PUBLICATIONS MATHÉMATIQUES ET INFORMATIQUES DE RENNES

PIERRE MICHEL

Minimal Sets Generated by a Substitution of Non Constant Length

Publications des séminaires de mathématiques et informatique de Rennes, 1975, fascicule S4
« International Conference on Dynamical Systems in Mathematical Physics », , p. 1-4

<http://www.numdam.org/item?id=PSMIR_1975___S4_A17_0>

© Département de mathématiques et informatique, université de Rennes, 1975, tous droits réservés.

L'accès aux archives de la série « Publications mathématiques et informatiques de Rennes » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

MINIMAL SETS GENERATED BY A SUBSTITUTION OF NON CONSTANT LENGTH

Pierre MICHEL

(Rennes)

Let P denote the alphabet $\{0,1\}$ and let

$$\Omega = P^{\mathbb{Z}} = \{\omega = \ldots\omega_{-1}\ \omega_0\ \omega_1\ \omega_2 \ldots \mid \omega_i \in P\}$$

A substitution θ is a map from P to $\bigcup_{n\geq 2} P^n$. If card $\theta(0)$ = card $\theta(1$ the substitution is said of constant length. If not, it is said of non constant length.

For suitable i and j we can generate (1) a bisequence by the following:

$$\omega[-\text{card}\ \theta^{2n}(i),\ \text{card}\ \theta^{2n}(i)+\text{card}\ \theta^{2n}(j)] = \theta^{2n}(i)\ \theta^{2n}(j)\ ,\forall n \in \mathbb{N}$$

where $\omega(a,k)$ is the k-block of ω which begin at ω_a.

Under a condition of non-degeneracy, the dynamical system (O_ω,T) where O_ω is the orbit closure of ω, and T the shift, is minimal (1) and strictly ergodic (2).

We can prove that the dynamical system associated with the substitution θ_1 defined by

$$\begin{cases} \theta_1(0) = 0^{n+1-p}\ 1\ 0^p \\ \theta_1(1) = 1\ 0^n \end{cases} \qquad \text{for } 0\leq p\leq n\ ,$$

has a purely discrete spectrum. (i^k, $i \in P$, $k \in \mathbb{N}$ means $\underbrace{i\ i \ldots i}_{n \text{ times}}$).

---

(1) GOTTSCHALK. Substitution minimal sets. Trans. Amer. Math. Soc. 109 (1963)

(2) MICHEL. Stricte ergodicité d'ensembles minimaux de substitution. CRAS Paris, 278, (1974).

For this, we introduce the notion of "coïncidence-value of the substitution θ" by the following :

$$\text{if} \quad \theta^{\infty}(0) = \lim_{n\to\infty} \theta^{2n}(0) = \omega_1^0\, \omega_2^0 \ldots \omega_n^0 \ldots$$

$$\theta^{\infty}(1) = \lim_{n\to\infty} \theta^{2n}(1) = \omega_1^1\, \omega_2^1 \ldots \omega_n^1 \ldots \quad ,$$

we say that n is a "coïncidence value of θ " if and only if

$$\omega_n^0 = \omega_n^1 \ .$$

If we denote the density of the coïncidence values of θ in $\mathbb{N}$ by $\delta_c(\theta)$, we can prove that $\delta_c(\theta_1)$ is equal to one.

Conversely if we consider the substitution θ_2 :

$$\begin{cases} \theta_2(0) = 0\ 1 \\ \theta_2(1) = 1\ 1\ 0\ 0 \end{cases}$$

we have $\delta_c(\theta_2) = \frac{1}{3}$ and we can prove that there is a continuous part in the spectrum of θ_2 .

Moreover, if we consider the two closely related substitutions θ_2' and θ_2'' defined by

$$\theta_2'(0) = \theta_2''(0) = 0\ 1$$

and

$$\begin{cases} \theta_2'(1) = 1\ 0\ 1\ 0 \\ \theta_2''(1) = 1\ 0\ 0\ 1 \end{cases} .$$

we can prove that θ_2' and θ_2'' have purely discrete spectra and that

$$\delta_c(\theta_2') = \delta_c(\theta_2'') = 1 \ .$$

More generally, we conjecture that the substitution θ_n :

$$\theta_n(0) = 0^n\, 1^n = \underbrace{0\ldots0}_{n \text{ times}}\ \underbrace{11\ldots1}_{n \text{ times}}$$

$$\theta_n(1) = 1^{2n}\, 0^{2n} = \underbrace{11\ldots1}_{2n \text{ times}}\ \underbrace{00\ldots0}_{2n \text{ times}} \quad ,$$

has :

1) a purely discrete spectrum if n is even (then $\delta_c=1$)

2) a mixed spectrum (partly continuous and partly discrete) if n is odd (then $\delta_c=\frac{1}{3}$).

At last, if we consider the constant-length case (3), we know that the so-called

- regular Toeplitz substitution generates a dynamical system with discrete spectrum. In this case $\delta_c = 1$.
- generalized Morse substitution generates a dynamical system with mixed spectrum and in this case $\delta_c = 0$.

Thus we can draw the following table, and we state the question

Is the following true :

$\delta_c = 1 \Longrightarrow$ discrete spectrum

$\delta_c < 1 \Longrightarrow$ mixed spectrum ?

(see the following page)

---

(3) COVEN and KEANE. The structure of substitution minimal sets. (1971)

	SUBSTITUTION		δ_c	SPECTRUM
constant length	Regular Toeplitz substitution $(\theta(0) \neq \widetilde{\theta(1)})$ (1)		1	Discrete
	Generalized Morse substitution $(\theta(0) = \widetilde{\theta(1)})$ (1)		0	Mixed
non constant length	$\begin{cases}\theta(0) = 0^{n+1-p}\ 1\ 0^{p} \\ \theta(1) = 1\ 0^{n}\end{cases}$ $0 \leq p \leq n$		1	Discrete
	$\begin{cases}\theta(0)=01 \\ \theta(1)=1010\end{cases}$ and $\begin{cases}\theta(0)=001 \\ \theta(1)=1001\end{cases}$		1	Discrete
	$\begin{cases}\theta(0) = 0\ 1 \\ \theta(1) = 1\ 1\ 0\ 0\end{cases}$		$\frac{1}{3}$	Mixed
	$\begin{cases}\theta(0) = 0^{n}\ 1^{n} \\ \theta(1) = 1^{2n}\ 0^{2n}\end{cases}$	n even	1	Discrete
		n odd	$\frac{1}{3}$	Mixed
	other cases		?	?

(1) with $\widetilde{\omega_k} = 1 - \omega_k$