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GROUPS ACTING ON TREES: FROM LOCAL TO GLOBAL STRUCTURE

by Marc BURGER *and* Shahar MOZES

Introduction

The group of automorphisms Aut T of a locally finite tree T is a locally compact group. In this work we study a large class of groups of automorphisms of a locally finite tree which exhibit a rich structure theory, analogous to that of semisimple Lie groups. Recall that a rank one simple algebraic group G over a locally compact non-archimedean field acts on the associated Bruhat-Tits tree Δ. Thus G is a closed subgroup of Aut Δ. Moreover its action on Δ is locally ∞-transitive, that is, the stabilizer of every vertex x acts transitively on all spheres of finite radius centered at x. In this paper we study the structure of closed non-discrete subgroups of Aut T satisfying various local properties. Of particular interest will be the class of locally primitive groups, namely subgroups $H < \mathrm{Aut}\,T$ such that for every vertex x, its stabilizer $H(x)$ acts as a primitive permutation group on the set of neighbouring edges.

Given a totally disconnected locally compact group H, we define two topologically characteristic subgroups:

- $H^{(\infty)}$ which is the intersection of all normal cocompact subgroups of H,
- QZ(H) which is the subgroup consisting of all elements with open centralizer in H.

In our setting these groups will play a role analogous to the one played by the connected component of the identity and the kernel of the adjoint representation in Lie group theory.

A corollary of the structure theory developed in this paper is that for any closed, non-discrete, locally 2-transitive group $H < \mathrm{Aut}\,T$, the group $H^{(\infty)}/QZ(H^{(\infty)})$ is topologically simple.

In Chapter 1 we study the structure of closed non-discrete locally primitive groups of automorphisms of trees and more generally of graphs. The main results show that $H^{(\infty)}$ is (minimal) cocompact in H whereas QZ(H) is a maximal discrete normal subgroup of H. Next we study the structure of $H^{(\infty)}/QZ(H^{(\infty)})$ and show that this group is the product of finitely many topologically simple groups. The structure theorem of locally primitive groups of automorphisms of a graph is complemented by a result (see Section 1.7) showing that such a graph may be obtained via a certain fiber product of graphs associated to the simple factors of $H^{(\infty)}/QZ(H^{(\infty)})$. Examples of graphs and locally primitive groups of automorphisms may be obtained by considering actions of algebraic groups over nonarchimedean local fields on certain graphs "drawn" equivariantly on the associated Bruhat-Tits building, or more generally, the action of appropriate subgroups. Considering the action of a higher rank simple algebraic group G on subgraphs of the 1-skeleton of the corresponding Bruhat-Tits building leads to an action of an extension of G on a tree. In (1.8) a special case of this construction is analyzed in detail; namely we construct a "graph of diagonals" $\mathscr{D}$ in the product of trees associated with the group $L = PSL(2, \mathbf{Q}_p) \times PSL(2, \mathbf{Q}_p)$. The semidirect product G of L with the automorphism switching the factors acts on $\mathscr{D}$ as a locally primitive group of automorphisms, and an extension H of G by the fundamental group $\pi_1(\mathscr{D})$ acts on the corresponding universal covering tree $\widetilde{\mathscr{D}}$; we show then that QZ(H) coincides with $\pi_1(\mathscr{D})$ and $H^{(\infty)}$ with the inverse image of L in H.

In Chapter 2, the structure of small neighbourhoods of the identity (i.e. stabilizers of balls in T) in a locally primitive group H is studied. When the group H is discrete there has been a lot of interest in the structure and classification of such groups for example in connection with the Goldschmidt-Sims conjecture which asserts the finiteness of the number of conjugacy classes of discrete locally primitive groups acting on a given tree. A basic result concerning the structure of *discrete* locally primitive groups is the Thompson-Wielandt Theorem which shows that the stabilizers of certain 2-balls in the tree are p-groups for some prime p. In general, when considering a non-discrete locally primitive group one cannot expect the stabilizer of some fixed size ball to be a pro-p-group. However, we obtain (Proposition 2.1.2) a substitute which holds also in the non-discrete case saying that when the stabilizer of a vertex contains a non trivial closed normal pro-p-subgroup, then already the stabilizer of a ball of radius 2 is a pro-p-group.

In Chapter 3 we consider vertex transitive groups, in particular 2-transitive groups and show that under quite general conditions these are often either discrete or already ∞-transitive; on the way, the results of Chapter 2 are used among other things to deduce the non-discreteness of certain subgroups of Aut T. To every permutation group $F < S_d$ on d elements we associate a closed subgroup $U(F) < \mathrm{Aut}\, T_d$ acting on the d regular tree T_d. Every vertex transitive subgroup $H < \mathrm{Aut}\, T_d$ is conjugate to a subgroup of U(F), where F is permutation isomorphic to the permutation group induced by a vertex stabilizer in H on the neighbouring edges. We show that when F is a 2-transitive group, then U(F) is ∞-transitive; if moreover the stabilizer in F of an edge is simple the only closed non-discrete vertex transitive subgroup of Aut T acting locally like F is U(F). These results go towards a classification of closed, non-discrete, 2-transitive subgroups of Aut T.

The study of discrete subgroups of Aut T satisfying local transitivity properties can be traced back to the work of W. T. Tutte ([Tu]), who introduced the concept of "s-transitive graphs" and initiated their classification. Via the fundamental results of

Thompson and Wielandt alluded to above, this culminated in the classification of locally primitive discrete groups of automorphisms of the 3-regular tree by D. Goldschmidt ([Go]) and the classification of s-transitive groups of automorphisms of general regular trees, for $s \geqslant 4$, by R. Weiss ([We]). From another viewpoint, the analogy between groups of automorphisms of a tree and Lie groups have led to various works exploiting the Bass-Serre theory to study general lattices in Aut T, see [Ba-Ku], [Ba-Lu], [Lu]. In the case of semisimple Lie groups, irreducible lattices in higher rank groups have a very rich structure theory and one encounters many deep and interesting phenomena such as (super)rigidity and arithmeticity. It turns out that in order to develop a structure theory for irreducible lattices in groups of the form $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ one needs to consider lattices whose projections satisfy various "largeness" conditions such as being locally primitive. The results of the present paper are used in an essential way in $[\text{B-M}]_2$ where we study the normal subgroup structure of irreducible cocompact lattices in $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ and in [B-M-Z] where the linear representation theory of such lattices is studied. Some of the results presented in this paper were announced in $[\text{B-M}]_1$.

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0. Notations, terminology

0.1. A permutation group $F < \mathrm{Sym}(\Omega)$ of a set Ω is quasiprimitive if it is transitive and if every nontrivial normal subgroup $e \neq N \lhd F$ acts transitively on Ω. Let $F^+ = \langle F_\omega : \omega \in \Omega \rangle$ denote the normal subgroup of F generated by the stabilizers F_ω of points $\omega \in \Omega$. We have the following implications:

F is 2-transitive $\Rightarrow$ F is primitive $\Rightarrow$ F is quasiprimitive $\Rightarrow$

$$\begin{cases} F = F^+ \\ \text{or} \\ F \text{ is simple and regular (that is simply transitive) on } \Omega. \end{cases}$$

Recall that a permutation group $F < \mathrm{Sym}\Omega$ is called *primitive* if it is transitive and if every F-invariant partition of Ω is either the partition into points or the trivial partition $\{\Omega\}$. An equivalent condition which is often used in the sequel is that F is transitive and the stabilizer F_ω of a point $\omega \in \Omega$ is a maximal subgroup of F. See [Di-Mo] Chapt. 4 for the structure of primitive and [Pr] §5 for the structure of quasiprimitive groups.

0.2. For notations and notions pertaining to graph theory we adopt the viewpoint of Serre ([Se]). Let $\mathfrak{g} = (X, Y)$ be a graph with vertex set X and edge set Y, let $E(x) = \{y \in Y : o(y) = x\}$ denote the set of edges with origin x; for a subgroup $H < \operatorname{Aut} \mathfrak{g}$ let $H(x) = \operatorname{Stab}_H(x)$ and $\underline{H}(x) < \operatorname{Sym}(E(x))$ be the permutation group obtained by restricting to $E(x)$ the action of $H(x)$ on Y. We say that H is locally "P" if for every $x \in X$, the permutation group $\underline{H}(x) < \operatorname{Sym}(E(x))$ satisfies one of the following properties "P": transitive, quasiprimitive, primitive, 2-transitive. We say that H is locally n-transitive ($n \geqslant 3$) if, for every $x \in X$, the group $H(x)$ acts transitively on the set of reduced paths (i.e. without back-tracking) of length n and origin x. Observe that H is locally 2-transitive iff, for every $x \in X$, $H(x)$ acts transitively on the set of reduced paths of length 2 and origin x. We say that $H < \operatorname{Aut} \mathfrak{g}$ is n-transitive, $n \geqslant 1$, if H acts transitively on the set of oriented paths of length n without back-tracking; $H < \operatorname{Aut} \mathfrak{g}$ is locally ∞-transitive if it is locally n-transitive for all $n \geqslant 1$.

For a connected graph $\mathfrak{g}$ and $H < \operatorname{Aut} \mathfrak{g}$, we have $H < \operatorname{Sym}(Y)$, and H^+ denotes the subgroup generated by edge stabilizers; ${}^+H$ denotes the subgroup generated by all vertex-stabilizers. If $\mathfrak{g} = (X, Y)$ is connected and locally finite, the group $\operatorname{Aut} \mathfrak{g} < \operatorname{Sym}(Y)$ is locally compact for the topology of pointwise convergence on Y.

Let d denote the combinatorial distance on X, $n \geqslant 1$ and $x_1, \ldots, x_k \in X$;

$$H_n(x_1, \ldots, x_k) = \left\{ g \in H : \begin{array}{c} g(y) = y \text{ for all } y \in X \\ \text{with } d(y, \{x_1, \ldots, x_k\}) \leqslant n \end{array} \right\}$$

and for $x \in X$, we set

$$\underline{H}_n(x) = H_n(x)/H_{n+1}(x).$$

For $x, y \in X$ adjacent vertices, set $H(x, y) := H(x) \cap H(y)$.

1. The structure of locally primitive groups

1.1. Let H be a locally compact, totally disconnected group. Define

$$H^{(\infty)} := \bigcap_{L<H} L$$

where the intersection is taken over all open subgroups $L < H$ of finite index, and

$$QZ(H) = \{h \in H : Z_H(h) \text{ is open}\}.$$

Then $H^{(\infty)}$ and $QZ(H)$ are topologically characteristic subgroups of H, and $H^{(\infty)}$ is closed. These definitions are motivated by the following:

Example 1.1.1. — Let $G = \mathbf{G}(\mathbf{Q}_p)$, where $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbf{Q}_p$. Then $G^{(\infty)}$ coincides with the subgroup $\mathbf{G}(\mathbf{Q}_p)^+$ generated by all unipotent elements in $\mathbf{G}(\mathbf{Q}_p)$, and QZ(G) coincides with the kernel of the adjoint representation of the p-adic Lie group G. The first statement follows from [Ti]$_1$ (main theorem) and [Bo-Ti] 6.14, while the second follows from the fact that $\mathrm{Ad}(g)$, $g \in G$, is the identity if and only if g centralizes a small neighbourhood of the identity $e \in G$.

As any compact totally disconnected group K is profinite (Cor. 1.2.4 [Wil]), we have $K^{(\infty)} = (e)$ and hence $H^{(\infty)} = \bigcap_{N \lhd H} N$, where the intersection is taken over all closed, cocompact normal subgroups of H. Thus, every normal cocompact subgroup of H contains $H^{(\infty)}$; at the other extreme, every discrete normal subgroup of H is contained in QZ(H). While $H^{(\infty)}$ is closed, this need not be so for QZ(H), as is seen in the following:

Example 1.1.2. — Let $K = F^{\mathbf{N}}$, where F is finite, centerless. Then QZ(K) coincides with the direct sum $\bigoplus_{\mathbf{N}} F$, which is a countable and dense subgroup of K.

Let $\mathfrak{g}$ be a locally finite graph and $H < \mathrm{Aut}\,\mathfrak{g}$ a closed subgroup; then H is locally compact totally disconnected and the most basic issue concerning its structure is the control over closed normal subgroups of H. In this problem, $H^{(\infty)}$ and QZ(H) are relevant objects since they control cocompact normal, resp. discrete normal subgroups. However, at this level of generality not much can be said about the size of $H^{(\infty)}$ and $QZ(H^{(\infty)})$.

The general theme of this chapter is that in requiring local transitivity properties of $H < \mathrm{Aut}\,\mathfrak{g}$, one obtains a good control over normal closed subgroups of H, and ends up with a class of locally compact groups behaving in many respects like semisimple Lie groups over local fields.

In Sections 1.1 to 1.5 we develop the basic structure theory of locally quasiprimitive groups. Our main goal is then the decomposition theorem (Thm. 1.7.1, Cor. 1.7.3) which describes how locally primitive groups are built up from topologically simple pieces; this result is reminiscent of the O'Nan-Scott theorem about (finite) primitive permutation groups.

1.2. In this section $\mathfrak{g} = (X, Y)$ denotes a locally finite connected graph with vertex set X and edge set Y.

Proposition **1.2.1.** — *Let* $H < \mathrm{Aut}\,\mathfrak{g}$ *be a closed subgroup. We assume that* H *is non-discrete and locally quasiprimitive.*

1) $H/H^{(\infty)}$ *is compact.*
2) QZ(H) *acts freely on* X*; it is a discrete non cocompact subgroup of* H.
3) *For any closed normal subgroup* $N \lhd H$, *either* N *is non-discrete cocompact and* $N > H^{(\infty)}$ *or* N *is discrete and* $N \subset QZ(H)$.

Moreover, the subgroup $H^{(\infty)}$ *enjoys the following properties:*

4) $QZ(H^{(\infty)})$ *acts freely without inversions on* $\mathfrak{g}$*; more precisely,* $QZ(H^{(\infty)}) = QZ(H) \cap H^{(\infty)}$.

5) *For any open normal subgroup* $N \lhd H^{(\infty)}$, *we have* $N = H^{(\infty)}$.
6) $H^{(\infty)}$ *is topologically perfect, that is* $H^{(\infty)} = \overline{[H^{(\infty)}, H^{(\infty)}]}$.

In other words, there exists a unique minimal normal cocompact subgroup $H^{(\infty)}$, and a unique maximal normal discrete subgroup QZ(H); and every closed normal subgroup of H is either cocompact or discrete. As we shall see in the course of the proof, $H^{(\infty)}$ admits an edge or a complete star as precise fundamental domain. However, $H^{(\infty)}$ needs not be locally quasiprimitive, thus 4) and 5) are not formal consequences of 1) and 2). Moreover, the group $H^{(\infty)}/QZ(H^{(\infty)})$ usually fails to be topologically simple; informations about its normal subgroups will be obtained in Section 1.5. In order to illustrate the objects occuring in Proposition 1.2.1, we present the following:

Example 1.2.1. — Let Δ_p be the Bruhat-Tits building associated to $PSL(3, \mathbf{Q}_p)$, $\mathfrak{g}_p$ the subgraph of the 1-skeleton of Δ_p consisting of all edges of a fixed given label and T the universal covering tree of $\mathfrak{g}_p$, which is regular of degree $\#\mathbf{P}^2(\mathbf{F}_p) = p^2 + p + 1$ (the number of points in the projective plane over a field of p elements). Let

$$1 \longrightarrow \pi_1(\mathfrak{g}_p) \longrightarrow H_p \longrightarrow PSL(3, \mathbf{Q}_p) \longrightarrow 1$$

be the exact sequence associated to the universal covering projection $T \to \mathfrak{g}_p$. Then $PSL(3, \mathbf{Q}_p) < \mathrm{Aut}\,\mathfrak{g}_p$ and $H_p < \mathrm{Aut}\,T$ are non-discrete, locally 2-transitive (since the linear group acts 2-transitively on the points of the projective plane); in particular, Proposition 1.2.1 applies. Notice that there is a natural map from $PSL(3, \mathbf{Z}_p)$ (the stabilizer of a fixed base vertex of the building Δ_p) into H_p so that its composition with the projection $H_p \to$, $PSL(3, \mathbf{Q}_p)$ is the identity. Let us denote the image of $PSL(3, \mathbf{Z}_p)$ by $K_p < H_p$. It turns out that

(a) $H_p^{(\infty)} = H_p$
(b) $QZ(H_p) = \pi_1(\mathfrak{g}_p)$

and the extension H_p of $PSL(3, \mathbf{Q}_p)$ has the following algebraic connectedness property:

(c) $[H_p, \pi_1(\mathfrak{g}_p)] = \pi_1(\mathfrak{g}_p)$, in particular H_p is perfect.

Let us briefly outline the argument for showing (a)-(c), see also 1.8.1 for similar considerations. Let $N \lhd H_p$ be a finite index closed normal subgroup, let $\Lambda = N \cap \pi_1(\mathfrak{g}_p)$. Since Λ is of finite index in $\pi_1(\mathfrak{g}_p)$ there is some $g \in \Lambda$ represented by a loop in a fixed apartment $\mathscr{A}$ of Δ_p, and such that the intersection of this loop with a certain half apartment $\mathscr{A}^+$ consists of 3 edges forming half a basic hexagon (recall that the graph $\mathfrak{g}_p$ can be viewed as consisting of small hexagons glued together). Now one can verify that the commutator $[h, g]$ of g with an element $h \in K_p$ whose action on the apartment $\mathscr{A}$ fixes the complement of $\mathscr{A}^+$ and moves $\mathscr{A}^+$, is an element of $\Lambda < \pi_1(\mathfrak{g}_p)$ which up to conjugacy is represented by a closed basic hexagon. It follows now easily, using the transitivity of the action of H_p, that $[H_p, \Lambda] = \pi_1(\mathfrak{g}_p)$. This in particular proves (c).

It follows also that $N \supset \pi_1(\mathfrak{g}_p)$ and since $H_p/\pi_1(\mathfrak{g}_p) = PSL(3, \mathbf{Q}_p)$ is simple it follows that N must be all of H_p and (a) is proved. To see (b) observe first that any element $g \in \pi_1(\mathfrak{g}_p)$ commutes with the open neighbourhood of the identity consisting of elements stabilizing the lift in T of a path representing g. Hence $\pi_1(\mathfrak{g}_p) \subset QZ(H_p)$. The reverse inequality follows easily by considering the structure of the action of elements in $PSL(3, \mathbf{Q}_p)$ on the building Δ_p.

Example 1.2.2. (see 3.2).— Let $F < S_d$ be a finite permutation group on d letters and $U(F) < \mathrm{Aut} T_d$ the associated universal group; then at every vertex x of T_d, $\underline{U(F)(x)} < \mathrm{Sym}\, E(x)$ is permutation isomorphic to $F < S_d$; if F is transitive and generated by stabilizers, then $U(F)^+$ is of index 2 in U(F) and simple. One has then: $U(F)^{(\infty)} = U(F)^+$ and $QZ(U(F)) = (e)$. This applies for instance to $F < S_d$, quasiprimitive, non-regular.

Example 1.2.3. — For the class of locally primitive groups obtained via the graph of diagonals, we refer to 1.8.

Example 1.2.4. — The group $H = PGL(2, \mathbf{F}_2((x)))$, considered as a closed subgroup of the group of automorphisms $\mathrm{Aut}\, T_3$ of the 3-regular tree, is locally primitive (in fact ∞-transitive). We have $H^{(\infty)} = PSL(2, \mathbf{F}_2((x)))$; since

$$H/H^{(\infty)} \simeq \mathbf{F}_2((x))^*/\mathbf{F}_2((x))^{*^2},$$

this gives an example where $H^{(\infty)}$ is of infinite index in H.

We turn now to some corollaries. As an immediate consequence of Proposition 1.2.1: 1), 5), 6), we obtain:

Corollary **1.2.2.** — *Let* $H < \mathrm{Aut}\, \mathfrak{g}$ *be as in Proposition 1.2.1 and* $G < H$ *a closed subgroup containing* $H^{(\infty)}$.

1) $G/\overline{[G, G]}$ *is compact.*
2) *For any open normal subgroup* $N \lhd G$, *one has* $N \supset H^{(\infty)}$. *In particular,* $G^{(\infty)} = H^{(\infty)}$.

For the next corollary, observe that if $H < \mathrm{Aut}\, \mathfrak{g}$ is locally quasiprimitive, then ${}^+H$ is of index at most two in H (see 1.3.0), and locally quasiprimitive as well since

$$H(x) = {}^+H(x), \quad \forall x \in X.$$

Corollary **1.2.3.** — *Let* $H < \mathrm{Aut}\, \mathfrak{g}$ *be as in Proposition 1.2.1.*

1) *If* $H \gneq {}^+H$, *and* x, y *are adjacent vertices, we have* ${}^+H = H(x, y) \cdot H^{(\infty)}$.
2) *If* $H = {}^+H$, *there exists* $v \in X$, *with*

$${}^+H = H(v) \cdot H^{(\infty)}.$$

1.3. In this section we collect a few general facts used in the proof of Proposition 1.2.1.

1.3.0. Let $\mathfrak{g} = (X, Y)$ be a connected graph, $H < \operatorname{Aut}\mathfrak{g}$ be a locally transitive subgroup. Then ${}^{+}H$ coincides with the kernel of the homomorphism $\chi : H \to \mathbf{Z}/2\mathbf{Z}$, $\chi(g) = d(gx, x) \bmod 2$. The group ${}^{+}H$ is transitive on the set of geometric edges and thus equals $H(x) *_{H(x,y)} H(y)$.

1.3.1. Let $\mathfrak{g} = (X, Y)$ be a connected graph, $H < \operatorname{Aut}\mathfrak{g}$, $\mathfrak{g}' = (X', Y')$ a connected subgraph of $\mathfrak{g}$ and $R \subset H$ such that, for all $x' \in X'$ and $y \in E(x')$, there is $r \in R$ with $ry \in Y'$; then $\Lambda := \langle R \rangle < H$ satisfies $\bigcup_{\lambda \in \Lambda} \lambda\mathfrak{g}' = \mathfrak{g}$.

Fact 1.3.1 implies

1.3.2. Let $\mathfrak{g} = (X, Y)$ be connected, locally finite and $H < \operatorname{Aut}\mathfrak{g}$ with $H \backslash \mathfrak{g}$ finite. Then there exists a finitely generated subgroup $\Lambda \subset H$ with $\Lambda \backslash \mathfrak{g}$ finite.

1.3.3. Let $\mathfrak{g} = (X, Y)$ be connected, locally finite and $\Lambda < \operatorname{Aut}\mathfrak{g}$ with $\Lambda \backslash \mathfrak{g}$ finite. Then $Z_{\operatorname{Aut}\mathfrak{g}}(\Lambda)$ is a discrete subgroup of $\operatorname{Aut}\mathfrak{g}$.

Indeed, let $B \subset Y$ be a finite subset with $\bigcup_{\lambda \in \Lambda} \lambda B = Y$ and $U < Z_{\operatorname{Aut}\mathfrak{g}}(\Lambda)$ an open subgroup with $u(b) = b$, $\forall u \in U$, $\forall b \in B$. Since U commutes with Λ, it acts like the identity on $\bigcup_{\lambda \in \Lambda} \lambda B = Y$, which implies that $U = (e)$ and that $Z_{\operatorname{Aut}\mathfrak{g}}(\Lambda)$ is discrete.

1.3.4. Let $\mathfrak{g} = (X, Y)$ be connected, locally finite; let Λ_1, Λ_2 be subgroups of $\operatorname{Aut}\mathfrak{g}$ such that $\Lambda_1 \backslash \mathfrak{g}$ is finite and $[\Lambda_1, \Lambda_2] < \operatorname{Aut}\mathfrak{g}$ is discrete. Then Λ_2 is discrete.

Indeed, pick $R \subset \Lambda_1$ finite such that (see 1.3.2) $\langle R \rangle \backslash \mathfrak{g}$ is finite; take $U \subset \Lambda_2$ open such that $[r, U] = e$, $\forall r \in R$. Thus $U \subset Z_{\operatorname{Aut}\mathfrak{g}}(\langle R \rangle)$, and hence U is discrete by (1.3.3). This implies that Λ_2 is discrete.

1.3.5. Let $\mathfrak{g} = (X, Y)$ be connected, locally finite, and $G < \operatorname{Aut}\mathfrak{g}$ non-discrete. Then, $QZ(G) \backslash \mathfrak{g}$ is not finite. Indeed assume that $QZ(G) \backslash \mathfrak{g}$ is finite and pick (using 1.3.2) a finitely generated subgroup $\Lambda < QZ(G)$ with $\Lambda \backslash \mathfrak{g}$ finite. Since Λ is finitely generated there is $U < G$ open with $U < Z_{\operatorname{Aut}\mathfrak{g}}(\Lambda)$ which by 1.3.3 implies that U, and hence G, is discrete; a contradiction.

1.3.6. Let $\mathfrak{g} = (X, Y)$ be locally finite, connected, and $\Gamma < \operatorname{Aut}\mathfrak{g}$, discrete with $\Gamma \backslash \mathfrak{g}$ finite. Then $N_{\operatorname{Aut}\mathfrak{g}}(\Gamma)$ is a discrete subgroup of $\operatorname{Aut}\mathfrak{g}$.

Indeed, apply 1.3.4 with $\Lambda_1 = \Gamma$ and $\Lambda_2 = N_{\operatorname{Aut}\mathfrak{g}}(\Gamma)$.

1.4. In this section $\mathfrak{g} = (X, Y)$ is a locally finite, connected graph. Let Λ, H be closed subgroups of $\operatorname{Aut}\mathfrak{g}$ with $\Lambda \lhd H$. We define,

$$\mathcal{N}_{nf}(H, \Lambda) = \left\{ N < H : \begin{array}{l} \Lambda < N \lhd H, N \text{ is closed} \\ \text{and does not act freely on } X \end{array} \right\}.$$

The set $\mathcal{N}_{nf}$ (H, Λ) is ordered by inclusion; let $\mathcal{M}_{nf}$ (H, Λ) be the set of minimal elements.

Lemma **1.4.1.** — *Assume moreover that* $H\backslash\mathfrak{g}$ *is finite and that* H *does not act freely on* X. *Then* $\mathcal{M}_{nf}(H, \Lambda) \neq \emptyset$.

Proof. — We use Zorn's Lemma. Observe that $\mathcal{N}_{nf}$ $(H, \Lambda) \neq \emptyset$. Let $\mathscr{C} \subset \mathcal{N}_{nf}(H, \Lambda)$ be a chain and F a finite set of representatives of H\X. For every $N \in \mathscr{C}$, the set $F_N = \{ f \in F : \underline{N}(f) < \mathrm{SymE}(f)$ is not trivial $\}$ is non void. Since F is finite and $\mathscr{C}$ is a chain it follows that $\cap_{N\in\mathscr{C}} F_N$ is non void. Thus there exists $f \in F$ such that for every $N \in \mathscr{C}$, $\underline{N}(f)$ is nontrivial which implies that $M := \cap_{N\in\mathscr{C}} \underline{N}(f)$ is not trivial. For $g \in M, g \neq e$, and $N \in \mathscr{C}$, the set $N^g := \{n \in N(f) : n|_{E(f)} = g\}$ is a compact non-void subset of $H(f)$ and, since $\mathscr{C}$ is a chain any finite subset of $\{N^g : N \in \mathscr{C}\}$ has nonempty intersection. Thus $\bigcap_{N\in\mathscr{C}} N^g \neq \emptyset$, and therefore $N_{\mathscr{C}} := \bigcap_{N\in\mathscr{C}} N$ does not act freely on X which shows that $\mathscr{C}$ admits a lower bound in $\mathcal{N}_{nf}(H, \Lambda)$. □

We turn now to locally quasiprimitive subgroups of Aut $\mathfrak{g}$.

Lemma **1.4.2.** — *Let* $H < \mathrm{Aut}\,\mathfrak{g}$ *be a locally quasiprimitive subgroup,* $N \lhd H$ *a normal subgroup and*

$$X_1(N) := \{x \in X : \underline{N}(x) \text{ acts transitively on } E(x)\}$$
$$X_2(N) := \{x \in X : \underline{N}(x) = e\}.$$

One of the following holds:

1) $X_2(N) = X$ *and* N *acts freely on* X.
2) $X_1(N) = X$ *and* N *acts transitively on the set of geometric edges of* $\mathfrak{g}$.
3) $X = X_1(N) \sqcup X_2(N)$ *gives an* H*-invariant* 2*-colouring of* $\mathfrak{g}$*; for any* $x_2 \in X_2(N)$, *the* 1*-neighbourhood* $\mathscr{V}(x_2, 1)$ *is a precise fundamental domain for the* N*-action on* $\mathfrak{g}$.

Proof. — The subsets $X_1(N), X_2(N)$ are H-invariant and, since H is locally quasiprimitive, $X = X_1(N) \sqcup X_2(N)$. If N does not act freely on X, there exists $z \in X$ with $N(z) \neq \{e\}$ and, since $\mathfrak{g}$ is connected, there must exist an $N(z)$-fixed vertex $x \in X$, with $\underline{N}(x) \neq (e)$. Then $\underline{N}(x)$ acts transitively on $E(x)$ and $X_1(N) \neq \emptyset$. Either $X_2(N) = \emptyset$, N is locally transitive and we are in Case 2), or $X_2(N) \neq \emptyset$. Since H acts transitively on the set of geometric edges (see 1.3.0) it has at most 2 orbits in X. Since both $X_1(N)$, $X_2(N)$ are non void and H-invariant, they are exactly these 2 orbits. Since any pair of adjacent vertices $\{x_1, x_2\}$ is a fundamental domain for the H-action on X, we conclude that if $x_2 \in X_2(N)$ then $x_1 \in X_1(N)$. Thus every terminal vertex of $\mathscr{V}(x_2, 1)$ is in $X_1(N)$ and we are in case 3 by 1.3.1. □

Lemmas 1.4.1 and 1.4.2 play an important role in our study of the structure of locally quasiprimitive groups, and in the context of Proposition 1.2.1 we shall use them to identify $H^{(\infty)}$ with the smallest closed normal subgroup of H whose action on X is not free.

Proof of Proposition 1.2.1

(1) Let $N \lhd H$ be closed, cocompact; since H is non-discrete, N is non-discrete (see 1.3.6) and hence $N \in \mathcal{N}_{nf}(H, e)$. Conversely, if $N \in \mathcal{N}_{nf}(H, e)$ then N is cocompact in H, by Lemma 1.4.2. Thus $H^{(\infty)} = \cap N$, where the intersection is over all N's in $\mathcal{N}_{nf}(H, e)$.
Take $M \in \mathcal{M}_{nf}(H, e)$ (Lemma 1.4.1) and $N \in \mathcal{N}_{nf}(H, e)$; assume that $N \not\supset M$. Then $N \cap M \supset [N, M]$ is discrete, hence N, M are discrete (see 1.3.4) and therefore $H \subset \mathcal{N}_{\mathrm{Aut}\,\mathfrak{g}}(N)$ is discrete (see 1.3.6), a contradiction. This shows that $H^{(\infty)} = M \in \mathcal{M}_{nf}(H, e)$ and proves 1).

(2) Follows from Lemma 1.4.2 and 1.3.5.

(3) Let $N \lhd H$ be a closed normal subgroup; either N acts freely on X, in particular N is discrete and hence contained in QZ(H), or N does not act freely on X, thus it is cocompact in H (Lemma 1.4.2) and hence contains $H^{(\infty)}$.

(4) The inclusion $QZ(H) \cap H^{(\infty)} \subset QZ(H^{(\infty)})$ follows from the definitions. The group $QZ(H^{(\infty)})$, being topologically characteristic in $H^{(\infty)}$, is normal in H; if $QZ(H^{(\infty)}) \not\subset QZ(H)$, then $QZ(H^{(\infty)})$ is not discrete, hence acts non-freely on X and $QZ(H^{(\infty)})\backslash\mathfrak{g}$ is finite (Lemma 1.4.2); but (1.3.5) implies then that $H^{(\infty)}\backslash\mathfrak{g}$ is not finite, contradicting 1). Thus $QZ(H^{(\infty)})$ is a discrete normal subgroup of H and therefore contained in QZ(H). This proves 4).

(5) Since $H^{(\infty)}$ is cocompact in $\mathrm{Aut}\,\mathfrak{g}$, and non discrete, $\mathcal{M}_{nf}(H^{(\infty)}, e) \neq \emptyset$ (Lemma 1.4.1); since $QZ(H^{(\infty)})$ acts freely on X, every $N \in \mathcal{N}_{nf}(H^{(\infty)}, e)$ is non-discrete. Given $O \lhd H^{(\infty)}$ open and $N \in \mathcal{M}_{nf}(H^{(\infty)}, e)$, the group $O \cap N$ is non-discrete normal in $H^{(\infty)}$, in particular $O \cap N$ acts non-freely on X hence $O \cap N = N$. Thus O contains the closed subgroup of $H^{(\infty)}$ generated by the elements of $\mathcal{M}_{nf}(H^{(\infty)}, e)$; this latter group being closed, normal in H and non-discrete, we conclude $O = H^{(\infty)}$.

(6) Proposition 1.2.1, 1) and 1.3.4 imply that $[H^{(\infty)}, H^{(\infty)}]$ is not discrete, thus $\overline{[H^{(\infty)}, H^{(\infty)}]}$ is non-discrete, closed and normal in H which implies

$$\overline{[H^{(\infty)}, H^{(\infty)}]} = H^{(\infty)}$$

by 3), above. □

Proof of Corollary 1.2.3. — Assume first that $\mathfrak{g} = T$ is a tree. Let $U_n \lhd H$ be a decreasing sequence of open finite index subgroups with $H^{(\infty)} \subset U_n \subset {}^{+}H$

and $\bigcap_{n\geqslant 1} U_n = H^{(\infty)}$; let $F_n = {}^+H/U_n$, $\pi_n : {}^+H \to F_n$ the canonical projection and $D_n := \pi_n(H(x)) *_{\pi_n(H(x,y))} \pi_n(H(y))$ which we endow with the discrete topology. Since ${}^+H = H(x) *_{H(x,y)} H(y)$ (see 1.3.0), the universal property of amalgams implies that π_n is the composition of a continuous homomorphism $\pi'_n : {}^+H \twoheadrightarrow D_n$ and the natural quotient map $D_n \twoheadrightarrow F_n$. Since D_n is discrete, $\mathrm{Ker}\,\pi'_n \lhd {}^+H$ is open, non-discrete, hence $\mathrm{Ker}\,\pi'_n \supset H^{(\infty)}$, which implies that D_n is compact and hence finite. This implies that $\pi_n(H(x,y))$ equals either $\pi_n(H(x))$ or $\pi_n(H(y))$ and hence $\pi_n(H(y))$ or $\pi_n(H(x))$ equals F_n. If $H \gneq {}^+H$, the group H acts transitively on X and $H(x)$, $H(y)$ are H-conjugate; it follows then from the equivariance of π_n w.r.t. conjugation by H, that $\pi_n(H(x,y)) = F_n$. Since this holds for all $n \geqslant 1$ and $H(x,y)$ is compact, we obtain ${}^+H = H(x,y)\cdot H^{(\infty)}$. In the former case, there is a vertex $v \in X$ and a sequence $n_k \to \infty$, for which $\pi_{n_k}(H(v)) = F_{n_k}$, which implies ${}^+H = H(v)\cdot H^{(\infty)}$. In general, let $p : T \to \mathfrak{g}$ be the universal covering of $\mathfrak{g}$ and $1 \to \pi_1(\mathfrak{g}) \to G \xrightarrow{\pi} H \to 1$ the associated exact sequence. The corollary follows then from the facts: $\pi^{-1}(H^{(\infty)}) \supset G^{(\infty)}$, $\pi(G(x)) = H(p(x))$, $\forall\, x \in \mathrm{Vert}\,T$. □

1.5. In this section we turn our attention to normal subgroups of $H^{(\infty)}$. Our main result is:

Proposition **1.5.1.** — *Let* $H < \mathrm{Aut}\,\mathfrak{g}$ *be a closed subgroup. Assume that* H *is non-discrete, locally quasiprimitive. Let* $\Lambda \lhd H$, *with* $\Lambda \subset QZ(H^{(\infty)})$.

1) a) H *acts transitively on* $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$.
   b) *The set* $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ *is finite.*
2) *Let* $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$;
   a) M/Λ *is topologically perfect.*
   b) $QZ(M)$ *acts freely on* X; $QZ(M) = QZ(H^{(\infty)}) \cap M$.
   c) $M/QZ(M)$ *is topologically simple.*
3) *For every* $N \in \mathscr{N}_{nf}(H^{(\infty)}, \Lambda)$ *there is* $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ *with* $N \supset M$.

Corollary **1.5.2.** — *Minimal normal closed nontrivial subgroups of* $H^{(\infty)}/QZ(H^{(\infty)})$ *exist; they are all* H*-conjugate, finite in number and topologically simple.*

Proof of Proposition 1.5.1. — Since every discrete normal subgroup of $H^{(\infty)}$ is contained in $QZ(H^{(\infty)})$ and the latter acts freely on X (Proposition 1.2.1, 4)), it follows that every element of $\mathscr{N}_{nf}(H^{(\infty)}, \Lambda)$ is non-discrete; this and 1.3.4 implies

(1) For every $N \in \mathscr{N}_{nf}(H^{(\infty)}, \Lambda)$, we have $[H^{(\infty)}, N] \not\subset QZ(H^{(\infty)})$.

For $\mathscr{E} \subset \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$, let $M_{\mathscr{E}} := \langle M : M \in \mathscr{E} \rangle$, that is the subgroup of $H^{(\infty)}$ generated by $\bigcup_{M\in\mathscr{E}} M$.

(2) H acts transitively on $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$.

Otherwise, let $\mathscr{E} \subset \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ be an H-orbit and $M \notin \mathscr{E}$. For every $N \in \mathscr{E}$, the subgroup $[N, M] \subset N \cap M$ acts freely on X, is discrete and normal in $H^{(\infty)}$.

Hence $[N, M] \subset QZ(H^{(\infty)})$, which implies $[\overline{M_{\mathscr{E}}}, M] \subset QZ(H^{(\infty)})$. On the other hand, since $\mathscr{E}$ is an H-orbit, $\overline{M_{\mathscr{E}}}$ is closed, normal in H and non-discrete, hence (Prop. 1.2.1, 3)) $\overline{M_{\mathscr{E}}} = H^{(\infty)}$. We conclude $[H^{(\infty)}, M] \subset QZ(H^{(\infty)})$, which contradicts (1).

(3) We have $\overline{[M, M]\cdot\Lambda} = M, \forall M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$.
Otherwise, there exists $M_0 \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ with $\overline{[M_0, M_0]\cdot\Lambda} \lneq M_0$, hence $\overline{[M_0, M_0]\cdot\Lambda}$ acts freely on X, is discrete and thus $[M_0, M_0] \subset QZ(H^{(\infty)})$; (2) implies then $[M, M] \subset QZ(H^{(\infty)}), \forall M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$. Since on the other hand $[M, M'] \subset QZ(H^{(\infty)})$ for all $M \neq M'$ in $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$, one concludes easily that $[H^{(\infty)}, H^{(\infty)}] \subset QZ(H^{(\infty)})$, contradicting (1).

(4) For every $N \in \mathscr{N}_{nf}(H^{(\infty)}, \Lambda)$, there is $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ with $N \supset M$.
Indeed, let $\mathscr{E} = \{M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda) : N \not\supset M\}$. Then we have $[\overline{M_{\mathscr{E}}}, N] \subset QZ(H^{(\infty)})$; on the other hand, for $\mathscr{F} = \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$, the group $\overline{M_{\mathscr{F}}} \subset H^{(\infty)}$ is closed, non-discrete and normal in H, thus $\overline{M_{\mathscr{F}}} = H^{(\infty)}$. Using (1) we conclude that $\mathscr{E} \neq \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$, which establishes (4).

(5) Let $\mathscr{E}, \mathscr{E}'$ be disjoint subsets of $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$. Then $\overline{M_{\mathscr{E}}} \cap \overline{M'_{\mathscr{E}}} \subset QZ(H^{(\infty)})$.
Indeed, otherwise we have $\overline{M}_{\mathscr{E}} \cap \overline{M}_{\mathscr{E}'} \in \mathscr{N}_{nf}(H^{(\infty)}, \Lambda)$ and there exists (by (4)) $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ with $M \subset \overline{M}_{\mathscr{E}} \cap \overline{M}_{\mathscr{E}'}$. But this implies $[M, M] \subset [\overline{M}_{\mathscr{E}}, \overline{M}_{\mathscr{E}'}] \subset QZ(H^{(\infty)})$, which contradicts (3).

(6) $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ is finite.
Let $G = \cup \overline{M}_{\mathscr{E}}$, where the union is over all finite subsets $\mathscr{E} \subset \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$; then G is non-discrete, normal in H hence $\overline{G} = H^{(\infty)}$. Since $H^{(\infty)}$ is a separable metric space, the same holds for G and therefore there exists a countable dense subgroup $L \subset G$. Fix an exhaustion $F_n \subset F_{n+1} \subset \dots \subset L$ of L by finite subsets and let $\mathscr{E}_n \subset \mathscr{E}_{n+1}, |\mathscr{E}_n| < +\infty$, be such that $F_n \subset \overline{M_{\mathscr{E}_n}}$, for all $n \geqslant 1$. Thus $L \subset \overline{M_{\bigcup_{n=1}^{\infty} \mathscr{E}_n}}$ and hence $\overline{M_{\bigcup_{n=1}^{\infty} \mathscr{E}_n}} = H^{(\infty)}$, which by (5) and (1) implies $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda) = \bigcup_{n=1}^{\infty} \mathscr{E}_n$ and hence $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ is countable. Fix $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$; the closed subgroup $N_H(M)$ is then of countable index in H, hence has non-void interior. Thus $N_H(M)$ is open in H, contains $H^{(\infty)}$, and hence is of finite index in H. Since H acts transitively on $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ this implies that the latter is finite.

So far we have proved assertions 1 a), b), 2 a), and 3) of Proposition 1.5.1. We turn now to the remaining assertions.

2 b) Let $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda) = \{M_1, \dots, M_r\}$ (see 1 b)) and define

$$\Omega := QZ(M_1) \cdot \dots \cdot QZ(M_r).$$

Then Ω is a normal subgroup of H; if Ω does not act freely on X, $\Omega\backslash\mathfrak{g}$ is finite (Lemma 1.4.2), and (1.3.2) there exist $\lambda_1, \ldots, \lambda_k$ in Ω such that for $\Omega' := \langle\lambda_1, \ldots, \lambda_k\rangle$, $\Omega'\backslash\mathfrak{g}$ is finite. Let $\lambda_i = a_i \cdot b_i$, $a_i \in QZ(M_1)$, $b_i \in QZ(M_2) \cdot \ldots \cdot QZ(M_r)$, $1 \leqslant i \leqslant k$; let $U_1 < M_1$ be an open subgroup with $[a_i, U_1] = e$, $1 \leqslant i \leqslant k$; since $[M_2 \cdot \ldots \cdot M_r, M_1] \subset QZ(H^{(\infty)})$, there exists an open subgroup $U_2 < M_1$ such that $[b_i, U_2] = e$, $1 \leqslant i \leqslant k$. The open subgroup $U := U_1 \cap U_2 < M_1$ is therefore contained in $Z_{\mathrm{Aut}\,\mathfrak{g}}(\Omega')$ which implies (1.3.3) that U and hence M_1 is discrete, a contradiction. This shows that Ω acts freely on X, is discrete and hence $\Omega \subset QZ(H^{(\infty)})$, that is, $QZ(M_i) \subset QZ(H^{(\infty)}) \cap M_i$; the opposite inclusion follows from the definitions. This shows 2 b).

2 c) Let $M \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ and $N \lhd M$ a closed subgroup with $N \supset QZ(M)$. For any $M' \in \mathscr{M}_{nf}(H^{(\infty)}, \Lambda)$ with $M' \neq M$, we have $[M', M] \subset M' \cap M \subset QZ(H^{(\infty)})$, which implies that $[M', N] \subset QZ(H^{(\infty)}) \cap M = QZ(M) \subset N$; thus M' normalizes N. Since $N \lhd M$, this implies that $N \lhd H^{(\infty)}$ and hence, by minimality of M, we have either $N = M$ or N acts freely on X and $N \subset QZ(H^{(\infty)}) \cap M = QZ(M)$. $\square$

1.6. *Fiber products*

1.6.1. In the theory of finite permutation groups, wreath products are used to build new primitive actions out of old ones; fiber products of graphs, whose basic properties we look at now, play a similar role in the theory of locally primitive groups.

Let $\varphi_i : \mathfrak{g}_i \to \mathfrak{h}$, $1 \leqslant i \leqslant r$, be surjective morphisms of graphs, where $\mathfrak{g}_i = (X_i, Y_i)$, $\mathfrak{h} = (V, E)$; the fibered product of the graphs $\mathfrak{g}_i$ relative to the morphisms φ_i is the graph $\prod_{\varphi_i} \mathfrak{g}_i = (X, Y)$ where

$$X = \prod_{\varphi_i} X_i = \left\{(x_1, \ldots, x_r) \in \prod_{i=1}^r X_i : \varphi_i(x_i) = \varphi_j(x_j),\ \forall\, i, j\right\}$$

$$Y = \prod_{\varphi_i} Y_i = \left\{(y_1, \ldots, y_r) \in \prod_{i=1}^r Y_i : \varphi_i(y_i) = \varphi_j(y_j),\ \forall\, i, j\right\}$$

and the origin, terminus maps are given by

$$o\big((y_1, \ldots, y_r)\big) = (o(y_1), \ldots, o(y_r)),$$

$$t\big((y_1, \ldots, y_r)\big) = (t(y_1), \ldots, t(y_r)).$$

This graph $\mathfrak{p} = \prod_{\varphi_i} \mathfrak{g}_i$ comes with projection morphisms $p_i : \mathfrak{p} \to \mathfrak{g}_i$ and a product morphism $\varphi : \mathfrak{p} \to \mathfrak{h}$, such that the diagram

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\varphi} & \mathfrak{h} \\ {\scriptstyle p_i}\big\downarrow & \nearrow {\scriptstyle \varphi_i} & \\ \mathfrak{g}_i & & \end{array}$$

commutes for all i, $1 \leqslant i \leqslant r$. This latter property could be used to give a categorial definition of fibered products; we leave this point to the reader.

Let $\mathrm{Aut}_{\varphi_i}\mathfrak{g}_i$ be the group of those automorphisms of $\mathfrak{g}_i$ which permute the fibers of φ_i, and $\pi_i : \mathrm{Aut}_{\varphi_i}\mathfrak{g}_i \to \mathrm{Aut}\,\mathfrak{h}$ the corresponding homomorphism, with respect to which φ_i is equivariant. The fiber product $G = \prod_{\pi_i} \mathrm{Aut}_{\varphi_i}\mathfrak{g}_i$ is a subgroup of $\mathrm{Aut}\,\mathfrak{p}$; it comes with projection homomorphisms $\rho_i : G \to \mathrm{Aut}\,\mathfrak{g}_i$ with respect to which $p_i : \mathfrak{p} \to \mathfrak{g}_i$ is equivariant, a product homomorphism $\pi : G \to \mathrm{Aut}\,\mathfrak{h}$, w.r.t. which $\varphi : \mathfrak{p} \to \mathfrak{h}$ is equivariant and such that $\pi = \pi_i \circ \rho_i$, for all $1 \leqslant i \leqslant r$.

When $\mathfrak{g}_1 = \ldots = \mathfrak{g}_r = \mathfrak{g}$, $\varphi_1 = \ldots = \varphi_r = \varphi$, which is our main case of interest, we denote by $\prod_{\varphi}^{(r)} \mathfrak{g}$ the r-fold fiber product of $\mathfrak{g}$ relative to φ, $\prod^{(r)} \varphi : \prod_{\rho}^{(r)} \mathfrak{g} \to \mathfrak{h}$ the product morphism, $\pi : \mathrm{Aut}_{\varphi}\mathfrak{g} \to \mathrm{Aut}\,\mathfrak{h}$ the natural homomorphism, $\prod_{\pi}^{(r)} \mathrm{Aut}_{\varphi}\mathfrak{g} < \mathrm{Aut}\,(\prod_{\varphi}^{(r)} \mathfrak{g})$ the r-fold fiber product of the group $\mathrm{Aut}_{\varphi}\mathfrak{g}$ relative to π, and finally, $\prod^{(r)} \pi : \prod_{\pi}^{(r)} \mathrm{Aut}_{\varphi}\mathfrak{g} \to \mathrm{Aut}\,\mathfrak{h}$ the associated product homomorphism. The permutation of factors realizes the symmetric group S_r on r letters as subgroup of $\mathrm{Aut}\left(\prod_{\varphi}^{(r)} \mathfrak{g}\right)$; given any subgroup $H < \mathrm{Aut}_{\varphi}\mathfrak{g}$, $\prod_{\pi}^{(r)} H$ is a subgroup of $\prod_{\pi}^{(r)} \mathrm{Aut}_{\varphi}\mathfrak{g}$, normalized by S_r.

We turn now to the following special situation: $\mathfrak{g} = (X, Y)$ is a locally finite, connected graph, $H < \mathrm{Aut}\,\mathfrak{g}$ is closed, non-discrete, locally primitive, $M \lhd H$ is a closed normal subgroup which acts non-freely on X and without inversions on $\mathfrak{g}$; let $\mathfrak{q} = M\backslash\mathfrak{g}$ be the quotient graph and $\pi : \mathrm{Aut}_{\varphi}\mathfrak{g} \to \mathrm{Aut}\,\mathfrak{q}$ the canonical homomorphism. Observe that $H \subset \mathrm{Aut}_{\varphi}\mathfrak{g}$, let $r \geqslant 1$ and $W \subset S_r$ be a transitive subgroup.

Proposition **1.6.1.** — *The semidirect product*

$$G = \left(\prod_{\pi}^{(r)} H\right) \rtimes W < \mathrm{Aut}\left(\prod_{\varphi}^{(r)} \mathfrak{g}\right)$$

is closed, non-discrete; it is locally primitive if and only if, $\underline{H}(x) < \mathrm{Sym}\,E(x)$ *is non regular whenever* $\underline{M}(x) < \mathrm{Sym}\,E(x)$ *is non-trivial. In this case,*

$$G^{(\infty)} = M^{(\infty)} \times \ldots \times M^{(\infty)}$$
$$QZ(G) = QZ(H) \times_{\pi} \ldots \times_{\pi} QZ(H)$$
$$QZ(G^{(\infty)}) = QZ(M^{(\infty)}) \times \ldots \times QZ(M^{(\infty)}).$$

Remark 1.6.1. — The main point of Proposition 1.6.1 is the one concerning (local) primitivity; the other statements are formal consequences of the definitions and left to the reader.

Remark 1.6.2. — If in the above context $\mathfrak{g}$ is a tree and $M = H^{(\infty)}$, then (see Corollary 1.7.7) $\underline{H}(x) < \mathrm{Sym}\,E(x)$ is non-regular whenever $\underline{M}(x) < \mathrm{Sym}\,E(x)$ is non-trivial, and the conclusions of Proposition 1.6.1 hold.

Using 1.4.2 we compute now the local permutation groups of G; let $\prod_{\varphi}^{(r)} \mathfrak{g} = (V, E)$; we distinguish two cases:

1) The quotient $\mathfrak{q}=M\backslash\mathfrak{g}$ is an edge; thus $^+H \subset \operatorname{Ker}\pi$, and for any vertex $v=(x,\dots,x)$ of diagonal type, the group $\underline{G}(v) < \operatorname{Sym} E(v)$ is permutation isomorphic to the wreath product $\underline{H}(x)^r \rtimes W$ on $E(x)^r$, the latter is primitive if and only if W is transitive and $\underline{H}(x) < \operatorname{Sym} E(x)$ is primitive, non-regular (see [Di-Mo] or Lemma 1.6.2 below). Finally, observe that any vertex of $\prod_\varphi^{(r)} \mathfrak{g}$ is G-conjugate to a vertex of diagonal type.

2) The quotient $\mathfrak{q}=M\backslash\mathfrak{g}$ is a star. Let $v=(x,\dots,x)$ be a vertex of $\prod_\varphi^{(r)} \mathfrak{g}$, such that $\underline{M}(x)$ acts regularly on $E(x)$; then $\underline{G}(v) < \operatorname{Sym} E(v)$ is permutation isomorphic to $\underline{H}(x) < \operatorname{Sym} E(x)$.

Let $v=(z,\dots,z)$ such that $\underline{M}(z)$ acts transitively on $E(z)$; let $\underline{\pi} : \underline{H}(z) \to \underline{H}(z)/\underline{M}(z)$ be the canonical projection; then $\underline{G}(v) < \operatorname{Sym} E(v)$ is permutation isomorphic to the action of $\left(\underline{H}(z)\times_{\underline{\pi}}\dots\times_{\underline{\pi}}\underline{H}(z)\right) \rtimes W$ on $E(z)^r$; the Proposition 1.6.1 follows then from

Lemma **1.6.2.** — *Let* $F < \operatorname{Sym}\Omega$, $|\Omega| \geqslant 2$, $r \geqslant 2$, *and* $W < S_r$ *be finite transitive permutation groups; let* $N \lhd F$, $D(F) < F^r$ *the diagonal subgroup and*

$$G = D(F)\cdot N^r \rtimes W.$$

Then, the G*-action on* Ω^r *is primitive if and only if* $N_\omega \neq (e)$, $\omega \in \Omega$, *and the* F*-action on* Ω *is primitive.*

Proof. — If $G < \operatorname{Sym}(\Omega^r)$ is primitive, then $F^r \rtimes W$ is primitive and hence $F < \operatorname{Sym}\Omega$ is primitive; for $x=(\omega,\dots,\omega)$ we have $G_x = D(F_\omega)\cdot N_\omega^r \rtimes W$; if $N_\omega=(e)$ and $N \neq e$, then $L := D(F) \rtimes W$ satisfies, $G_x \lneq L \lneq G$ and G is not primitive; if $N=e$, then G is not transitive. This shows the necessity of the above conditions. Conversely, let $L < G$ with $G_x < L < G$. Let $N^{(i)}$ be the ith factor of N^r viewed as subgroup of N^r and $p_i : F^r \to F$, the projection on the ith factor. We have $N_\omega^{(i)} \subset L\cap N^{(i)} \lhd L\cap N^r$, and thus: $N_\omega \subset p_i(L\cap N^{(i)}) \lhd p_i(L\cap N^r) \subset N$. Since $D(F_\omega)$ normalizes $L\cap N^{(i)}$ and $L\cap N^r$, F_ω normalizes $p_i(L\cap N^{(i)}) \lhd p_i(L\cap N^r)$.

We observe now that if $N_\omega < U < N$, and F_ω normalizes U, then either $U=N_\omega$ or $U=N$; indeed $F_\omega\cdot U(> F_\omega)$ is a subgroup of F and F_ω is maximal in F. We have thus two cases:

1) $p_i(L\cap N^{(i)})=N$ for one, and hence by transitivity of W, every i. Thus $L \supset N^r$, hence $D(F_\omega)\cdot N^r = D(F_\omega\cdot N)\cdot N^r = D(F)\cdot N^r$ is contained in L which implies $L=G$.

2) $p_i(L\cap N^{(i)})=N_\omega$, for one and hence all i's. If $p_i(L\cap N^r)=N$, we get $N_\omega \lhd N$ and hence $N_\omega=(e)$ since N acts transitively. This is a contradiction; thus $p_i(L\cap N^r)=N_\omega$ for all i's and hence $L\cap N^r=(N_\omega)^r$. Since $D(F)\cdot N^r = D(F_\omega)\cdot N^r$, we have $L\cap F^r = D(F_\omega)\cdot(L\cap N^r) = D(F_\omega)(N_\omega)^r$, and hence $L=G_x$. □

1.6.2. Let $\mathfrak{g}=(X,Y)$ be a graph and $G < \operatorname{Aut}\mathfrak{g}$. In this subsection we will show that when G is a direct product, then under certain conditions, $\mathfrak{g}$ is in a natural way a fiber product. More precisely, assume that there are normal subgroups $M_1, \ldots, M_r$ in G such that

(i) $M_i \cap M_j = (e)$, $\forall i \neq j$.
(ii) $G = M_1 \cdot \ldots \cdot M_r$.
(iii) $G(x) = M_1(x) \cdot \ldots \cdot M_r(x)$, $\forall x \in X$.

Let M_i' be the product of all factors $M_1, \ldots, M_r$ except M_i, $\mathfrak{g}_i = M_i' \backslash \mathfrak{g}$, $q_i : \mathfrak{g} \to \mathfrak{g}_i$, $p_i : \mathfrak{g}_i \to H_i \backslash \mathfrak{g}_i = G \backslash \mathfrak{g}$, where H_i is the image of G in Aut $\mathfrak{g}_i$ and

$$q : \mathfrak{g} \to \mathfrak{g}_1 \times_{p_1} \ldots \times_{p_r} \mathfrak{g}_r$$

defined by

$$q(x) = (q_1(x), \ldots, q_r(x)), \; x \in X,$$
$$q(y) = (q_1(y), \ldots, q_r(y)), \; y \in Y.$$

Proposition **1.6.3.** — *The morphism of graphs*

$$q : \mathfrak{g} \to \prod_{p_i} \mathfrak{g}_i$$

is an isomorphism; it is equivariant with respect to the isomorphism $G \to \prod_{i=1}^r H_i$.

Proof. — This follows immediately from the analogous statement where $G < \operatorname{Sym} X$ is a permutation group of a set X; thus let $X_i = M_i' \backslash X$,

$$\Delta = G \backslash X, \quad q_i : X \longrightarrow X_i, \quad p_i : X_i \longrightarrow \Delta,$$

and $q(x) = (q_1(x), \ldots, q_r(x))$.

1) *q is injective:* Let $x, y \in X$ with $q_i(x) = q_i(y)$, $\forall i$, that is $M_i'x = M_i'y$. For every i, we have then $y = m_1^{(i)} \cdot \ldots \cdot m_r^{(i)} x$, where $m_j^{(i)} \in M_j$ and $m_i^{(i)} = e$. It follows then that $\left(\left(m_1^{(j)}\right)^{-1} m_1^{(i)}\right) \ldots \left(\left(m_r^{(j)}\right)^{-1} m_r^{(i)}\right) x = x$, $\forall i, j$, which by (iii) implies $\left(m_\ell^{(j)}\right)^{-1} m_\ell^{(i)} \in M_\ell(x)$, and hence for $\ell = j$, $m_j^{(i)} \in M_j(x)$, for all i, j which implies $y = x$.
2) *q is surjective:* Let $x_i \in X_i$, with $p_i(x_i) = p_j(x_j)$; pick $\widetilde{x}_i \in X$ with $x_i = M_i' \widetilde{x}_i$; observe that $G\widetilde{x}_i = G\widetilde{x}_j$ $\forall i, j$ and thus writing $G\widetilde{x}_i = G\widetilde{x}$, we find $m_i \in M_i$ such that $x_i = M_i' m_i \widetilde{x}$. Define now $h_i = m_1 \cdot \ldots \cdot \widehat{m}_i \cdot \ldots \cdot m_r$, where m_i is omitted in the product; then $h_i \in M_i'$, $h_i m_i \widetilde{x} = h_j m_j \widetilde{x}$, $\forall i, j$ and thus defining $x = h_i m_i \widetilde{x}$, we get $q_i(x) = x_i$, for all i. □

1.7. *The structure of locally primitive groups*

The results of 1.5 and 1.6 suggest that a locally primitive group $G < \mathrm{Aut}\,\mathfrak{g}$ is built out of several copies of a locally primitive "almost-simple" group $M < \mathrm{Aut}\,\mathfrak{M}$, and that the graph $\mathfrak{g}$ is related to an appropriate fiber product of copies of $\mathfrak{M}$. The purpose of this section is to show that this is indeed the case when $\mathfrak{g}$ is a tree.

1.7.1. Let $T = (X, Y)$ be a locally finite tree, $H < \mathrm{Aut}\,T$ a closed, non-discrete, locally quasiprimitive group and $\Lambda < QZ(H^{(\infty)})$ with $\Lambda \lhd H$. Then we know (see 1.5) that the set $\mathcal{M}_{nf}(H^{(\infty)}, \Lambda)$ of minimal closed non-discrete normal subgroups of $H^{(\infty)}$ containing Λ is non-void, finite and that H acts transitively on it by conjugation. We have

Theorem **1.7.1.** — *Assume moreover that* $H < \mathrm{Aut}\,T$ *is locally primitive; let*

$$\mathcal{M}_{nf}(H^{(\infty)}, \Lambda) = \{M_1, \ldots, M_r\}.$$

Then

$$H^{(\infty)} = M_1 \cdot \ldots \cdot M_r$$
$$H^{(\infty)}(x) = M_1(x) \cdot \ldots \cdot M_r(x), \quad \forall x \in X,$$

and the latter product is direct.

The following corollary justifies our claim that non-discrete, locally primitive groups behave like semisimple groups.

Corollary **1.7.2.** — *Let* $T = (X, Y)$ *be a locally finite tree and* $H < \mathrm{Aut}\,T$ *a closed, non-discrete, locally primitive group. Then,* $H^{(\infty)}/QZ(H^{(\infty)})$ *is a direct product of topologically simple groups.*

We turn now to a more precise geometric version of Corollary 1.7.2: let $\mathfrak{g} := QZ(H^{(\infty)})\backslash T$ be the quotient graph, $\varphi : T \to \mathfrak{g}$ the covering map and $\omega : \mathrm{Aut}_\varphi T \to \mathrm{Aut}\,\mathfrak{g}$ the associated homomorphism. We have the inclusion $H \subset \mathrm{Aut}_\varphi T$ and the group $G := \omega(H) \simeq H/QZ(H^{(\infty)})$ is closed and non-discrete in $\mathrm{Aut}\,\mathfrak{g}$; moreover, for every $x \in X$, the homomorphism ω induces an isomorphism $H(x) \simeq G(\varphi(x))$ via which $\underline{H}(x) < \mathrm{Sym}\,E(x)$ is permutation isomorphic to $\underline{G}(\varphi(x)) < \mathrm{Sym}\,E(\varphi(x))$. In particular, G is locally primitive as well, $G^{(\infty)} \simeq H^{(\infty)}/QZ(H^{(\infty)})$ and $QZ(G^{(\infty)}) = (e)$. Let $\{M_1, \ldots, M_r\}$ be the set of minimal closed normal subgroups of $G^{(\infty)}$ (see Corollary 1.5.2); we know (Corollary 1.7.2) that $G^{(\infty)} = M_1 \cdot \ldots \cdot M_r$, this product being direct.

Let

$$\mathfrak{m} := M_2 \cdot \ldots \cdot M_r \backslash \mathfrak{g}$$
$$M := \text{image of } M_1 \text{ in } \operatorname{Aut} \mathfrak{m},$$
$$\mathfrak{q} := M \backslash \mathfrak{m}\,,\ p : \mathfrak{m} \to \mathfrak{q} \text{ the canonical projection}$$
$$\pi : N_{\operatorname{Aut} \mathfrak{m}}(M) \longrightarrow \operatorname{Aut} \mathfrak{q}\,, \text{ the canonical homomorphism.}$$

Corollary **1.7.3.**

1) $M < \operatorname{Aut} \mathfrak{m}$ *is closed, topologically simple, non-abelian.*
2) $N_{\operatorname{Aut} \mathfrak{m}}(M) < \operatorname{Aut} \mathfrak{m}$ *is locally primitive, and*

$$N_{Aut\, \mathfrak{m}}(M)^{(\infty)} = M.$$

3) *There is an isomorphism*

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{m} \times_p \ldots \times_p \mathfrak{m},$$

of $\mathfrak{g}$ *with the r-fold fiber product of* $\mathfrak{m}$ *with respect to p.*

4) *Identifying* $\mathfrak{g}$ *with* $\prod_p^{(r)} \mathfrak{m}$ *under this isomorphism, we have:*

a) $G^{(\infty)} = M^r$
b) $M^r < G < N_{\operatorname{Aut} \mathfrak{g}}(M^r)$
c) $N_{\operatorname{Aut} \mathfrak{g}}(M^r) = (N_{\operatorname{Aut} \mathfrak{m}}(M) \times_\pi \ldots \times_\pi N_{\operatorname{Aut} \mathfrak{m}}(M)) \rtimes S_r.$

Applying Corollary 1.7.3 to locally 2-transitive groups, we obtain

Corollary **1.7.4.** — *Let* $H < \operatorname{Aut} T$ *be a closed, non-discrete, locally* 2*-transitive subgroup. Then* $H^{(\infty)}/QZ(H^{(\infty)})$ *is topologically simple.*

This corollary is an analogue of the celebrated Theorem of Burnside, saying that the socle (that is the subgroup generated by all minimal normal subgroups) of a doubly transitive finite permutation group is either non abelian simple or elementary abelian (see [Di-Mo], Thm. 4.1B); observe moreover that in our case $H^{(\infty)}/QZ(H^{(\infty)})$ is never abelian.

1.7.2. Here we collect two facts from Bass-Serre theory, and a corollary concerning locally quasiprimitive groups, which are of independent interest. For proofs of Lemmas 1.7.5 and 1.7.6 see the discussion in [Se] 4.4.

Lemma **1.7.5.** — *Let* $T = (X, Y)$ *be a tree,* $a, b \in X$ *adjacent vertices,* K_a , K_b *subgroups of* $\operatorname{Aut} T$ *fixing* a, b*, respectively, and acting transitively on* $E(a), E(b)$*, respectively. Assume that* $K_a(b) = K_b(a) = K_a \cap K_b$*, let* $G := \langle K_a, K_b \rangle$*. Then* $G(a) = K_a$, $G(b) = K_b$.

Lemma **1.7.6.** — *Let* $T = (X, Y)$ *be a tree,* $x \in X$, $\{y_1, ..., y_d\}$ *the set of vertices adjacent to x,* $K_{y_1}, ..., K_{y_d}$ *subgroups of* $\operatorname{Aut} T$ *such that* K_{y_i} *fixes* y_i *and acts transitively on* $E(y_i)$, $1 \leqslant i \leqslant d$. *Let* $K_x < \operatorname{Aut} T$, *fixing pointwise* $E(x)$. *Assume that* $K_{y_i}(x) = K_x$, $1 \leqslant i \leqslant d$; *let* $G := \langle K_{y_1}, ..., K_{y_d}\rangle$. *Then,* $G(y_i) = K_{y_i}$, $1 \leqslant i \leqslant d$, *and* $G(x) = K_x$.

Corollary **1.7.7.** — *Let* $T = (X, Y)$ *be a locally finite tree and* $H < \operatorname{Aut} T$ *a closed, non-discrete locally quasiprimitive subgroup. Assume that for some* $x \in X$, $\underline{H}(x)$ *acts regularly on* $E(x)$. *Then* $T(x, 1)$ *is a precise fundamental domain for* $H^{(\infty)}$, *in particular* $H^{(\infty)}\backslash T$ *is a star.*

Proof. — Let $\{y_1, ..., y_d\}$ be the set of vertices adjacent to x, $K_{y_i} := H(y_i)$, $1 \leqslant i \leqslant d$, $K_x := H_1(x)$ and $y \in \{y_1, ..., y_d\}$. Then $H\{x, y\} = X$. We claim that $\underline{H}(y)$ does not act regularly on $E(y)$; indeed, otherwise $\underline{H}(z)$ would act regularly on $E(z)$ for all $z \in X$ which would imply that $H_1(x) = \{e\}$ and H is discrete, a contradiction. Thus $X = Hx \sqcup Hy$ is the partition of X into H-orbits. For the subgroup $G = \langle K_{y_1}, ..., K_{y_d}\rangle$, we obtain using 1.3.1. that $X = G\{y_1, ..., y_d, x\}$. Since $Gx \subset Hx$ and $G\{y_1, ..., y_d\} \subset Hy$ we conclude that $G\{y_1, ..., y_d\} = Hy$. This implies $G = \langle K_z : z \in Hy\rangle$ and shows that G is normal in H. On the other hand, G is nondiscrete, thus $G \supset H^{(\infty)}$. Applying Lemma 1.7.6 to G, we obtain in particular that $G(x)$ and hence $H^{(\infty)}(x)$ acts trivially on $E(x)$; this proves the corollary using Lemma 1.4.2 and Proposition 1.2.1). □

1.7.3

Proof of Theorem 1.7.1. — For $z \in X$, let $K_z := M_1(z) \cdot ... \cdot M_r(z)$; observe that $M_i \cap M_j \subset QZ(H^{(\infty)})$, $\forall i \neq j$, thus $M_i(z) \cap M_j(z) = (e)$, $\forall i \neq j$, and hence $[M_i(z), M_j(z)] = (e)$, $\forall i \neq j$.

For adjacent vertices x, y, define:

$$N_{x,y} := N_{M_1(x)}(M_1(x, y)) \cdot ... \cdot N_{M_r(x)}(M_r(x, y)).$$

One verifies easily that $N_{x,y}$ is a subgroup of $H(x)$, normalized by $H(x, y)$.

Given a vertex $x \in X$, we claim that either

(1) $N_{x,y} \subset H(x, y)$ and $K_x(y) = M_1(x, y) \cdot ... \cdot M_r(x, y)$ for all vertices y adjacent to x, or
(2) $N_{x,y} \cdot H(x, y) = H(x)$ for all y adjacent to x, K_x acts transitively on $E(x)$ and $\underline{M}_i(x)$ acts regularly on $E(x)$, $\forall 1 \leqslant i \leqslant r$.

Indeed, since $H(x, y)$ is maximal in $H(x)$ (see 0.1), we have either $N_{x,y} \cdot H(x, y) = H(x, y)$, or $N_{x,y} \cdot H(x, y) = H(x)$, for one and hence, by transitivity of $H(x)$ on $E(x)$, all y adjacent to x.

In the first case, $N_{x,y} \subset H(x, y)$ implies $N_{M_i(x)}(M_i(x, y)) = M_i(x, y)$, and $N_{x,y} = M_1(x, y) \cdot ... \cdot M_r(x, y)$. Since on the other hand, $K_x(y) = K_x \cap H(x, y) \subset N_{x,y}$, we obtain $K_x(y) = M_1(x, y) \cdot ... \cdot M_r(x, y)$, for all y adjacent to x. In the second case, $N_{x,y}$ and hence K_x acts transitively on $E(x)$. Moreover, $N_{x,y}$ normalizes $M_i(x, y)$ and hence leaves invariant the fixed point set of $M_i(x,y)$ in $E(x)$, thus $M_i(x,y)$ acts trivially on $E(x)$ and since this holds for all y adjacent to x, we deduce that $\underline{M}_i(x)$ acts regularly on $E(x)$.

Assume that (2) holds for some $x \in X$. If (2) holds for some y adjacent to x, then, since all vertices are H-conjugate to x or y, we get that $\underline{M}_i(z)$ acts regularly on $E(z)$ for all $z \in X$ and therefore M_i is discrete, a contradiction. Thus, if $\{y_1, ..., y_d\}$ denotes the set of vertices adjacent to x, (1) holds for $y_1, ..., y_d$. In particular, we have

$$K_{y_i}(x) = M_1(y_i, x) \cdot ... \cdot M_r(y_i, x), \quad 1 \leqslant i \leqslant d.$$

Since $\underline{M}_\ell(x)$ acts regularly on $E(x)$, $1 \leqslant \ell \leqslant r$, we get $M_\ell(y_i, x) = M_\ell(y_j, x)$, $\forall i,j$ and hence $K_{y_i}(x) = K_{y_j}(x)$, $\forall i,j$. Set $B = K_{y_i}(x)$ and define $G := \langle K_{y_1}, ..., K_{y_d} \rangle$. If K_{y_i} acts trivially on $E(y_i)$, then $\underline{M}_\ell(y_i) < \operatorname{Sym} E(y_i)$ is trivial and $\underline{M}_\ell(x) < \operatorname{Sym} E(x)$ is regular. Since any vertex is H-conjugate to x or y_i, we deduce that M_ℓ is discrete, a contradiction. Thus K_{y_i} does not act trivially on $E(y_i)$ and, being normal in $H(y_i)$, acts transitively on $E(y_i)$. We are thus in position to apply Lemma 1.7.6 and obtain $G(y_i) = K_{y_i}$, $1 \leqslant i \leqslant d$, and $G(x) = B$; in particular $G < \operatorname{Aut} T$ is locally closed and hence closed. Since $T(x, 1)$ is a precise fundamental domain of G and ${}^+H$ preserves the bipartite structure of T, we have $G\{y_1, ..., y_d\} = {}^+H\{y_1, ..., y_d\}$ (see the proof of Cor. 1.7.7); thus, we have the equality $G = \langle K_z : z \in {}^+H\{y_1, ..., y_d\} \rangle$, which implies that G is closed, normal in ${}^+H$ and hence $G = H^{(\infty)}$, since G acts non-freely on X. In particular $B = G(x) = H^{(\infty)}(x) \supset K_x$ which is a contradiction since B acts trivially on $E(x)$ and K_x acts transitively on $E(x)$.

We conclude that $N_{x,y} \subset H(x,y)$ and

$$K_x(y) = M_1(x,y) \cdot ... \cdot M_r(x,y)$$

for all pairs x, y of adjacent vertices. There are two cases:

(1) At some vertex $x \in X$, K_x induces the identity on $E(x)$. By the same argument as above, we conclude that K_{y_i} acts transitively on $E(y_i)$ and since $K_{y_i}(x) = K_x(y_i) = K_x(y_j) = K_{y_j}(x)$, we obtain, applying Lemma 1.7.6 to $G = \langle K_{y_1}, ..., K_{y_d} \rangle = H^{(\infty)}$, that $G(z) = K_z = H^{(\infty)}(z)$, $\forall z \in X$.
(2) K_x acts transitively on $E(x)$, $\forall x \in X$. Then, for $G = \langle K_x, K_y \rangle$, we have $G(z) = K_z$ by Lemma 1.7.5, thus $G = H^{(\infty)}$ and again $H^{(\infty)}(z) = K_z$, $\forall z \in X$.

Finally we obtain that

$$H^{(\infty)}(x) = M_1(x) \cdot ... \cdot M_r(x), \ \forall x \in X;$$

this implies that $M_1 \cdot ... \cdot M_r$ is open, hence closed in $H^{(\infty)}$; on the other hand as this subgroup $M_1 \cdot ... \cdot M_r$ is normal in H, closed and non-discrete it must contain $H^{(\infty)}$, we conclude that $M_1 \cdot ... \cdot M_r = H^{(\infty)}$. □

Proof of Corollary 1.7.2. — This is a direct consequence of Theorem 1.7.1 and Proposition 1.5.1 applied to $\Lambda = QZ(H^{(\infty)})$. □

Proof of Corollary 1.7.3. — The fact that M is topologically simple, non abelian, follows from Prop. 1.5.1 2a, 2c; this shows 1). We apply Theorem 1.7.1 with $\Lambda = QZ(H^{(\infty)})$; let $\mathscr{M}_{nf}(H^{(\infty)}, \Lambda) = \{N_1, \ldots, N_r\}$ and $M_i = N_i/\Lambda$. Then $\mathscr{M}_{nf}(G^{(\infty)}, e) = \{M_1, \ldots, M_r\}$. Theorem 1.7.1 implies then that the hypotheses of Prop. 1.6.3 are satisfied. Using moreover that all $M_1, \ldots, M_r$ are G-conjugate we obtain by Prop. 1.6.3 an isomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{m} \times_p \ldots \times_p \mathfrak{m}$$

equivariant w.r.t. the isomorphism $G^{(\infty)} \longrightarrow M^r$. Identifying $\mathfrak{g}$ with $\prod_p^{(r)} \mathfrak{m}$ and $G^{(\infty)}$ with M^r, we obtain $M^r < G < N_{\mathrm{Aut}\mathfrak{g}}(M^r)$, which shows 4a) and 4b. To show 4c) we observe that, since M is topologically simple and non-abelian, any continuous automorphism of M^r permutes the factors. Given $g \in N_{\mathrm{Aut}\mathfrak{g}}(M^r)$ we may thus find $\sigma \in S_r < N_{\mathrm{Aut}\mathfrak{g}}(M^r)$ such that $h = \sigma g$ fixes every factor of M^r. Let $\mathfrak{q} = G^{(\infty)} \backslash \mathfrak{g}$, let M_i' be the product of all the factors except the ith one, let $p_i : \mathfrak{g} \longrightarrow \mathfrak{m} = M_i' \backslash \mathfrak{g}$ and $q : \mathfrak{m} \longrightarrow \mathfrak{q}$ be the natural projection maps and $\pi : N_{\mathrm{Aut}\mathfrak{m}}(M) \longrightarrow \mathrm{Aut}\mathfrak{q}$ the induced homomorphism. The element $h = \sigma g$ induces via p_i an element $h_i \in N_{\mathrm{Aut}\mathfrak{m}}(M)$ and we have obviously $\pi(h_i) = \pi(h_j)$, $\forall i, j$. The injection of $\prod_\pi^{(r)} N_{\mathrm{Aut}\mathfrak{m}}(M) \longrightarrow N_{\mathrm{Aut}\mathfrak{g}}(M^r)$ sends then $(h_1, \ldots, h_r)$ to $h = \sigma g$. This shows 4c). Concerning the assertion 2) we observe that G and hence $N_{\mathrm{Aut}\mathfrak{g}}(M^r) = (\prod_\pi^{(r)} N_{\mathrm{Aut}\mathfrak{m}}(M)) \rtimes S_r$ is locally primitive which implies, using Lemma 1.6.3 and the discussion preceding it, that $N_{\mathrm{Aut}\mathfrak{m}}(M)$ is locally primitive. □

Proof of Corollary 1.7.4. — Using the notations of Corollary 1.7.3, assume that $r \geqslant 2$, and let $x = (v, \ldots, v) \in \mathrm{Vert}\,\mathfrak{g}$, $v \in \mathrm{Vert}\,\mathfrak{m}$, such that $\underline{G}^{(\infty)}(x) \neq (e)$. Then, we have the inclusion $\underline{G}(x) < F^r \rtimes S_r$, where $F = \underline{N(M)}(v)$; this implies that the wreath product $F^r \rtimes S_r < \mathrm{Sym}(E(v)^r)$ is 2-transitive, this is impossible when $r \geqslant 2$. Indeed given distinct points a, b in $E(v)$ it is clear that no element of $F^r \rtimes S_r$ sends the pair $((a, a, a, \ldots, a), (a, b, b, \ldots, b))$ to the pair $((b, b, b, \ldots, b), (a, b, b, \ldots, b))$. Thus $r = 1$, and we conclude using Proposition 1.5.1 c). □

1.8. *Examples, the graph of diagonals*

Let $T = (X, Y)$ be a tree, $\mathfrak{q}$ the tree consisting of an edge and $\varphi : T \longrightarrow \mathfrak{q}$ a morphism; the fiber product $\mathscr{D} := T \times_\varphi T$ is our graph of diagonals of T. Observe that Aut T permutes the fibers of φ and, let $\pi : \mathrm{Aut}\,T \longrightarrow \mathrm{Aut}\,\mathfrak{q}$ denote the associated canonical homomorphism. Assume now that T is locally finite, let $H < \mathrm{Aut}\,T$ be a closed, non-discrete, locally primitive subgroup and consider

$$L := (H \times_\pi H) \rtimes \langle \tau \rangle$$

where $\tau \in \mathrm{Aut}(T \times_\varphi T)$ is the automorphism given by the switch of factors.

Let $\mathscr{T} = \widetilde{\mathscr{D}}$ be the universal covering tree and

$$1 \longrightarrow \pi_1(\mathscr{D}) \longrightarrow G \stackrel{\omega}{\longrightarrow} L \longrightarrow 1,$$

be the associated exact sequence. If $\underline{H}(x) < \mathrm{Sym}\,E(x)$ is non regular $\forall x \in X$, we know (see Proposition 1.6.1) that L, and hence G is locally primitive; moreover $L^{(\infty)} = H^{(\infty)} \times H^{(\infty)}$, and $\pi_1(\mathscr{D}) \subset QZ(G)$.

Proposition **1.8.1.** — *Assume that* $H < \mathrm{Aut}\,T$ *is closed, non-discrete, vertex transitive and that* $H^{(\infty)}$ *is locally 2-transitive. Then*

(1) $G < \mathrm{Aut}\,\mathscr{T}$ *is closed, non-discrete, locally primitive, and vertex transitive.*
(2) $[G^{(\infty)}, \pi_1(\mathscr{D})] = \pi_1(\mathscr{D})$.

Remark 1.8.1. — It follows from (2) that $G^{(\infty)} \supset \pi_1(\mathscr{D})$ and hence $\omega^{-1}(L^{(\infty)}) = G^{(\infty)}$.

The following is an interesting special case of the above situation: let p be an odd prime, $\mathscr{T}_{p+1}$ the Bruhat-Tits tree associated to $PGL(2, \mathbf{Q}_p)$ and

$$H_p := \left\{ g \in PGL(2, \mathbf{Q}_p) : \frac{\det g}{|\det g|} \in \left(\mathbf{Z}_p^*\right)^2 \right\}.$$

Then $H_p < \mathrm{Aut}\,\mathscr{T}_{p+1}$ is vertex transitive, $H_p^{(\infty)} = H_p^+ = PSL(2, \mathbf{Q}_p)$ is locally 2-transitive and we may apply Proposition 1.8.1. Notice that the image of a matrix $g \in GL(2, \mathbf{Q}_p)$ in $PGL(2, \mathbf{Q}_p)$ is actually in $PSL(2, \mathbf{Q}_p)$ exactly when the determinant of g is a square in $\mathbf{Q}_p$, hence H_p^+ is exactly $PSL(2, \mathbf{Q}_p)$ and since $PSL(2, \mathbf{Q}_p)$ is simple it follows that this is exactly $H_p^{(\infty)}$. In particular, we obtain an extension,

$$1 \longrightarrow \pi_1(\mathscr{D}_p) \longrightarrow G_p \longrightarrow (H_p \times_\pi H_p) \rtimes \langle\tau\rangle \longrightarrow 1$$

where $\mathscr{D}_p = \mathscr{T}_{p+1} \times_\varphi \mathscr{T}_{p+1}$ is the graph of diagonals of $\mathscr{T}_{p+1}$; this graph is regular of degree $(p+1)^2$ and $G_p < \mathrm{Aut}(\mathscr{T}_{(p+1)^2})$. It follows from Proposition 1.8.1 that $\pi_1(\mathscr{D}_p) \subset G_p^{(\infty)}$, hence $G_p^{(\infty)} = G_p^+$ since $H_p^{(\infty)} = H_p^+$; we obtain therefore an extension of $PSL(2, \mathbf{Q}_p)^2$

$$1 \longrightarrow \pi_1(\mathscr{D}_p) \longrightarrow G_p^+ \longrightarrow PSL(2, \mathbf{Q}_p) \times PSL(2, \mathbf{Q}_p) \longrightarrow 1$$

by the discrete group $\pi_1(\mathscr{D}_p)$, with the property

$$[G_p^+, \pi_1(\mathscr{D}_p)] = \pi_1(\mathscr{D}_p).$$

In particular, since $PSL(2, \mathbf{Q}_p)^2$ is perfect, we conclude that G_p^+ is perfect. Finally, it is easy to see that

$$QZ(G_p) = QZ(G_p^+) = \pi_1(\mathscr{D}_p).$$

To show this last assertion, observe that any $g \in QZ(G_p)$ is, modulo $\pi_1(\mathscr{D}_p)$, of the form (h_1, h_2) with $h_i \in QZ(H_p) = (e)$, which shows $QZ(G_p) \subset \pi_1(\mathscr{D}_p)$. We conclude by observing that $\pi_1(\mathscr{D}_p) \subset QZ(G_p^+) \subset QZ(G_p)$.

Proof of Proposition 1.8.1. — Let $v = (x, x)$ be a vertex of $\mathscr{D} := T \times_\varphi T$; the group G acts by conjugation on $\pi_1(\mathscr{D}, v)$ and, since T is a tree, the group $\pi_1(\mathscr{D}, v)$ is generated by all $\omega^{-1}(H \times_\pi H)$-conjugates of elements of $\pi_1(\mathscr{D}, v)$ represented by cycles of length four based at v. Such a cycle is given by a sequence of four consecutive edges $(e_1, e_1'), (e_2, e_2'), (e_3, e_3'), (e_4, e_4')$, where either

(i) $$e_1 = \bar{e}_2 , \ e_3 = \bar{e}_4 , \ e_1' = \overline{e_4'} , \ e_2' = \overline{e_3'}$$

or

(ii) $$e_1 = \bar{e}_4 , \ e_2 = \bar{e}_3 , \ e_1' = \overline{e_2'} , \ e_3' = \overline{e_4'}.$$

Let α be a cycle of type (i) and $e \in Y$ with $o(e) = x$, $e \notin \{e_1, e_3\}$; let β be the cycle given by $(e, e_1'), (\bar{e}, e_2'), (e_3, \overline{e_2'}), (\bar{e}_3, \overline{e_1'})$ and choose $h = (h_1, \mathrm{id}) \in H^{(\infty)}(x) \times H^{(\infty)}(x)$ with $h_1 e_3 = e_1$, $h_1 e_1 = e_3$, $h_1 e = e$. Observe that, since $\overline{\omega(G^{(\infty)})}$ is normal and cocompact in L, it contains $H^{(\infty)} \times H^{(\infty)}$. Thus, we may choose $g \in G^{(\infty)}$ such that $\omega(g)$ and h coincide on a ball of radius 2 centered at v. A computation gives then

$$g\beta^{-1}g^{-1}\beta = \alpha,$$

thus $[G^{(\infty)}, \pi_1(\mathscr{D})]$ contains all paths of type (i) and, by a similar argument, all paths of type (ii).

Since $[G^{(\infty)}, \pi_1(\mathscr{D})]$ is normal in $G \supset \omega^{-1}(H \times_\pi H)$, we obtain $[G^{(\infty)}, \pi_1(\mathscr{D})] = \pi_1(\mathscr{D})$. □

2. Thompson-Wielandt, revisited

2.1. Let T be a uniform tree, that is a locally finite tree such that $\mathrm{Aut}\, T \backslash T$ is finite and Aut T is unimodular. Then, T admits uniform lattices ([Ba-Ku]) and, guided by the analogy between trees and (rank one) symmetric spaces, one is lead to the question whether, given a Haar measure on Aut T, there exists a constant $c > 0$ such that

$$\mathrm{Vol}(\Lambda \backslash \mathrm{Aut}\, T) \geqslant c$$

for all uniform lattices $\Lambda \subset \mathrm{Aut}\, T$.

As observed by Bass-Kulkarni ([Ba-Ku]), the answer for regular trees $\mathcal{T}_n$ turns out to be negative, provided $n \geqslant 3$. In fact, if n is not a prime, one can even find an increasing sequence $\Lambda_i \subset \Lambda_{i+1}$ of (cocompact) lattices in $\mathrm{Aut}\,\mathcal{T}_n$ such that $\Lambda_i\backslash\mathcal{T}_n$ is an edge, and $\mathrm{Vol}(\Lambda_i\backslash\mathrm{Aut}\,\mathcal{T}_n) \to 0$, for $i \to +\infty$. There is, however,

Conjecture (Goldschmidt-Sims). — Given a locally finite tree T, there are only finitely many Aut T-conjugacy classes of locally primitive discrete subgroups.

Equivalently, there should exist $c > 0$, such that $\mathrm{Vol}(\Lambda\backslash\mathrm{Aut}\,T) \geqslant c$ for every locally-primitive lattice $\Lambda \subset \mathrm{Aut}\,T$.

This conjecture was settled for the 3-regular tree by Goldschmidt ([Go]).

A result supporting this conjecture is the theorem of Thompson-Wielandt which says that there exists a neighbourhood U of the identity in Aut T such that the intersection $\Lambda \cap U$ is a nilpotent group for any locally primitive lattice $\Lambda \subset \mathrm{Aut}\,T$; notice the analogy with the Margulis Lemma (see [Bal-G-S]). More precisely,

Theorem **2.1.1** (Thompson [Th], Wielandt [Wi]). — *Let* $T = (X, Y)$ *be a locally finite tree and* $\Lambda < \mathrm{Aut}\,T$ *a discrete, locally primitive subgroup. Then*

a) $\Lambda_2(z)$ *is a p-group for some vertex* $z \in X$ *and some prime p.*

b) *If* Λ *is vertex transitive and* x, y *are adjacent vertices, then* $\Lambda_1(x,y)$ *is a p-group for some prime p. If moreover* $\Lambda_1(x,y) \neq (e)$, *then* $O_p(\Lambda_1(x)) \not\subset \Lambda_1(x,y)$.

This theorem is used in an essential way in Section 3, to ensure non-discreteness of certain locally primitive groups whose action on a sphere of radius 2 are known. For a proof of Thm. 2.1.1, the reader may consult [B-C-N] 7.2.

In this section we will deduce Theorem 2.1.1 from a statement which is valid for all closed locally primitive groups. We introduce now a few notions and notations concerning profinite groups (see Lemma 2.2.0). For a profinite group K and a prime p, $O_p(K)$ denotes the unique maximal closed normal pro-p-subgroup of K, and $O^p(K)$ is the intersection of all closed normal subgroups $N \lhd K$, such that K/N is a pro-p-group. Thus $K/O^p(K)$ is the largest pro-p-quotient of K. Similarly, $O_\infty(K)$ denotes the unique maximal closed normal pro-solvable subgroup of K, while $O^\infty(K)$ is the intersection of all closed normal subgroups $N \lhd K$ such that K/N is pro-solvable. Thus $K/O^\infty(K)$ is the maximal prosolvable quotient of K. A finite subgroup $F < K$ is a component if it is quasi simple and subnormal (see [B-C-N] 7.1); finally E(K) is the closed subgroup generated by all components of K.

Proposition **2.1.2.** — *Let* $T = (X, Y)$ *be a locally finite tree and* $H < \mathrm{Aut}\,T$ *a closed, locally primitive subgroup.*

(1) a) *If* $H_2(x) \neq (e)$, *for some* $x \in X$, *then there exists* $y \in X$ *such that* $E(H(y)) = (e)$.

b) *If* H *acts transitively on* X *and* $H_1(x,y) \neq (e)$ *for some pair of adjacent vertices, then* $E(H(z)) = (e)$, $\forall z \in X$.

(2) a) *If* $O_p(H(x)) \neq (e)$ *for some* $x \in X$, *then there exists* $y \in X$ *such that* $H_2(y)$ *is a pro-p group.*

b) *If* $O_\infty(H(x)) \neq (e)$ *for some* $x \in X$, *then there exists* $y \in X$ *such that* $H_2(y)$ *is pro-solvable.*

(3) *Assume that* H *is vertex transitive and let* $x, y \in X$ *be adjacent vertices.*

a) *If* $O_p(H(x)) \neq (e)$, *then* $H_1(x,y)$ *is pro-p. If moreover* $H_1(x,y) \neq (e)$, *then* $O_p(H_1(x)) \not\subset H_1(x,y)$.

b) *If* $O_\infty(H(x)) \neq e$, *then* $H_1(x,y)$ *is pro-solvable. If moreover* $H_1(x,y) \neq (e)$, *then* $O_\infty(H_1(x)) \not\subset H_1(x,y)$.

Remark 2.1.1 (Prop. 2.1.2 implies Thm. 2.1.1). — If $\Gamma < \operatorname{Aut} T$ is discrete locally primitive, $\Gamma(x)$ is a finite group for every $x \in X$. Assume that Γ is vertex transitive; we show how to deduce Thm. 2.1.1 b). If $\Gamma_1(x,y) \neq (e)$, then (1) b) $E(\Gamma(z)) = (e)\ \forall z \in X$; on the other hand, $\Gamma(x)$ is non-trivial, hence its generalized Fitting subgroup $E(\Gamma(x)) \cdot \prod_p O_p(\Gamma(x))$ is non-trivial as well, implying the existence of a prime p with $O_p(\Gamma(x)) \neq (e)$. Thm. 2.1.1 b) follows then from Prop. 2.1.2.3 a).

Remark 2.1.2. — If K is a profinite, non-finite group it can happen that $E(K) \cdot \prod_p O_p(K) = (e)$, and indeed there are (non-discrete) locally primitive groups $H < \operatorname{Aut} T$ such that $H_2(x)$ is not pro-p for any p (see Section 3).

2.2. For ease of reference, we collect a few preliminary lemmas concerning profinite groups.

Lemma **2.2.0.** — *Let* K *be a profinite group.*

a) K *admits a unique maximal closed normal pro-p-subgroup* $O_p(K)$.

b) *If* $O^p(K)$ *denotes the intersection of all closed normal subgroups* N *of* K *with pro-p-quotient* K/N, *then* $K/O^p(K)$ *is the largest pro-p-quotient of* K.

Proof

a) For any normal pro-p-subgroup $N \lhd K$ and any open normal subgroup $D \lhd K$ the image of N in the finite group K/D is a normal p-subgroup and hence is contained in $O_p(K/D)$, where for any finite group H, one denotes by $O_p(H)$ the maximal normal p-subgroup of H (see [B-C-N] for a proof of the existence of $O_p(H)$ for finite groups). Define $O_p(K) := \varphi^{-1}(\prod_{O \lhd K \text{ open}} O_p(K/O))$ where $\varphi : K \to \prod_{O \lhd K \text{ open}} O_p(K/O)$ is given by $\varphi(k) = (kO)_{O \lhd K}$. Since φ is an isomorphism onto its image we deduce that $O_p(K)$ is a closed normal pro-p-subgroup. Moreover from the argument above it follows that it contains any closed normal pro-p-subgroup.

b) We have to show that $K/O^p(K)$ is a pro-p-group. Let $\mathscr{F}$ denote the set of all closed normal subgroups $N \lhd K$ with K/N a pro-p-group. The homomorphism $\psi : K \to \prod_{N\in\mathscr{F}} K/N$, $\psi(k) = (kN)_{N\in\mathscr{F}}$ induces an isomorphism from $K/O^p(K)$ onto a closed subgroup of the pro-p-group $\prod_{N\in\mathscr{F}} K/N$, which implies that $K/O^p(K)$ is a pro-p-group. □

Lemma **2.2.1.** — *Let* $H_i < K$, $1 \leqslant i \leqslant n$, *be closed subgroups with*

$$H_1 \lhd H_2 \lhd \dots \lhd H_n = K.$$

a) *If* H_i/H_{i-1} *is pro-p for all* $2 \leqslant i \leqslant n$, *then*

$$O^p(K) = O^p(H_1).$$

b) *If* H_i/H_{i-1} *is pro-solvable for all* $2 \leqslant i \leqslant n$, *then* $O^\infty(K) = O^\infty(H_1)$.

Proof. — Let us prove a), the proof of b) follows the same argument. Since H_i/H_{i-1} is a pro-p group we have $H_{i-1} \supset O^p(H_i)$ and since $H_{i-1}/O^p(H_i)$ is a pro-p group we obtain that $O^p(H_i) \supset O^p(H_{i-1})$. Since $O^p(H_{i-1})$ is a topologically characteristic subgroup in H_{i-1}, we have $O^p(H_{i-1}) \lhd H_i$; as H_i/H_{i-1} and $H_{i-1}/O^p(H_{i-1})$ are pro-p groups we deduce that $H_i/O^p(H_{i-1})$ is a pro-p group and hence that $O^p(H_{i-1}) \supset O^p(H_i)$ and a) follows. □

Lemma **2.2.2.** — *Let* $H_i < K$, $1 \leqslant i \leqslant n$, *be closed subgroups with* $H_1 \lhd H_2 \lhd \dots \lhd H_n = K$.

a) $O_p(H_1) \subset O_p(K)$.
b) $O_\infty(H_1) \subset O_\infty(K)$.

Proof. — We shall establish a) (The proof of b) follows the same argument and is omitted). The subgroup $O_p(H_i)$ is a topologically characteristic subgroup of H_i, hence it is normal in H_{i+1} which implies $O_p(H_i) \subset O_p(H_{i+1})$ and hence $O_p(H_1) \subset O_p(K)$. □

For a profinite group K, the subgroups $O_p(K)$, $O^p(K)$, $O_\infty(K)$, $O^\infty(K)$, $E(K)$ are all topologically characteristic, and we have

$$O^p(O^p(K)) = O^p(K),$$
$$O^\infty(O^\infty(K)) = O^\infty(K).$$

The following lemma is crucial and due to Wielandt in the context of finite groups.

Lemma **2.2.3.** — *Let* $H_i < K$, $1 \leqslant i \leqslant n$, *be closed subgroups with*

$$H_1 \lhd H_2 \lhd \dots \lhd H_n = K.$$

Then

a) $O_p(K) \subset N_K(O^p(H_1))$.
b) $O_\infty(K) \subset N_K(O^\infty(H_1))$.
c) $E(K) \subset N_K(H_1)$.

Proof. — We begin with a general observation: let $X = A \cdot B$ be a group where A, B are subgroups of X, $A \lhd X$, and $Y < X$ with $B \lhd Y$; then the homomorphism

$$\begin{aligned} Y \cap A &\longrightarrow Y/B \\ y &\longmapsto yB \end{aligned}$$

is surjective: indeed, for $y \in Y$, $y = a \cdot b$ with $a \in A$, $b \in B$, thus $a = yb^{-1} \in Y \cap A$.

a) Define $X = O_p(K) \cdot O^p(H_1)$, and $L_i := X \cap H_i$, $1 \leqslant i \leqslant n$. We have

$$O^p(H_1) \lhd L_1 \lhd L_2 \lhd \ldots \lhd L_n = X.$$

Since $L_1 \cap O_p(K) \to L_1/O^p(H_1)$ is surjective, $L_1/O^p(H_1)$ is pro-p, and thus, by Lemma 2.2.1a) $O^p(O^p(H_1)) = O^p(L_1)$, but $O^p(O^p(H_1)) = O^p(H_1)$, thus $O^p(H_1) = O^p(L_1)$ and $X = O^p(K) \cdot O^p(L_1)$. Now we prove by recurrence that $O^p(L_1) = O^p(L_{i+1})$. Thus, assume $O^p(L_1) = \ldots = O^p(L_i)$, so $X = O_p(K) \cdot O^p(L_i)$; since $L_i \lhd L_{i+1}$, $O_p(L_i) \lhd L_{i+1}$; since

$$L_{i+1} \cap O_p(K) \longrightarrow L_{i+1}/O^p(L_i)$$

is surjective, $L_{i+1}/O^p(L_i)$ is pro-p, and hence $O^p(O^p(L_i)) = O^p(L_{i+1})$, thus $O^p(L_i) = O^p(L_{i+1})$.
Thus we conclude that $O^p(H_1) = \ldots = O^p(X)$ and in particular $O^p(H_1) \lhd X$, which implies

$$O_p(K) \subset N_K(O^p(H_1)).$$

b) Same argument, for "$p = \infty$".
c) See 7.1.3 (iv) in [B-C-N]. □

2.3. *Proof of Proposition* 2.1.2. — We first prove 3 a). Let $x, y \in X$ be adjacent vertices and assume that $O^p(H_1(x,y)) \neq (e)$; we have to show that $O_p(H(x)) = e$.

Claim. — For every $z \in X$ adjacent to y, we have

$$O_p(H(y)) \cup O_p(H(x,y)) \subset H(z). \tag{1}$$

Indeed, pick a $z \equiv y$ violating (1) and let $g \in O_p(H(y)) \cup O_p(H(x,y))$, with $gz \neq z$. We have

$$\begin{aligned} H_1(z,y) &\lhd H_1(y) \lhd H(y) \\ H_1(z,y) &\lhd H_1(y) \lhd H(x,y) \end{aligned} \tag{2}$$

Lemma 2.2.3 a) implies that $O_p(H(y))$ and $O_p(H(x,y))$ normalize $O^p(H_1(z,y))$, in particular,

$$O^p(H_1(z,y)) = gO^p(H_1(z,y))g^{-1} = O^p(H_1(gz,y)).$$

But

$$O^p(H_1(z,y)) \lhd H(z,y),$$
$$O^p(H_1(gz,y)) \lhd H(gz,y),$$

which implies, $O^p(H_1(z,y)) \lhd \langle H(z,y), H(gz,y)\rangle = H(y)$. Using an element $\tau \in H$, interchanging z and y we get $O^p(H_1(z,y)) \lhd H(z)$ and therefore

$$O^p(H_1(z,y)) \lhd \langle H(z), H(y)\rangle = {}^+H \text{ (see 1.3.0 for the last equality)}$$

which implies that $O^p(H_1(z,y)) = (e)$. Indeed we have that ${}^+H = \langle H(z), H(y)\rangle = H(z) *_{H(z,y)} H(y)$ is a faithful amalgam. This proves the claim. Taking now intersection over all vertices z adjacent to y in (1), we obtain

$$O_p(H(y)) \cup O_p(H(x,y)) \subset H_1(y),$$

in particular,

$$O_p(H(y)) \subset H(x,y)$$
$$O_p(H(x,y)) \lhd H_1(y) \lhd H(y)$$

and thus by Lemma 2.2.2 and (1),

$$O_p(H(y)) \subset O_p(H(x,y)) \subset O_p(H_1(y)) \subset O_p(H(y)),$$

which implies $O_p(H(y)) = O_p((H(x,y))$ and $O_p(H(x,y)) = O_p(H(x))$. Thus

$$O_p(H(y)) = O_p(H(x)) \lhd \langle H(x), H(y)\rangle = {}^+H$$

and therefore $O_p(H(x)) = (e)$. This proves the first part of 3 a). For the second part, assume that

$$O_p(H_1(y)) \subset H_1(x,y).$$

Since $H_1(x,y) \lhd H_1(y)$ and $H_1(x,y)$ is a pro-p-group, we have

$$H_1(x,y) = O_p((H_1(x,y)) \subset O_p(H_1(y))$$

and thus $H_1(x,y) = O_p(H_1(y))$, hence $H_1(x,y) \lhd H(y)$. Using an element exchanging x, y we obtain $H_1(x,y) \lhd \langle H(x), H(y)\rangle = {}^+H$ which implies $H_1(x,y) = (e)$ and concludes the proof of 3 a).

The proof of 2 a) is the same, modulo replacing $H_1(y,z)$ by $H_2(z)$ in (2).

The proof of 2 b) and 3 b) works for "$p = +\infty$" as above.

The proof of 1 b) is analogous to the proof of 3 a) and is obtained by replacing $O_p(H_1(x,y))$ by $H_1(x,y)$, $O_p(H(y))$ by $E(H(y))$ and $O_p(H(x,y))$ by $E(H(x,y))$. □

3. From 2-transitivity to ∞-transitivity

In this chapter we consider vertex transitive, in particular 2-transitive groups of automorphisms of a (regular) tree and show that under additional assumptions of local nature, they are ∞-transitive; this is carried out in 3.3. In 3.1 we obtain some elementary properties of locally ∞-transitive groups; we mention in passing that locally ∞-transitive groups are quite well understood from the point of view of their unitary representation theory (see [Ne], [F-Ne], $[\text{B-M}]_1$, $[\text{B-M}]_2$). In 3.2 we associate to every permutation group $F < S_d$ the unique (up to conjugation) maximal vertex transitive subgroup $U(F)$ of $\operatorname{Aut} T_d$ acting locally like F; if F is 2-transitive these groups provide examples of ∞-transitive groups.

3.1. We establish now a few basic properties of locally ∞-transitive groups.

Lemma **3.1.1.** — *Let* $T = (X, Y)$ *be a locally finite tree. For a closed subgroup* $H < \operatorname{Aut} T$, *the following are equivalent:*

(1) H *is locally* ∞*-transitive.*
(2) $H(x)$ *is transitive on* $T(\infty)$, $\forall x \in X$.
(3) H *is non-compact and transitive on* $T(\infty)$.
(4) H *is* 2*-transitive on* $T(\infty)$.

Any of these properties imply,

(5) $\underline{H}(x) < \operatorname{Sym} E(x)$ *is* 2*-transitive and* H *is non-discrete.*

Proof. — The equivalence 1) $\Leftrightarrow$ 2) and the implication 4) $\Rightarrow$ 3) are clear. For the equivalence 2) $\Leftrightarrow$ 3), see [Ne]. Assume that 1) holds; let $r : \mathbf{N} \to X$ be a geodesic ray and $F_n < \operatorname{Sym}(E(r(0))\backslash e)$, $e = (r(0), r(1))$, be the transitive permutation group defined by the restriction of $\bigcap_{k=0}^{n} H(r(k))$ to $E(r(0))$. The decreasing sequence $F_n > F_{n+1}$ of finite transitive permutation groups stabilizes and hence $F := \bigcap_{k=1}^{\infty} F_n$ is transitive on $E(r(0))\backslash e$. A compactness argument implies that the restriction to $E(r(0))\backslash e$ of $\bigcap_{k=0}^{\infty} H(r(k))$ coincides with F. This implies that the stabilizer in H of the end $\xi \in T(\infty)$ defined by r acts transitively on $T(\infty)\backslash\{\xi\}$. This shows 1) $\Rightarrow$ 4). Finally, the implication 1) $\Rightarrow$ 5) is clear. □

In view of the preceding lemma, locally ∞-transitive groups are locally primitive and we may apply Proposition 1.2.1. This leads to

Proposition **3.1.2.** — *Let* T *be a locally finite tree and* H < Aut T *a closed, locally* ∞*-transitive group. Then,*

(1) $QZ(H) = (e)$.
(2) $H^{(\infty)}$ *is locally* ∞*-transitive and topologically simple.*

Proof. — 1) Let $S \subset T(\infty)$ be the set of fixed points of hyperbolic elements in QZ(H); then S is countable and H-invariant, hence $S = \emptyset$. Every $g \in QZ(H) \cap H^+$, $g \neq e$, being hyperbolic, we deduce $QZ(H) \cap H^+ = (e)$ and hence $|QZ(H)| \leqslant 2$, which finally implies $QZ(H) \subset Z(H) = (e)$.

2) Since H is 2-transitive on $T(\infty)$ and $(e) \neq H^{(\infty)}$ is normal in H, we conclude that $H^{(\infty)}$ is transitive on $T(\infty)$; being also closed and non-compact, Lemma 3.1.1 implies that $H^{(\infty)}$ is locally ∞-transitive. Thus, by 1), $QZ(H^{(\infty)}) = (e)$; applying prop. 1.2.1, we have that $(H^{(\infty)})^{(\infty)}$ is a cocompact characteristic subgroup of $H^{(\infty)}$. In particular it is cocompact and normal in H. Hence $(H^{(\infty)})^{(\infty)} = H^{(\infty)}$. Applying again Prop. 1.2.1 we deduce that $H^{(\infty)}$ is topologically simple. □

3.2. — *The universal group* U(F)

Let $d \geqslant 3$ and $\mathcal{T}_d = (X, Y)$ be the d-regular tree. A legal coloring is a map

$$i : Y \longrightarrow \{1, 2, \ldots, d\},$$

such that:

(1) $i(y) = i(\bar{y})$, $\forall y \in Y$.
(2) $i|_{E(x)} : E(x) \to \{1, 2, \ldots, d\}$, is a bijection $\forall x \in X$.

Given a permutation group $F < S_d$ and a legal coloring i, the group

$$U_{(i)}(F) = \left\{ g \in \operatorname{Aut} \mathcal{T}_d : i|_{E(gx)} g \, i|_{E(x)}^{-1} \in F \, , \ \forall x \in X \right\}$$

is a closed subgroup of $\operatorname{Aut} \mathcal{T}_d$. It enjoys the following basic properties:

(1) $\underline{U_{(i)}(F)(x)} < \operatorname{Sym} E(x)$ is permutation isomorphic to $F < S_d$, $\forall x \in X$.
(2) The group $U_{(i)}(F)$ acts transitively on X.
(3) Given legal colorings i, i', the corresponding subgroups $U_{(i)}(F)$ and $U_{(i')}(F)$ are conjugate in $\operatorname{Aut} \mathcal{T}_d$.

These properties follow from the fact (see [L-M-Z]) that a quadruple (i_1, i_2, x_1, x_2) consisting of legal colorings i, i' and vertices $x_1, x_2 \in X$ determines a (unique) automorphism $g \in \operatorname{Aut} \mathcal{T}_d$ with $g(x_1) = x_2$ and $i_2 = i_1 \circ g$.

Henceforth we shall write U(F) without explicit reference to a legal coloring.

The definition of U(F) implies readily that this group enjoys Tits' independence property (see $[\mathrm{Ti}]_2$ 4.2); using $[\mathrm{Ti}]_2$ 4.2 this observation implies then:

Proposition **3.2.1.** — 1) *The group* $\mathrm{U(F)}^+$ *is trivial or simple.*

2) *The group* $\mathrm{U(F)}^+$ *is of finite index in* U(F) *if and only if* $\mathrm{F} < \mathrm{S}_d$ *is transitive and generated by its point stabilizers; in this case,* $\mathrm{U(F)}^+ = \mathrm{U(F)} \cap (\mathrm{Aut}\,\mathcal{T}_d)^+$ *and is of index 2 in* U(F).

An important property is that, among all vertex transitive groups which are locally permutation isomorphic to $\mathrm{F} < \mathrm{S}_d$, the group U(F) is maximal; more precisely,

Proposition **3.2.2.** — *Let* $\mathrm{F} < \mathrm{S}_d$ *be a transitive permutation group and* $\mathrm{H} < \mathrm{Aut}\,\mathcal{T}_d$ *a vertex transitive subgroup such that* $\underline{\mathrm{H}}(x) < \mathrm{Sym}\,\mathrm{E}(x)$ *is permutation isomorphic to* $\mathrm{F} < \mathrm{S}_d$*. Then, for some suitable legal coloring, we have*

$$\mathrm{H} < \mathrm{U(F)}.$$

Proof. — We construct an appropriate legal coloring: fix $b \in \mathrm{X}$ and, for every $x \in \mathrm{X} \backslash \{b\}$, let $e_x \in \mathrm{Y}$ be the unique edge defined by $o(e_x) = x$, $d(x, b) = d(t(e_x), b) + 1$; for every $y \in \mathrm{Y}$, choose $\sigma_y \in \mathrm{H}$ with $\sigma_y(y) = \bar{y}$. We define a legal coloring $i : \mathrm{Y} \to \{1, 2, ..., d\}$ inductively on $\mathrm{E}_n := \bigcup_{x \in \mathrm{S}(b,\, n-1)} \mathrm{E}(x)$ as follows: for $n = 1$, $\mathrm{E}_1 = \mathrm{E}(b)$, and we choose a bijection $i_b : \mathrm{E}(b) \to \{1, 2, ..., d\}$ such that $i_b \underline{\mathrm{H}}(b) i_b^{-1} = \mathrm{F}$. Assume that i is defined in E_n; for every $x \in \mathrm{S}(b, n)$, define $i|_{\mathrm{E}(x)} := i|_{\mathrm{E}(y)} \sigma_{e_x}|_{\mathrm{E}(x)}$, where $y = t(e_x)$. One verifies easily that $i : \mathrm{Y} \to \{1, ..., d\}$ is a legal coloring and that $\mathrm{H} < \mathrm{U}_{(i)}(\mathrm{F})$. □

Now we turn to the structure of $\mathrm{U(F)}(x)$, $x \in \mathrm{X}$, in the case where $\mathrm{F} < \mathrm{S}_d$ is transitive. Let $\Delta := \{1, 2, ..., d\}$, $\mathrm{D} := \{2, ..., d\}$, $\Delta_n := \Delta \times \mathrm{D}^{n-1}$, $n \geqslant 1$, and $\mathrm{F}_1 := \mathrm{Stab}_{\mathrm{F}}(1)$. Using the legal coloring i, one obtains a family of bijections

$$b_n : \mathrm{S}(x, n) \longrightarrow \Delta_n\,,\quad n \geqslant 1,$$

uniquely determined by the following two properties:

a) $b_1 : \mathrm{S}(x, 1) \to \Delta$, $b_1(y) = i((x, y))$.

b) $\pi_n b_n = b_{n-1} p_n$ where $p_n : \mathrm{S}(x, n) \to \mathrm{S}(x, n-1)$, $\pi_n : \Delta_n = \Delta_{n-1} \times \mathrm{D} \to \Delta_{n-1}$ denote the canonical projection maps.

Define inductively $\mathrm{F}(n) < \mathrm{Sym}\,\Delta_n$ as follows: $\mathrm{F}(1) = \mathrm{F} < \mathrm{Sym}\,\Delta$, and for $n \geqslant 2$, $\mathrm{F}(n) = \mathrm{F}(n-1) \ltimes \mathrm{F}_1^{\Delta_{n-1}}$ is the wreath product for the action of $\mathrm{F}(n-1)$ on Δ_{n-1}. The bijection b_n induces then a surjective homomorphism $\mathrm{U(F)}(x) \to \mathrm{F}(n)$ with kernel

$$\mathrm{U(F)}_n(x) := \{ g \in \mathrm{U(F)}(x) : g|_{\mathrm{S}(x,\, n)} = id \}\,.$$

The homeomorphism $b_\infty : \mathfrak{T}(\infty) \to \varprojlim_n \Delta_n$ induces then an isomorphism

$$U(F)(x) \xrightarrow{\sim} \varprojlim_n F(n)$$

of topological groups.

3.3. In this section $F < S_d$ denotes a permutation group on d letters and $H < \mathrm{Aut}\,\mathcal{T}_d$ a closed vertex-transitive subgroup, such that $\underline{H}(x) < \mathrm{Sym}\,E(x)$ is permutation isomorphic to $F < S_d$. Let F_1 be the stabilizer in F of the letter 1.

In this section, $F < S_d$ will at least be 2-transitive. For a list of all 2-transitive (finite) permutation groups, we refer to [Ca]; explicit realizations of most of them can be found in [Di-Mo], we have used these more detailed descriptions for obtaining the lists appearing in the examples in this section. In the following propositions and examples we use the notations of the Atlas [At] of finite groups.

Proposition **3.3.1.** — *Assume that* F *is* 2*-transitive and* F_1 *is simple non-abelian. Then,*

$$\underline{H}_1(x) \simeq F_1^a\,, \quad \textit{where } a \in \{0, 1, d\}.$$

Moreover,

$$a \in \{0, 1\} \Leftrightarrow H \textit{ is discrete.}$$

$$a = d \Longleftrightarrow H = U(F).$$

Remark. — Let us note the following useful fact which we shall establish in the proof of Prop. 3.3.1: if x and y are adjacent vertices, $H_1(x,y)/H_2(x) \simeq F_1^b$ with $b \in \{0, d-1\}$ and

$$b = 0 \Leftrightarrow H \text{ is discrete.}$$

$$b = d-1 \Longleftrightarrow H = U(F).$$

Example **3.3.1.** — *a)* $F < S_d$, *2-transitive with non-abelian socle, and* F_1, *simple non-abelian.*

d	F	F_1	
n	A_n	A_{n-1}	$(n \geqslant 6)$
11	$L_2(11)$	A_5	
12	M_{11}	$L_2(11)$	
12	M_{12}	M_{11}	
15	A_7	$L_2(7)$	
22	M_{22}	$L_3(4)$	
23	M_{23}	M_{22}	
24	M_{24}	M_{23}	

b) $F < S_d$, *2-transitive of affine type, and* F_1 *simple non-abelian.*

d	F_1
2^4	A_6
2^4	A_7
2^6	$PSU(3, 3)$
2^{6m}	$G_2(2^m)$

Proposition **3.3.2.** — *Assume that* $F < S_d$ *is 2-transitive and* $F_1 = S : 2$ *or* $S{\cdot}2$, *where* S *is transitive, simple, non-abelian. Then,*

$$E(\underline{H}_1(x)) \simeq S^a \text{ where } a \in \{0, 1, d\}.$$

Moreover,

$$a \in \{0, 1\} \Leftrightarrow H \text{ is discrete.}$$
$$a = d \Longleftrightarrow H \text{ is } \infty\text{-transitive.}$$

Example **3.3.2.**

d	F	F_1	
n	S_n	S_{n-1}	$(n \geqslant 6)$
22	$M_{22}{\cdot}2$	$L_3(4) : 2_2$	
176	HS	$U_3(5) : 2$	
276	Co_3	McL : 2	

Proposition **3.3.3.** — *Assume that* $F < S_d$ *is 2-transitive and* F_1 *is quasisimple. Then,*

$$H_1(x)/Z(\underline{H}_1(x)) \simeq (F_1/Z(F_1))^a, \text{ where } a \in \{0, 1, d\}.$$

Moreover,

$$a \in \{0, 1\} \Longleftrightarrow H_1(x, y)/H_2(x) \text{ is abelian.}$$
$$a = d \Longleftrightarrow H = U(F).$$

Example **3.3.3.** — $F < S_d$ *is of affine type.*

d	F_1	
q^n	$SL(n, q)$	$(n, q) \neq (2, 2), (2, 3)$
q^{2n}	$Sp(n, q)$	$(n, q) \neq (2, 2), (2, 3), (4, 2)$
3^6	$SL(2, 13)$	

3.4. In this section we collect a few simple facts used in the proof of the results of Section 3.3.

3.4.1. Let $T = (X, Y)$ be a locally finite tree, $H < \mathrm{Aut}\, T$ a closed, vertex-transitive, locally primitive subgroup and $x, y, z \in X$ with $x \neq z$, adjacent to y. If $H_1(x, y) \subset H_1(z)$, then $H_1(x, y) = (e)$ and H is discrete.

Since ${}^+H$ has index 2 in H and ${}^+H\backslash T$ is an edge, the locally compact group H is unimodular ([BK]). From $H_1(x, y) \subset H_1(z)$ we deduce $H_1(x, y) \subset H_1(y, z)$; these groups are conjugate in H, thus have the same Haar measure, which implies $H_1(x, y) = H_1(y, z)$; this in turn implies $H_1(x, y) \lhd \langle H(x, y), H(y, z)\rangle = H(y)$, the latter equality following from the fact that H is locally primitive unless $\underline{H}(y)$ acts regularly on $E(y)$, in which case $\underline{H}(t)$ acts regularly on $E(t)$ for all $t \in X$ implying that $H(t)$ is isomorphic to the finite group $\underline{H}(t)$ and hence H is discrete. Using an inversion in H exchanging x, y we get $H_1(x, y) \lhd \langle H(x), H(y)\rangle = {}^+H$ which implies $H_1(x, y) = (e)$.

3.4.2. Let $H < \mathrm{Aut}\, T$ be a 1-transitive (in particular vertex transitive) subgroup and $n \geqslant 1$ with $H_n(x) = H_{n+1}(x)$. Then $H_n(x) = (e)$.

Indeed, for $y \in X$ adjacent to x we have $H_n(x) = H_{n+1}(x) \subset H_n(y)$, and applying an inversion which exchanges x, y we get $H_n(x) = H_n(y)$. This in turn implies $H_n(x) \lhd \langle H(x), H(y)\rangle$; the latter group acts transitively on geometric edges, hence $H_n(x) = (e)$.

3.4.3. Let $S \lhd L$ be finite groups, where L/S is solvable and S simple non-abelian. Let $U < L^n$, $n \geqslant 1$, with $pr_i(U) \supset S$ for all $1 \leqslant i \leqslant n$. Then $pr_i(U \cap S^n) = S$ for all i, and $U \cap S^n$ is a (direct) product of subdiagonals of S^n. Here, and in the sequel, a product of subdiagonals is a subgroup of S^n of the form $\Delta_{I_1} \ldots \Delta_{I_r}$, where $\{I_j : 1 \leqslant j \leqslant r\}$ is a partition of $[1, n]$ and for any subset $J \subset [1, n]$, $\Delta_J = \{ (s_i) \in S^n : s_i = e \ \forall i \notin J, \quad s_l = s_k \ \forall l, k \in J\}$.

Indeed, take $k \geqslant 1$ such that $\mathscr{D}^k(L) = S$; we have then $\mathscr{D}^k(U) \subset S^n$ and $pr_i(\mathscr{D}^k(U)) \supset S$ for all i, which implies that $pr_i(U \cap S^n) = S$ for all i. The last assertion is then a well known fact concerning the cartesian powers of a non-abelian simple group.

3.5. In this section we state and prove the lemmas from which the results in 3.3 follow directly, namely Prop. 3.3.3 follows from Lemmas 3.5.2 and 3.5.4, Prop. 3.3.2 follows from Lemma 3.5.1 and 3.5.3, Prop. 3.3.1 follows from Lemmas 3.5.1 and 3.5.3. Assumptions: in this section $F < S_d$ denotes a transitive permutation group and $H < \mathrm{Aut}\, T_d$ a closed, vertex-transitive subgroup such that $\underline{H}(x) < \mathrm{Sym}\, E(x)$ is permutation isomorphic to F. In the sequel, $x, y \in X$ always denote a pair of adjacent vertices.

Lemma **3.5.1.** — *Assume that* $F < S_d$ *is primitive and* $F_1 = S$, $S : 2$ *or* $S{\cdot}2$, *where* S *is simple non-abelian. Then* $E(\underline{H}_1(x)) \simeq S^a$ *with* $a \in \{0, 1, d\}$ *and* $a \in \{0, 1\}$ *if and only if* H *is discrete.*

Proof. — For y adjacent to x, the inclusion $H_1(x) \subset H(x,y)$ gives rise to a homomorphism $\varphi_y : H_1(x) \to H(x,y)/H_1(y) := H_{x,y} \simeq F_1$ with normal image, and an injective homomorphism

$$\varphi : \underline{H}_1(x) \longrightarrow \prod\nolimits_{y \in S(x,1)} H_{x,y} \simeq F_1^d.$$

Since $\operatorname{Im} \varphi_y \lhd H_{x,y}$ and $H_1(x) \lhd H(x)$, we have either $\varphi_y(H_1(x)) = (e)$, $\forall y \in S(x,1)$ or $\varphi_y(H_1(x)) \supset E(H_{x,y}) =: S_y \simeq S$, $\forall y \in S(x,1)$. In the first case we obtain $\underline{H}_1(x) = (e)$, hence $H_1(x) = (e)$ (by 3.4.2) and H is discrete. In the second case, it follows from (3.4.3) that $\varphi(\underline{H}_1(x)) \cap \prod_{y \in S(x,1)} S_y$ is a product of subdiagonals; these subdiagonals determine a bloc decomposition for the $H(x)$-action on $S(x,1)$; since this action is primitive, there are two cases: in the first case we obtain the full diagonal; hence $E(\underline{H}_1(x)) \simeq S$ and $H_1(x,y)/H_2(x)$ is a 2-group. In particular, if $z \in S(x,1)$, $z \neq y$, the image of $H_1(x,y)$ in $H(x,z)/H_1(z) \simeq F_1$ is a subnormal 2-group, and hence trivial. Thus $H_1(x,y) \subset H_1(z)$ which by 3.4.1 implies $H_1(x,y) = (e)$ and H is discrete. Finally, in the second case we obtain $E(H_1(x)) \supset S^d$, in particular $H_1(x,y)/H_2(x)$ cannot be a p-group and hence H is not discrete by Thompson-Wielandt (Theorem 2.1.1). □

Lemma **3.5.2.** — *Assume that* $F < S_d$ *is primitive and* F_1 *is quasisimple. Then*

$$\underline{H}_1(x)/Z(\underline{H}_1(x)) \simeq (F_1/Z(F_1))^a, \quad \textit{with} \quad a \in \{0, 1, d\}.$$

Moreover,

$$a \in \{0, 1\} \iff H_1(x,y)/H_2(x) \textit{ is abelian.}$$

$$a = d \iff H_k(x) \textit{ is non-abelian } \forall k \geqslant 1.$$

Proof. — The proof of Lemma 3.5.2 is completely analogous to that of Lemma 3.5.1; the only point which needs to be verified is that $H_1(x,y)/H_2(x)$ is abelian, if and only if $\underline{H}_k(x)$ is abelian for some $k \geqslant 1$. Assume $H_1(x,y)/H_2(x)$ is abelian; this holds then for all pairs x, y of adjacent vertices. For every $z \in S(x,2)$ let $p(z) \in S(x,1)$ be such that z is adjacent to $p(z)$. Then, the injection $H_3(x) \to H_1(z, p(z))$, $z \in S(x,2)$ gives rise to an injective homomorphism

$$\underline{H}_3(x) \longrightarrow \prod_{z \in S(x,2)} (H_1(z, p(z))/H_2(z))$$

which shows that $\underline{H}_3(x)$ is abelian. Conversely, assume that $\underline{H}_k(x)$ is abelian for some $k \geqslant 1$. Then, for all $m \geqslant k$, $\underline{H}_m(x)$ injects into $\prod \underline{H}_k(z)$, where the product is over all $z \in S(x, m-k)$. This implies that $H_k(x)$ is prosolvable, thus $O_\infty(H(x)) \neq (e)$, and hence (Proposition 2.1.2 3) b)) $H_1(x,y)$ is prosolvable, in particular $H_1(x,y)/H_2(x)$ is solvable, and since F_1 is quasisimple, this implies that $H_1(x,y)/H_2(x)$ is abelian. □

In the next lemmas, $c(n)$ denotes the cardinality of the sphere $S(x,n)$ in $\mathscr{T}_d$.

Lemma **3.5.3.** — *Under the assumptions of Lemma 3.5.1 assume moreover that* S *is transitive on* $\{2, \ldots, d\}$ *and* $a = d$. *Then,*

$$E(\underline{H}_k(x)) \simeq S^{c(k)}, \quad \forall k \geqslant 1.$$

In particular H *is* ∞*-transitive.*

Proof.

Notation. — For $z \in S(x, n)$, set $p(z) = S(x, n-1) \cap S(z, 1)$, $H_{x,z} := H(z, p(z))/H_1(z)$; for $y \in S(x, 1)$, set $S_n(x, y) := \{z \in S(x, n) : d(z, x) = d(z, y) + 1\}$, $a(n) = |S_n(x, y)|$, $c(n) = |S(x, n)|$.

First we claim that H is locally ∞-transitive. Since H is non-discrete (Lemma 3.5.1), we have $H_n(x) \neq (e)$ and hence (3.3.2) $H_n(x) \neq H_{n+1}(x)$ for all $n \geqslant 1$. Thus, for every $n \geqslant 1$, there exists $z \in S(x, n)$ such that the image $I_{n,z}$ of $H_n(x)$ in $H_{x,z}$ is non-trivial; let $x_0 = x, x_1, \ldots, x_{n-1} = p(z)$, $x_n = z$, be the consecutive vertices of a geodesic path connecting x to z, then:

$$H_n(x) \lhd H_1(x_1, \ldots, x_{n-1}) \lhd H_1(x_{n-1}) \lhd H(p(z), z),$$

and therefore $I_{n,z}$ is a non-trivial, subnormal subgroup of $H_{x,z} \simeq F_1$, which implies $I_{n,z} \supset E(H_{x,z})$ and hence is transitive on $E(z) \backslash (z, p(z))$. Using that F is 2-transitive and recurrence on $n \geqslant 2$ one concludes that $H(x)$ is transitive on $S(x, n)$.

Next we claim that if $\{B_i\}_{1 \leqslant i \leqslant k}$ is a bloc decomposition for the $H(x)$-action on $S(x, n)$ $(n \geqslant 2)$ with $|B_i \cap S_n(x, y)| \leqslant 1$ for all i and $y \in S(x, 1)$, then $|B_i| = 1$ for all i. Since $H(x)$ acts transitively on $S(x, n)$, all blocs have the same cardinality. Assume $|B_1| \geqslant 2$ and choose $y \neq y'$ in $S(x, 1)$ such that $B_1 \cap S_n(x, y) = \{z\}$, $B_1 \cap S_n(x, y') = \{z'\}$; choose $t \in S_n(x, y')$ with $t \neq z'$, so that $t \in B_i$ for some $i \neq 1$. Since H is locally ∞-transitive we may choose $g \in H(x)$ with $g(z) = z$ and $g(z') = t$; this implies $g(B_1) = B_1$ and $g(B_1) = B_i$, contradicting $i \neq 1$. This establishes the claim.

Now we show by induction on $n \geqslant 1$ that $E(\underline{H}_n(x)) \simeq S^{c(n)}$; this holds for $n = 1$. Let $n \geqslant 2$; for $y \in S(x, 1)$, we have $H_n(x) \lhd H_{n-1}(y)$, and, if $I_{n,x,y}$ denotes the image of $H_{n-1}(y)$ in $\prod_{z \in S_n(x, y)} H_{x,z}$, we have $E(I_{n,x,y}) \simeq S^{a(n)}$ by the induction hypothesis; the same holds therefore for the image of $H_n(x)$ in $\prod_{z \in S_n(x,y)} H_{x,z}$.

For the image $\underline{H}_n(x)$ of $H_n(x)$ in $\prod_{y \in S(x,1)} \prod_{z \in S_n(x,y)} H_{x,z}$, we have that $E(\underline{H}_n(x))$ is a product of subdiagonals in $S^{c(n)}$; this induces then a bloc decomposition of $S(x, n)$ as above, and hence $E(\underline{H}_n(x)) \simeq S^{c(n)}$. □

Lemma **3.5.4.** — *Under the assumptions of Lemma 3.5.2, assume moreover that* $F < S_d$ *is 2-transitive and* $a = d$. *Then,* $\underline{H}_k(x) \simeq F_1^{c(k)}$, $\forall k \geqslant 1$.

Proof. — The proof proceeds as in Lemma 3.5.3, with the following additional remark:

By Lemma 3.5.2, there is for every $n \geqslant 1$, a $z \in \mathrm{S}(x, n)$ such that $\mathrm{I}_{n,z} \lhd \mathrm{H}_{x,z} \simeq \mathrm{F}_1$, is non-abelian, and therefore $\mathrm{I}_{n,z} = \mathrm{F}_1$. From this one concludes that H is locally ∞-transitive, and concludes the proof as in Lemma 3.5.3. □

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