$. $\square$

4. Remark.

If $h_\mu(T) + \mu f = P(T, f)$, then μ is called an equilibrium state for (T, f) (measure with maximal entropy in the case of $f = 0$). The above construction shows that if $P_\delta(T, f) = P(T, f)$ for some $\delta \in \mathcal{H}$, then there exists an equilibrium state for (T, f) . In the case of $N = 1$, $f = 0$, this can be reformulated as follows:

If there exists an open cover A such that $h(T, A) = h(T)$, then there exists a measure with maximal entropy.

This is a particular case of the theorem of Denker ([4]), but obtained without assuming X finite dimensional.

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