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ON TRANSVERSE FOLIATIONS

by ITIRO TAMURA *and* ATSUSHI SATO

The structure of foliations displays a high degree of variability, and is generally far less rigid in contrast to complex structures. Thus, it is virtually impossible to give a precise description which characterizes effectively all foliations on a manifold, and the consequent lack of appropriate classification theorems seems to constitute a barrier to the derivation of precise results in foliation theory. However, if we fix some foliation on a manifold, and restrict our considerations to foliations having a definite relation with the given foliation (i.e. a structure of foliations on a foliated manifold), then a characterization of this class of foliations can often be obtained.

This paper deals with subfoliations of, and foliations transverse to, a given foliation. We shall establish classification theorems for codimension one foliations transverse to the Reeb component of $S^1 \times D^2$, and to the Reeb foliation of S^3 respectively (Theorems 1, 2, 3, 4 and 5).

Furthermore, as an application of Theorem 1, we shall prove that the foliations of codimension one of S^3 constructed from fibred knots do not admit any transverse foliation of codimension one (Theorem 6).

In Section 1, we define subfoliations, superfoliations, and transverse foliations. In Section 2, we consider a generalization of a result due to Reinhart, Davis and Wilson; this constitutes the starting point of our work. In Sections 3, 4 and 5, we study foliations of codimension one transverse to the Reeb component $\mathscr{F}_R$, the set of which is denoted by $t_1(\mathscr{F}_R)$. The existence of the half Reeb component and the TS components in $\mathscr{F}' \in t_1(\mathscr{F}_R)$ are proved in Sections 3 and 4, respectively. In Section 5, we give classification theorems for $t_1(\mathscr{F}_R)$. As a direct consequence of these theorems, the classification for foliations of codimension one transverse to the Reeb foliation of S^3 is derived in Section 6. In Section 7, we prove the non-existence of a foliation of codimension one transverse to a foliation of S^3 constructed from a fibred knot. The problems raised by the results of this paper are given in Section 8.

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1. Subfoliations, superfoliations and transverse foliations

Let M^n be an n-dimensional C^∞ manifold with or without boundary. Denote by $\mathscr{F}^{(k)}$ a C^r foliation of codimension k of M^n $(r \geqq 0)$, where, in case $\partial M^n \neq \varnothing$,

$$\mathscr{F}^{(k)} \,|\, \partial M^n = \{L \cap \partial M;\ L \in \mathscr{F}^{(k)}\}$$

is a C^r foliation of codimension $k-1$ or k of ∂M^n. Two C^r foliations $\mathscr{F}_1^{(k)}$ and $\mathscr{F}_2^{(k)}$ of codimension k of M^n are called *isomorphic* if there exists a C^r diffeomorphism $f: M^n \to M^n$ which preserves the leaves of $\mathscr{F}_1^{(k)}$ and $\mathscr{F}_2^{(k)}$.

Let $\mathscr{F}^{(k)}$ and $\mathscr{F}'^{(k')}$ be C^r foliations of codimensions k and k' of M^n respectively. Then $\mathscr{F}'^{(k')}$ is called a *subfoliation* of $\mathscr{F}^{(k)}$ and $\mathscr{F}^{(k)}$ is called a *superfoliation* of $\mathscr{F}'^{(k')}$, denoted by $\mathscr{F}'^{(k')} \prec \mathscr{F}^{(k)}$, if the following conditions hold:

(i) $k \leqq k' \leqq n$.

(ii) For any leaf L' of $\mathscr{F}'^{(k')}$, there exists a leaf L of $\mathscr{F}^{(k)}$ such that $L' \subset L$, and the restriction of $\mathscr{F}'^{(k')}$ on a leaf L of $\mathscr{F}^{(k)}$ is a C^r foliation of codimension $k'-k$ of L.

In case $r \geqq 1$, it is obvious that, if $\mathscr{F}'^{(k')} \prec \mathscr{F}^{(k)}$ and $\mathscr{F}''^{(k'')} \prec \mathscr{F}'^{(k')}$, then $\mathscr{F}''^{(k'')} \prec \mathscr{F}^{(k)}$. Therefore the relation $\prec$ is an order in the set of C^r foliations of M^n $(r \geqq 1)$.

Two subfoliations $\mathscr{F}_1'^{(k')}$ and $\mathscr{F}_2'^{(k')}$ of $\mathscr{F}^{(k)}$ are called *strongly isomorphic*, if there exists a C^r diffeomorphism $f: M^n \to M^n$ which preserves $\mathscr{F}^{(k)}$ and maps $\mathscr{F}_1'^{(k')}$ onto $\mathscr{F}_2'^{(k')}$.

Let $\tau(\mathscr{F}^{(k)})$ denote the subbundle of the tangent bundle $\tau(M^n)$ of M^n consisting of vectors tangent to leaves of $\mathscr{F}^{(k)}$. In order that $\mathscr{F}^{(k)}$ has a C^r subfoliation of codimension k', it is necessary that $\tau(\mathscr{F}^{(k)})$ has a $(k'-k)$-dimensional subbundle if $r \geqq 1$.

A C^r foliation $\mathscr{F}'^{(k')}$ of codimension k' of M^n is called *transverse* to a C^r foliation $\mathscr{F}^{(k)}$ of M^n $(r \geqq 1)$, denoted by $\mathscr{F}'^{(k')} \pitchfork \mathscr{F}^{(k)}$, if the following conditions hold:

(i) $k + k' \leqq n$.

(ii) Any leaves L of $\mathscr{F}^{(k)}$ and L' of $\mathscr{F}'^{(k')}$ intersect transversely in case $L \cap L' \neq \varnothing$.

Let $\mathscr{F}^{(k)} \cap \mathscr{F}'^{(k')}$ denote $\{L \cap L'; L \in \mathscr{F}^{(k)}, L' \in \mathscr{F}'^{(k')}\}$, then it is clear that $\mathscr{F}^{(k)} \cap \mathscr{F}'^{(k')}$ is a C^r foliation of codimension $k+k'$ which is a common subfoliation of $\mathscr{F}^{(k)}$ and $\mathscr{F}'^{(k')}$.

Two C^r foliations $\mathscr{F}_1'^{(k')}$ and $\mathscr{F}_2'^{(k')}$ which are transverse to $\mathscr{F}^{(k)}$ are called *strongly isomorphic*, if there exists a C^r diffeomorphism $f: M^n \to M^n$ which preserves $\mathscr{F}^{(k)}$ and maps $\mathscr{F}_1'^{(k')}$ onto $\mathscr{F}_2'^{(k')}$.

We note that the transversality $\mathscr{F}^{(k)} \pitchfork \mathscr{F}'^{(k')}$ is invariant under a small perturbation of $\mathscr{F}^{(k)}$ and $\mathscr{F}'^{(k')}$ respectively.

In order that $\mathscr{F}^{(k)}$ admits a transverse C^r foliation of codimension k', it is necessary that $\tau(M^n)$ has an $(n-k')$-dimensional subbundle which is transverse to $\tau(\mathscr{F}^{(k)})$ at each point of M^n if $r \geqq 1$.

Example 1. — It is well known that a C^r foliation $\mathscr{F}^{(1)}$ of codimension 1 of M^n always admits a transverse C^r foliation of codimension $n-1$ $(r \geqq 1)$.

Example 2. — Let $\mathscr{F}^{(2)}$ be a C^r foliation of codimension 2 of the 3-sphere S^3 consisting of compact leaves $(r \geqq 1)$. Then there exists no C^r foliation of codimension one which is transverse to $\mathscr{F}^{(2)}$. Because, if there exists a C^r foliation of codimension one transverse to $\mathscr{F}^{(2)}$, say $\mathscr{F}^{(1)}$, then $\mathscr{F}^{(1)}$ contains a Reeb component by Novikov's theorem ([4]) which implies that $\mathscr{F}^{(2)}$ should contain non-compact leaves.

Example 3. — Let $\mathscr{F}^{(2)}$ be a C^r foliation of codimension 2 of S^3 which admits a superfoliation of codimension one, then $\mathscr{F}^{(2)}$ has a compact leaf. That is to say, the conjecture of Seifert holds in this case. Because a C^r foliation $\mathscr{F}^{(1)}$ of codimension one of S^3 having $\mathscr{F}^{(2)}$ as a subfoliation contains a Reeb component ([4]), and any subfoliation of a Reeb component has a compact leaf (see Proposition 2 of Section 2).

For a family $\{\mathscr{F}_\lambda^{(k)}\}_{\lambda\in\Lambda}$ of C^r foliations of codimension k of M^n, we denote by $t_j(M^n, \{\mathscr{F}_\lambda^{(k)}\})$ or simply by $t_j(\{\mathscr{F}_\lambda^{(k)}\})$ the family $\{\mathscr{F}_\sigma'^{(k')}\}_{\sigma\in\Sigma}$ of C^r foliations of codimension k' such that $j=n-k-k'$ and that there exists $\mathscr{F}_\lambda^{(k)}$ transverse to $\mathscr{F}_\sigma'^{(k')}$. Further we denote by $t_j^m(\{\mathscr{F}_\lambda^{(k)}\})$ the m fold iteration $t_j(t_j(\dots(t_j(\{\mathscr{F}_\lambda^{(k)}\})\dots)$ of t_j. It is obvious that the iterations of t_j have the property

$$t_j^m(\{\mathscr{F}_\lambda^{(k)}\}) \subset t_j^{m+2}(\{\mathscr{F}_\lambda^{(k)}\}) \qquad (m \geqq 1).$$

Now we give a sufficient condition for the existence of transverse plane fields for a C^r foliation:

Proposition 1. — *Let* M^n *be a compact orientable n-dimensional* C^∞ *manifold and* $\mathscr{F}^{(k)}$ *a* C^r *foliation of codimension* k $(r \geqq 1)$ *such that, in case* $\partial M^n \neq \varnothing$, $\mathscr{F}^{(k)} | \partial M^n$ *is a* C^r *foliation of codimension* $k-1$. *Then, in order that* M^n *admits a* $(k+1)$*-plane field transverse to* $\tau(\mathscr{F}^{(k)})$, *it is sufficient that*

$$H^j(M^n; \pi_{j-1}(S^{n-k-1})) = 0, \qquad j = 1, 2, \dots, n.$$

In particular, any C^r *foliation* $\mathscr{F}^{(1)}$ *of codimension one of* S^3 *admits a 2-plane field transverse to* $\tau(\mathscr{F}^{(1)})$.

Proof. — The obstruction to construct a non-zero cross section of $\tau(\mathscr{F}^{(k)})$ lies in $H^j(M^n; \pi_{j-1}(S^{n-k-1}))$ ([1; Theorem (1.1)]). The $(k+1)$-plane field generated by the vector field of $\tau(\mathscr{F}^{(k)})$ and a k-plane field transverse to $\tau(\mathscr{F}^{(k)})$ has the required property.

2. Subfoliations of a foliation of codimension one defined by a fibering over S^1

In the following sections, we fix an orientation on the circle S^1. The Reeb component of $S^1 \times D^2$ constructed by turbulizing ([4]) a collar of the boundary $S^1 \times \partial D^2$ in the minus (resp. plus) direction of S^1 is called the *plus Reeb component* (resp. the *minus Reeb component*) and denoted by $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$) (Fig. 1). That is, $S^1 \times \partial D^2$ has a contracting holonomy in the minus (resp. plus) direction of S^1 for $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$). We define the *plus Reeb component* $\overline{\mathscr{F}}_R^{(+)}$ (resp. the *minus Reeb component* $\overline{\mathscr{F}}_R^{(-)}$) of $S^1 \times D^1$ similarly (Fig. 2). We understand that $\mathscr{F}_R^{(\pm)}$, $\overline{\mathscr{F}}_R^{(\pm)}$ mean standard ones (i.e. leaves are "symmetric" with respect to an "axis" and $\{*\} \times D^2$ (resp. $\{*\} \times D^1$) is tangent to exactly one leaf of $\mathscr{F}_R^{(\pm)}$ (resp. $\overline{\mathscr{F}}_R^{(\pm)}$) at one point).

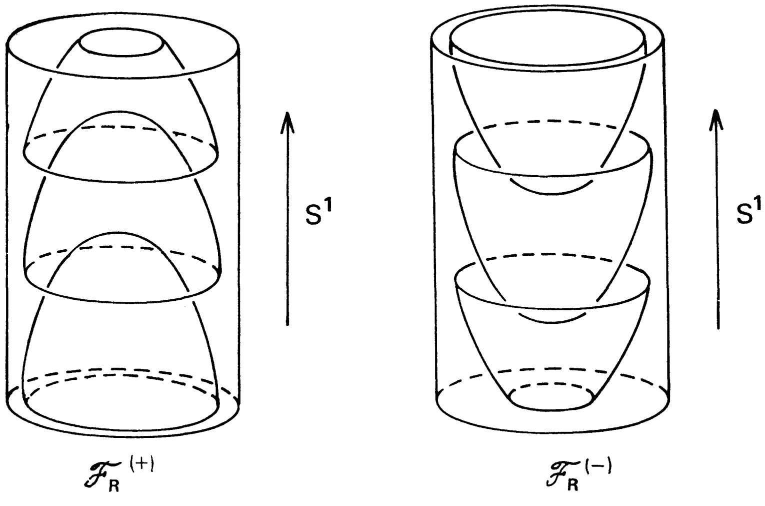

FIG. 1

A C^∞ foliation of codimension one of $S^1 \times D^1$ constructed by turbulizing a collar of $S^1 \times \{-1\}$ and $S^1 \times \{1\}$ in different directions is called a *slope component* and denoted by $\mathcal{F}_S$ (Fig. 2). The set of vertices (i.e. maximal or minimal points) of leaves of $\overline{\mathcal{F}}_R^{(+)}$ (resp. $\overline{\mathcal{F}}_R^{(-)}$) is denoted by $\Sigma^{(+)}$ (resp. $\Sigma^{(-)}$) (Fig. 2).

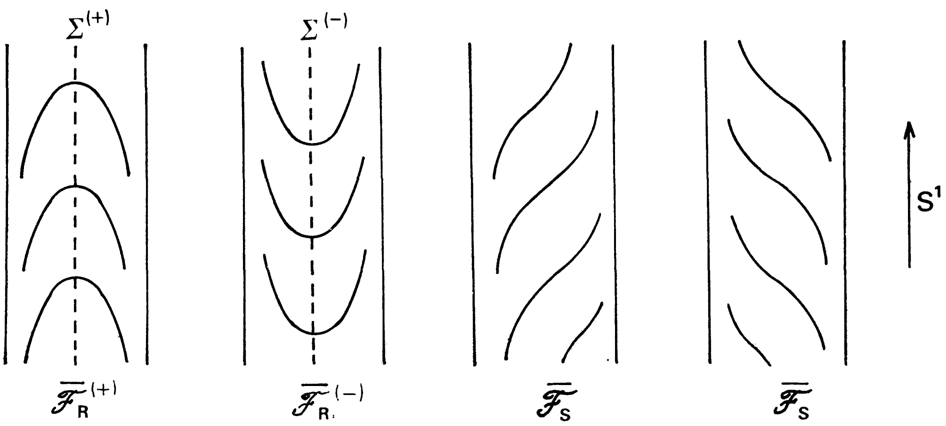

FIG. 2

Let E be a compact connected orientable 3-dimensional C^∞ manifold with boundary $\partial E = T^2$ (torus) and let $\pi : E \to S^1$ be a C^∞ fibering over S^1 with fibre $G - \text{Int } D^2$, where G is an orientable closed surface of genus g and D^2 is a 2-disc imbedded in G. The C^∞ foliation of codimension one of E constructed by turbulizing the fibers in a collar of the boundary ∂E in the minus (resp. plus) direction is denoted by $\mathcal{F}_\pi^{(+)}$ (resp. $\mathcal{F}_\pi^{(-)}$).

The following proposition is a generalization of a result of Reinhart, Davis and Wilson about tangent vector fields of the Reeb component ([1], [6]).

Proposition 2. — Let $\mathcal{F}^{(2)}$ *be a* C^∞ *foliation of codimension* 2 *of* E *which is a subfoliation of* $\mathcal{F}_\pi^{(+)}$. *Denote by* $\mathcal{F}^{(2)} \mid T^2$ *the* C^∞ *foliation of codimension one of* $\partial E = T^2$ *which is the restriction of* $\mathcal{F}^{(2)}$ *to the compact leaf* T^2 *of* $\mathcal{F}_\pi^{(+)}$. *Then the following holds*:

(i) $\mathcal{F}^{(2)} \mid T^2$ *has a compact leaf.*

(ii) $\mathcal{F}^{(2)} \mid T^2$ *is isomorphic to a* C^∞ *foliation consisting of p copies of the plus Reeb component, q copies of the minus Reeb component (with respect to the orientation induced naturally from that of* S^1*), a countable number of slope components, and compact leaves* (Fig. 3), *for which, letting the homology class* $[L_{\text{comp}}]$ *of* $H_1(T^2; \mathbf{Z})$ *represented by a compact leaf* L_{comp} *of* $\mathcal{F}^{(2)} \mid T^2$ *with a suitable orientation be* $a\alpha + b\beta$ $(a \geqq 0)$, *the equation*

$$a(p-q) = 2(1-2g)$$

holds, where α *(resp.* β*) is the homology class represented by a cross section of* π *with the orientation compatible with that of the base space* S^1 *(resp. by* $\partial(G - \operatorname{Int} D^2)$).

(iii) *In particular, if* $G = S^2$, *then we have*

$$a = 1, \quad p - q = 2 \quad \text{or} \quad a = 2, \quad p - q = 1,$$

and $\tau(\mathcal{F}^{(2)})$ *is orientable if and only if* $a = 1$.

The number *a* in Proposition 2, (ii) is called the *longitudinal number* of $\mathcal{F}^{(2)} \mid T^2$.

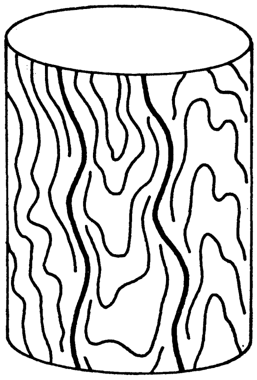

FIG. 3

Proof. — Let F be the line field on T^2 determined by $\mathcal{F}^{(2)} \mid T^2$, then F induces a homomorphism

$$F_*: H_1(T^2; \mathbf{Z}) \to H_1(P^1; \mathbf{Z})$$

([6]). If F_* is not a zero map, then $\mathcal{F}^{(2)} \mid T^2$ has a compact leaf ([6; Corollary 3]). Letting $c: T^2 \times I \to E$ be a collar of T^2 such that $c(x, 0) = x$ $(x \in T^2)$, we define a projection $P: c(T^2 \times I) \to T^2$ by $P(c(x, t)) = x$. Let L be a leaf of $\mathcal{F}_\pi^{(+)}$ and let

$$\iota: G - \operatorname{Int} D^2 \to L$$

be an imbedding such that

$$\iota(\partial(G-\mathrm{Int}\, D^2)) \subset c(T^2\times I).$$

Let F′ be the line field on $\iota(G-\mathrm{Int}\,D^2)$ determined by $\mathscr{F}^{(2)}|\iota(G-\mathrm{Int}\,D^2)$. If $F_*([P\circ\iota(\partial(G-\mathrm{Int}\,D^2))])=0$, then the line field $F'|\iota(\partial(G-\mathrm{Int}\,D^2))$ should be homotopic to the line field tangent to $\iota(\partial(G-\mathrm{Int}\,D^2))$. This implies that the Euler number $\chi(G)$ must be 1. This is a contradiction. Thus F_* is not a zero map and $\mathscr{F}^{(2)}|T^2$ has a compact leaf.

As is easily verified ([1]), the existence of a compact leaf implies that $\mathscr{F}^{(2)}|T^2$ is isomorphic to a C^∞ foliation consisting of p copies of the plus Reeb component, q copies of the minus Reeb component, a countable number of slope components, and compact leaves. Therefore we may choose the imbedding ι defined above so that it satisfies that $P\circ\iota(\partial(G-\mathrm{Int}\,D^2))$ intersects $\mathscr{F}^{(2)}$ transversely except at $a(p+q)$ points corresponding to $\Sigma^{(+)}$ or $\Sigma^{(-)}$. Let $\hat{G}$ be the double of $\iota(G-\mathrm{Int}\,D^2)$, then F′ defines a continuous line field on $\hat{G}$ with ap singular points of plus type and aq singular points of minus type. Therefore, by computing the Euler number $\chi(G-\mathrm{Int}\,D^2)$, we have

$$a(p-q)=2(1-2g).$$

Thus (ii) is proved. The proof of (iii) is obvious.

3. Half Reeb components

Let D^2_+ denote the half 2-disc $\{(x,y)\in D^2; y\geqq 0\}$. The restriction of the plus (resp. minus) Reeb component $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$) of $S^1\times D^2$ to $S^1\times D^2_+$ is called the *plus* (resp. *minus*) *half Reeb component* and denoted by $\mathscr{F}_{R/2}^{(+)}$ (resp. $\mathscr{F}_{R/2}^{(-)}$). Let $\mathscr{F}'_+$ (resp. $\mathscr{F}'_-$) denote the C^∞ foliation of codimension one of $S^1\times D^2$ obtained from two copies of $\mathscr{F}_{R/2}^{(+)}$ (resp. $\mathscr{F}_{R/2}^{(-)}$) by identifying their compact leaves (Fig. 4).

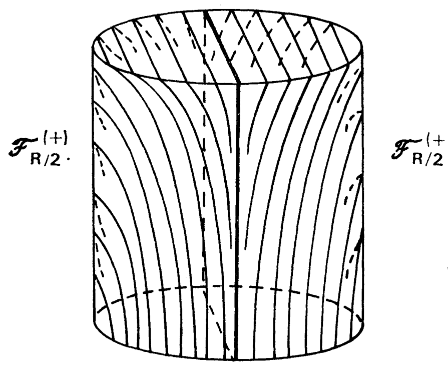

FIG. 4

It is well known that $\mathscr{F}'_+$ (resp. $\mathscr{F}'_-$) is transverse to $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$). (See, for example, [7].)

In Sections 3, 4 and 5, we let $\mathscr{F}'$ be a C^∞ foliation of codimension one of $S^1\times D^2$ transverse to $\mathscr{F}_R^{(+)}$:

$$\mathscr{F}'\in t_1(\mathscr{F}_R^{(+)}).$$

The C^∞ foliation $\mathscr{F}_R^{(+)}\cap\mathscr{F}'$ of codimension 2 of $S^1\times D^2$ is a subfoliation of $\mathscr{F}_R^{(+)}$ (Section 1). Denote by $\bar{\mathscr{F}}'$ the restriction of $\mathscr{F}_R^{(+)}\cap\mathscr{F}'$ to $S^1\times\partial D^2$. Then, by Proposition 2, the C^∞ foliation $\bar{\mathscr{F}}'=\{L\cap(S^1\times S^1)\,;\,L\in\mathscr{F}'\}$ is isomorphic to a C^∞ foliation $\bar{\mathscr{F}}$ consisting of p copies of the plus Reeb component, q copies of the minus Reeb component, a countable number of slope components, and compact leaves, for which Proposition 2, (iii) holds. Therefore there exists a C^∞ diffeomorphism $f: S^1\times S^1\to S^1\times S^1$ isotopic to the identity such that f maps $\bar{\mathscr{F}}$ to $\bar{\mathscr{F}}'$ and that, for any $x\in S^1$, $f(\{x\}\times S^1)$ intersects $\bar{\mathscr{F}}'$ transversely except at $a(p+q)$ points $f((\{x\}\times S^1)\cap(\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)}))$, where $\hat{\Sigma}^{(+)}$ (resp. $\hat{\Sigma}^{(-)}$) denotes the union of $\Sigma^{(+)}$ (resp. $\Sigma^{(-)}$) of each plus (resp. minus) Reeb component contained in $\bar{\mathscr{F}}$ (Fig. 5).

We fix a natural product Riemannian metric on $S^1\times D^2$. Let U be a neighborhood of $S^1\times S^1$ in $S^1\times D^2$ and let $V=\{V(z)\,;\,z\in U\}$ be a C^∞ vector field on U satisfying the following conditions:

(i) $|V(z)|=1$;

(ii) $V(z)$ is tangent to the leaf of $\mathscr{F}'$ containing z;

(iii) for $z\in S^1\times S^1$, $V(z)$ is inward and normal to the leaf of $\bar{\mathscr{F}}'$ containing z.

The existence of such a C^∞ vector field V is obvious.

For $z\in S^1\times S^1$, let $\varphi(t,z)$ $(0\leqq t<\varepsilon_z)$ denote the integral curve with the initial condition $\varphi(0,z)=z$. Let $\varepsilon>0$ be sufficiently small and let $\bar{\varepsilon}: S^1\to]0,\varepsilon[$ be a C^∞ function. Then, by a suitable choice of $\bar{\varepsilon}$, $\bigcup_{x\in S^1}\varphi(\bar{\varepsilon}(x),f(x,y_0))$ is transverse to $\mathscr{F}_R^{(+)}$. Denote by $L(x,y_0)$ $((x,y_0)\in S^1\times S^1)$ the leaf of $\mathscr{F}_R^{(+)}$ containing $\varphi(\bar{\varepsilon}(x),f(x,y_0))$. Then there exists a unique C^∞ function

$$\gamma_x: S^1\to]0,1[$$

such that

$$\gamma_x(y_0)=\bar{\varepsilon}(x),\qquad \varphi(\gamma_x(y),f(x,y))\in L(x,y_0)$$

and that $\bigcup_{y\in S^1}\varphi(\gamma_x(y),f(x,y))$ is a simple closed curve in $L(x,y_0)$. Now we define

$$A=S^1\times D^2-\{\varphi(t,f(x,y))\,;\,0\leqq t<\gamma_x(y),\ (x,y)\in S^1\times S^1\}.$$

Then A is a 3-dimensional C^∞ manifold diffeomorphic to $S^1\times D^2$, and $A\cap L$ is a closed 2-disk for each non-compact leaf L of $\mathscr{F}_R^{(+)}$. Let $\bar{\mathscr{F}}''=\{\partial A\cap L'\,;\,L'\in\mathscr{F}'\}$, then $\bar{\mathscr{F}}''$ is a C^∞ foliation of codimension one of ∂A. The C^∞ diffeomorphism $g: S^1\times S^1\to\partial A$ which maps (x,y) to $\varphi(\gamma_x(y),f(x,y))$ gives an isomorphism from $\bar{\mathscr{F}}$ to $\bar{\mathscr{F}}''$.

Denote by $A^{(x)}$ the intersection $A\cap L_x$, where L_x is the leaf of $\mathscr{F}_R^{(+)}$ containing $(x,0)\in S^1\times D^2$. By the construction above, $\partial A^{(x)}$ is a simple closed curve intersecting $\bar{\mathscr{F}}''$ transversely except at $a(p+q)$ points $\partial A^{(x)}\cap g(\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)})$.

Obviously there exists a C^∞ diffeomorphism from A to $S^1\times D^2$ which maps $A^{(x)}$ to $\{x\}\times D^2$. Thus, making use of the identification by this diffeomorphism, we may assume that

$$A=S^1\times D^2, \qquad A^{(x)}=\{x\}\times D^2,$$

and that the plus and the minus Reeb components in $\mathscr{F}''$ of $\partial A=S^1\times S^1$ are standard (as in Fig. 2). So we use the same notations $\hat{\Sigma}^{(+)}$, $\hat{\Sigma}^{(-)}$ for $\mathscr{F}''$ as for $\mathscr{F}$.

The intersection $A^{(x)}\cap L'$ ($L'\in\mathscr{F}'$) defines a family of C^∞ simple curves $\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda(x)}$ of $A^{(x)}$, where we understand that $\ell_\lambda^{(x)}$ is a closed set of $A^{(x)}$ and $\ell_\lambda^{(x)}\cap \operatorname{Int} A^{(x)}$ is connected. We note that there exists a C^∞ vector field on the manifold (with corner) obtained by cutting $S^1\times D^2$ at $\{x_0\}\times D^2$ such that integral curves are $\bigcup_x\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda(x)}$.

Lemma 1. — (i) $\ell_\lambda^{(x)}$ *is tangent to* $\partial A^{(x)}$ *at* (x,y) *if and only if* $y\in\partial A^{(x)}\cap\hat{\Sigma}^{(-)}$, $y\in\ell_\lambda^{(x)}$.

(ii) $\ell_\lambda^{(x)}$ *is reduced to a point at* $(x,y)\in\partial A^{(x)}\cap\hat{\Sigma}^{(+)}$.

(iii) $\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda(x)}$ *forms a family of concentric half circles with* (x,y) *as center near* $(x,y)\in\partial A^{(x)}\cap\hat{\Sigma}^{(+)}$ *and upper part of a family of confocal parabolas with* (x,y) *as focus near* $(x,y)\in\partial A^{(x)}\cap\hat{\Sigma}^{(-)}$ (Fig. 4 and 5).

This lemma is clear, because the situation of $\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda}$ near $(x,y)\in\partial A^{(x)}\cap\hat{\Sigma}^{(+)}$ (resp. $(x,y)\in\partial A^{(x)}\cap\hat{\Sigma}^{(-)}$) is similar as the situation of leaves of the plus (resp. minus) Reeb component $\overline{\mathscr{F}}_R^{(+)}$ (resp. $\overline{\mathscr{F}}_R^{(-)}$) near a point of $\hat{\Sigma}^{(+)}$ (resp. $\hat{\Sigma}^{(-)}$).

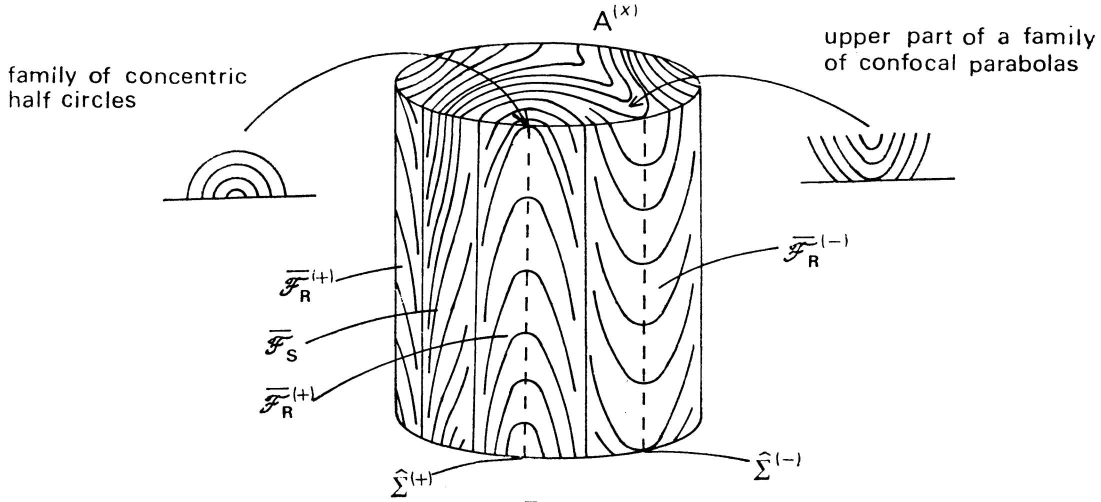

FIG. 5

For $y\in\partial D^2$, let $\ell^{(x)}(y)$ denote a simple curve of $\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda(x)}$ containing (x,y). If $(x,y)\notin\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)}$, $\ell^{(x)}(y)$ exists and is unique, and if $(x,y)\in\hat{\Sigma}^{(-)}$ there exist two kinds of $\ell^{(x)}(y)$, say $\ell_1^{(x)}(y)$ and $\ell_2^{(x)}(y)$. The following lemma is an immediate consequence of the Poincaré-Bendixson theorem:

Lemma 2. — *For* $(x,y)\notin\hat{\Sigma}^{(+)}$, *the simple curve* $\ell^{(x)}(y)$ (*resp.* $\ell_i^{(x)}(y)$ $(i=1, 2)$) *intersects* $\partial A^{(x)}$ *at exactly two points.*

We denote by $[\ell^{(x)}(y)]$ the intersection point different from (x,y).

Let $X=\{X(z); z\in A\}$ be a C^∞ vector field on $A=S^1\times D^2$ satisfying the following conditions:

(i) $X(z)\neq 0$ if $z\in A-(\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)})$; $X(z)=0$ if $z\in\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)}$.

(ii) $X(z)$ $(z\notin\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)},\ z=(x,y)\in S^1\times D^2)$ is transverse to $A^{(x)}=\{x\}\times D^2$ and lies in the positive direction of S^1.

(iii) X is tangent to $\mathscr{F}'$ and hyperbolic at each point of $\hat{\Sigma}^{(-)}$.

(iv) $X\,|\,\partial A$ is tangent to $\partial A=S^1\times\partial D^2$.

The existence of such a C^∞ vector field X is obvious.

Let $\tilde{\pi}:\mathbf{R}\times D^2\to S^1\times D^2$ denote the covering map such that $\tilde{\pi}^{-1}(\{*\}\times D^2)=\mathbf{Z}\times D^2$ and $\tilde{\pi}\,|\,(\mathbf{R}\times\{**\}):\mathbf{R}\to S^1$ is orientation-preserving with respect to the natural orientation of $\mathbf{R}$. Let $\tilde{X}=\{\tilde{X}(\tilde{z});\tilde{z}\in\mathbf{R}\times D^2\}$ be the C^∞ vector field on $\mathbf{R}\times D^2$ such that $\tilde{\pi}_*(\tilde{X}(\tilde{z}))=X(\tilde{\pi}(\tilde{z}))$, and $\tilde{\varphi}(t,\tilde{z})$ denote the integral curve of $\tilde{X}$ with the initial condition $\tilde{\varphi}(0,\tilde{z})=\tilde{z}$ (for $\tilde{z}\in\mathbf{R}\times D^2$).

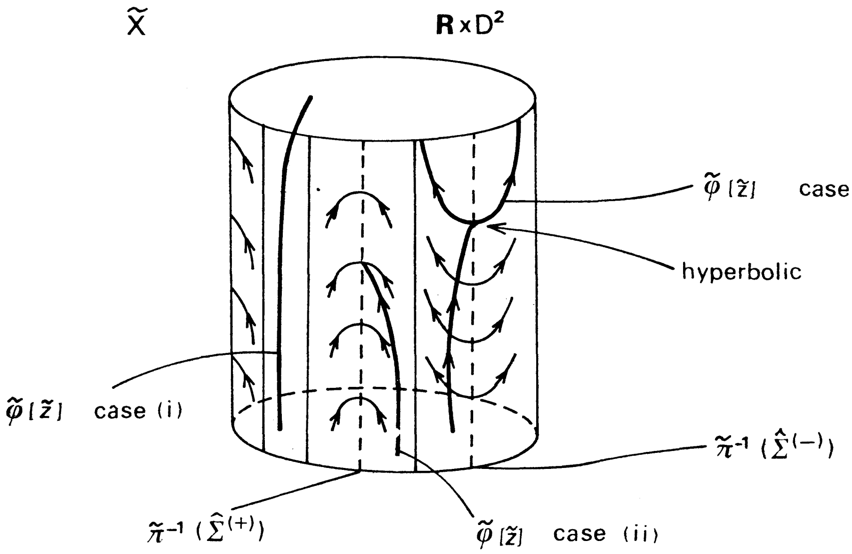

FIG. 6

For $\tilde{z}\in\mathbf{R}\times D^2-\tilde{\pi}^{-1}(\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)})$, we define a subset $\tilde{\varphi}[\tilde{z}]$ of $\mathbf{R}\times D^2$ as follows (Fig. 6):

(i) if $\tilde{\varphi}(t,\tilde{z})$ does not approach to a point of $\tilde{\pi}^{-1}(\hat{\Sigma}^{(+)}\cup\hat{\Sigma}^{(-)})$ for $t\geq 0$, we define

$$\tilde{\varphi}[\tilde{z}]=\{\tilde{\varphi}(t,\tilde{z});\ 0\leq t<\infty\};$$

(ii) if $\tilde{\varphi}(t, \tilde{z})$ approaches to a point of $\tilde{\pi}^{-1}(\hat{\Sigma}^{(+)})$ for $t \geqq 0$, we define

$$\tilde{\varphi}[\tilde{z}]=\{\tilde{\varphi}(t, \tilde{z})\,;\, 0 \leqq t<\infty\} \cup \{\lim_{t \to \infty} \tilde{\varphi}(t, \tilde{z})\};$$

(iii) if $\tilde{\varphi}(t, \tilde{z})$ approaches to a point of $\pi^{-1}(\hat{\Sigma}^{(-)})$ for $t \geqq 0$, we define

$$\tilde{\varphi}[\tilde{z}]=\{\tilde{\varphi}(t, \tilde{z})\,;\, 0 \leqq t<\infty\} \cup \{\lim_{t \to \infty} \tilde{\varphi}(t, \tilde{z})\} \cup \{\tilde{z}'\,;\, \lim_{t \to -\infty} \tilde{\varphi}(t, \tilde{z}')=\lim_{t \to \infty} \tilde{\varphi}(t, \tilde{z})\}.$$

For $s \geqq 0$, $z \in A=S^1 \times D^2$, we define a subset $\Phi_s(z)$ of A (possibly $\Phi_s(z)=\varnothing$) by

$$\Phi_s(z)=\tilde{\pi}((\tilde{\varphi}[\tilde{z}]) \cap (\{\tilde{x}+s\} \times D^2)),$$

where $\tilde{\pi}(\tilde{z})=z$ and $\tilde{z} \in \{\tilde{x}\} \times D^2$. $\Phi_s(z)$ consists of one or two points unless $\Phi_s(z)=\varnothing$.

If $\Phi_s(z) \subset \hat{\Sigma}^{(+)}$ for $z \in \ell^{(x)}(y)$, then, by Lemma 1, (iii), we have

$$\Phi_s(x, y)=\Phi_s([\ell^{(x)}(y)])=\Phi_s(z) \quad (s>0),$$

which implies that y and $[\ell^{(x)}(y)]$ belong to the interior of the same plus Reeb component. Thus, if one of the points $\ell_\lambda^{(x)} \cap \partial A^{(x)}$ is not contained in the interior of a plus Reeb component in $\mathscr{F}''$, then we have

$$\Phi_s(z) \cap \hat{\Sigma}^{(+)}=\varnothing \quad (z \in \ell_\lambda^{(x)},\ s \geqq 0),$$

that is, $\Phi_s(z) \neq \varnothing$ for $z \in \ell_\lambda^{(x)}$, $s \geqq 0$.

The image $\bigcup_{z \in \ell_\lambda^{(x)}} \Phi_s(z)$ of $\ell_\lambda^{(x)}$ with respect to Φ_s bifurcates at $\Phi_{s'}(z)$ if and only if $\Phi_{s'}(z) \subset A^{(\tilde{\pi}(\tilde{x}+s'))} \cap \hat{\Sigma}^{(-)}$ (Fig. 7). Thus, in general, the image of $\ell_\lambda^{(x)}$ with respect to Φ_s consists of a finite number of simple curves of $\{\ell_\lambda^{(\tilde{\pi}(\tilde{x}+s))}\}_{\lambda \in \Lambda(\tilde{\pi}(\tilde{x}+s))}$, because the number of values $s' \in [0, s]$ at which $\Phi_{s'}$ bifurcates are finite (Fig. 7).

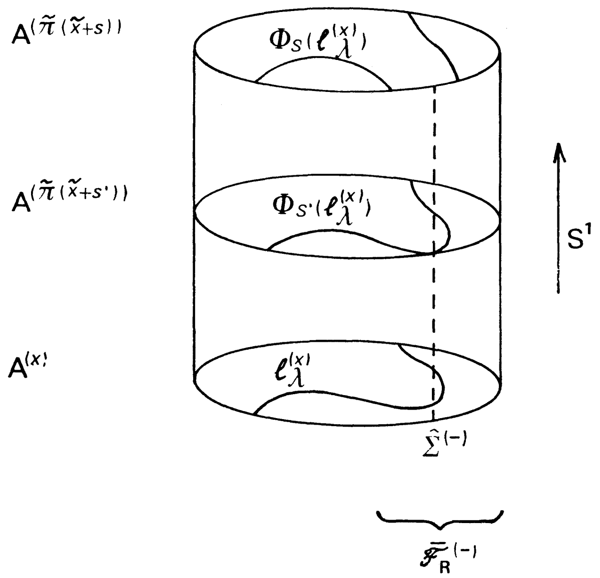

FIG. 7

Lemma 3. — *Let* $\hat{L}$ *be a compact leaf of* $\bar{\mathscr{F}}''$ *of* ∂A *and let* $(x, \hat{y}) \in \hat{L} \cap \partial A^{(x)}$, *then* $[\ell^{(x)}(\hat{y})]$ *is an intersection point of a compact leaf of* $\bar{\mathscr{F}}''$ *and* $\partial A^{(x)}$.

Proof. — Let us consider the case where the longitudinal number a of $\bar{\mathscr{F}}''$ is 1. Thus $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ is a family of integral curves of a C^∞ vector field on A.

First assume that the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_s does not bifurcate for $0 \leqq s \leqq 1$. Then $\Phi_s(z)$ moves continuously for $0 \leqq s \leqq 1$, $z \in \ell^{(x)}(\hat{y})$. Since $\hat{y}$ is a point of a compact leaf $\hat{L}$, we have $\Phi_1(x, \hat{y}) = \{(x, \hat{y})\}$. Thus, by the uniqueness of $\ell^{(x)}(\hat{y})$, the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_1 agrees with itself:

$$\Phi_1([\ell^{(x)}(\hat{y})]) = \{[\ell^{(x)}(\hat{y})]\}.$$

This shows that $[\ell^{(x)}(\hat{y})]$ is contained in a compact leaf of $\bar{\mathscr{F}}''$.

Suppose that the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_s bifurcates at a finite number of values of $0 \leqq s \leqq 1$, and the image of $\ell^{(x)}(\hat{y})$ with respect to Φ_1 is given by

$$\ell^{(x)}(\hat{y}_0) \cup \ell^{(x)}(\hat{y}_1) \cup \ldots \cup \ell^{(x)}(\hat{y}_m),$$

where $\hat{y}_0 = \hat{y}$ and $\ell^{(x)}(\hat{y}_i)$ $(i = 0, 1, \ldots, m)$ are simple curves in $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ such that $[\ell^{(x)}(\hat{y}_i)]$ and $(x, \hat{y}_{i+1})$ belong to the interior of the same minus Reeb component in $\bar{\mathscr{F}}''$ $(i = 0, 1, \ldots, m-1)$ (Fig. 7). Assume that $m \geqq 1$. Then, $[\ell^{(x)}(\hat{y})] = [\ell^{(x)}(\hat{y}_0)]$ should belong to the interior of a minus Reeb component in $\bar{\mathscr{F}}''$. However, according to properties of the minus Reeb component, it is easy to see that $\{[\ell^{(x)}(\hat{y}_m)]\} = \Phi_1([\ell^{(x)}(\hat{y})])$ and $[\ell^{(x)}(\hat{y}_1)]$ belong to different connected components of $A^{(x)} - \ell^{(x)}(\hat{y}_0)$. On the other hand, $[\ell^{(x)}(\hat{y}_1)]$ and $[\ell^{(x)}(\hat{y}_m)]$ should be connected by a connected continuous curve in $A^{(x)}$ oriented by the following order

$$\bar{\ell}_1 \cup \ell^{(x)}(\hat{y}_2) \cup \bar{\ell}_2 \cup \ell^{(x)}(\hat{y}_3) \cup \ldots \cup \bar{\ell}_{m-1} \cup \ell^{(x)}(\hat{y}_m)$$

such that $\bar{\ell}_i$ is contained in the interior of a minus Reeb component $(i = 1, 2, \ldots, m-1)$, and that, if $\bar{\ell}_i$ is contained in the minus Reeb component to which $\ell^{(x)}(\hat{y}_m)$ belongs, the orientation of $\bar{\ell}_i$ is consistent to $\overrightarrow{[\ell^{(x)}(\hat{y}_0)](x, \hat{y}_1)}$. This is a contradiction. Therefore $\ell^{(x)}(\hat{y})$ does not bifurcate for $0 \leqq s \leqq 1$. Thus this lemma is proved in case $a = 1$.

In case $a = 2$, the same arguments hold by considering the double covering of $A = S^1 \times D^2$. Thus Lemma 3 is proved. (See also [9; p. 61].)

Lemma 4. — *Let* $\hat{L}$ *be a compact leaf of* $\bar{\mathscr{F}}''$ *of* $\partial A = S^1 \times \partial D^2$ *and let* L *be the leaf of* $\mathscr{F}'$ *containing* $\hat{L}$. *Then* $L \cap A$ *is compact, and it is an annulus in case* $a = 1$ *and is an annulus or a Möbius band in case* $a = 2$, *where a is the longitudinal number of* $\bar{\mathscr{F}}''$. $L \cap A^{(x)}$ *consists of a simple arc in case* $a = 1$ *and of one or two simple arcs in case* $a = 2$.

Proof. — According to Lemma 3, it is easy to see that there exists a diffeomorphism from $\hat{L} \times I$ or Möbius band $\hat{L} \times I/\mathbf{Z}_2$ to $L \cap A$.

Lemma 5. — *Let* $\bar{\mathscr{F}}_R^{(+)}$ *be a plus Reeb component in* $\bar{\mathscr{F}}''$ *of* $\partial A = S^1 \times \partial D^2$ *and let* $|\bar{\mathscr{F}}_R^{(+)}|$ *denote the underlying submanifold of* $\bar{\mathscr{F}}_R^{(+)}$ *in* ∂A. *Denote by* $\hat{L}$, $\hat{L}'$ *the compact leaves of*

$\overline{\mathscr{F}}_{\mathrm{R}}^{(+)}$: $\partial|\overline{\mathscr{F}}_{\mathrm{R}}^{(+)}|=\hat{\mathrm{L}}\cup\hat{\mathrm{L}}'$, *where it may happen that* $\hat{\mathrm{L}}=\hat{\mathrm{L}}'$ *in case the longitudinal number of* $\mathscr{F}''$ *is* 2. *Then, for* $\{c\}\in\hat{\mathrm{L}}\cap\partial\mathrm{A}^{(x)}$, *we have*

$$[\ell^{(x)}(c)]\in\hat{\mathrm{L}}'\cap\partial\mathrm{A}^{(x)}.$$

Proof. — Let $\sigma\in|\overline{\mathscr{F}}_{\mathrm{R}}^{(+)}|\cap\partial\mathrm{A}^{(x)}\cap\hat{\Sigma}^{(+)}$. We denote $\hat{\mathrm{L}}'\cap\partial\mathrm{A}^{(x)}$ by c'. We fix an orientation on $\partial\mathrm{A}^{(x)}$ so that the oriented arcs $\widehat{c'\sigma}$, $\widehat{\sigma c}$ have the orientation compatible with that of $\partial\mathrm{A}^{(x)}$, where $\widehat{c'\sigma}\cap\widehat{\sigma c}=\{\sigma\}$ (Fig. 8).

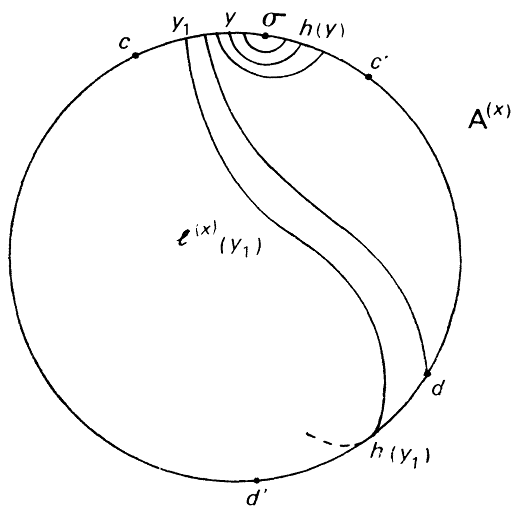

Fig. 8

Let $h:\widehat{\sigma c}\to\partial\mathrm{A}^{(x)}$ be the map (not necessary continuous) defined by

$$h(y)=[\ell^{(x)}(y)] \qquad (y\in\widehat{\sigma c}).$$

Then, by Lemma 1, (iii), there exists a neighborhood U_0 of σ in $\partial\mathrm{A}^{(x)}$ such that h is continuous on $\mathrm{U}_0\cap\mathrm{Int}\,\widehat{\sigma c}$.

Assume that there exists a point $y_1\in\widehat{\sigma c}$ such that h is continuous on $\widehat{\sigma y_1}-\{y_1\}$ and is not continuous at y_1. This is equivalent to that $\ell^{(x)}(y)$ intersects $\partial\mathrm{A}^{(x)}$ transversely at $h(y)=[\ell^{(x)}(y)]$ for $y\in\widehat{\sigma y_1}-\{y_1\}$ and $\ell^{(x)}(y_1)$ is tangent to $\partial\mathrm{A}^{(x)}$ at $h(y_1)=[\ell^{(x)}(y_1)]$ (Fig. 8, 9). Thus we have $h(y_1)\in\hat{\Sigma}^{(-)}$.

Denote by $\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}$ the minus Reeb component such that $h(y_1)\in|\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}|$, where $|\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}|$ is the underlying submanifold of $\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}$. Let d, d' denote the boundary points $\partial(|\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}|\cap\mathrm{A}^{(x)})$ such that oriented arcs $\widehat{d'h(y_1)}$ $\widehat{h(y_1)d}$ contained in $|\overline{\mathscr{F}}_{\mathrm{R}}^{(-)}|\cap\mathrm{A}^{(x)}$ have the orientation compatible with that of $\partial\mathrm{A}^{(x)}$ (Fig. 8).

Suppose that $\ell^{(x)}(y_1)$ is tangent to $\partial\mathrm{A}^{(x)}$ at $h(y_1)$ in the inverse direction of $\partial\mathrm{A}^{(x)}$, then, it is easy to see that

$$h|\widehat{\sigma y_1}:\ \widehat{\sigma y_1}\to\widehat{h(y_1)\sigma}$$

is an onto homeomorphism (Fig. 8). Thus $h^{-1}(d)$ exists in Int $\overset{\frown}{\sigma y_1}$ which should be contained in a compact leaf of $\mathscr{F}''$ by Lemma 3. This is a contradiction. Therefore $\ell^{(x)}(y_1)$ is tangent to $\partial A^{(x)}$ at $h(y_1)$ in the direction of $\partial A^{(x)}$ (Fig. 9).

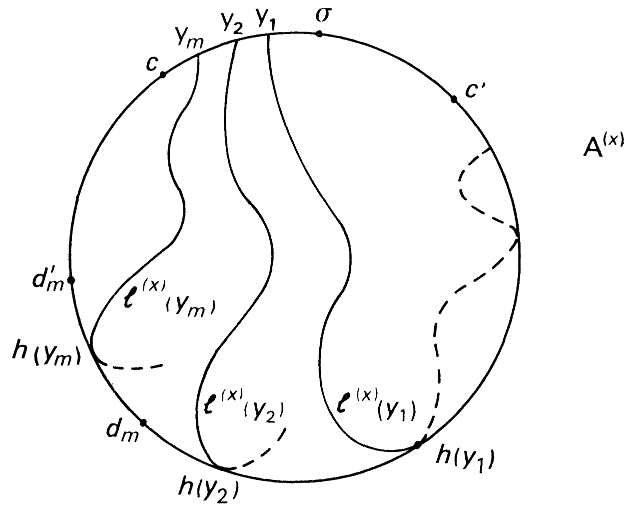

FIG. 9

Thus there exists a neighborhood U_1 of y_1 in $\partial A^{(x)}$ such that h is continuous on $U_1 \cap \mathrm{Int}\,\overset{\frown}{y_1 c}$. If h is not continuous at a point of $\overset{\frown}{y_1 c}$, then there exists $y_2 \in \mathrm{Int}\,\overset{\frown}{y_1 c}$ such that h is continuous on $\mathrm{Int}\,\overset{\frown}{y_1 y_2}$ and is not continuous at y_2. By the same argument used above, $\ell^{(x)}(y_2)$ is tangent to $\partial A^{(x)}$ at $h(y_2)$ in the direction of $\partial A^{(x)}$ (Fig. 9).

Since the number of minus Reeb components is finite, by repeating this process, there are a finite number of points $y_1, y_2, \ldots, y_m$ of Int $\overset{\frown}{\sigma c}$ situated in this order such that h is continuous on $\overset{\frown}{\sigma c} - \{\sigma\} - \bigcup_{i=1}^{m} y_i$ and discontinuous at y_i $(i = 1, 2, \ldots, m)$ and that $\ell^{(x)}(y_i)$ is tangent to $\partial A^{(x)}$ at $h(y_i) \in \hat{\Sigma}^{(-)}$ in the direction of $\partial A^{(x)}$ (Fig. 9). Suppose that $h(y_m)$ is contained in a minus Reeb component $'\mathscr{F}_{\mathrm{R}}^{(-)}$ and let $\overset{\frown}{d'_m d_m}$ be the arc $|'\mathscr{F}_{\mathrm{R}}^{(-)}| \cap A^{(x)}$ having the orientation compatible with that of $\partial A^{(x)}$:

$$h(y_m) \in \overset{\frown}{d'_m d_m}.$$

Then h maps $\overset{\frown}{y_m c}$ into $\overset{\frown}{c h(y_m)} - \{c\}$. If $h(\overset{\frown}{y_m c}) \subset \overset{\frown}{d'_m h(y_m)} - \{d'_m\}$, then $\overset{\frown}{d'_m h(y_m)} - \{d'_m\}$ must contain the point $h(c)$ of a compact leaf of $\mathscr{F}''$ by Lemma 3. This is a contradiction. Further, if $h(\mathrm{Int}\,\overset{\frown}{y_m c}) \supset \overset{\frown}{d'_m h(y_m)}$, then $\mathrm{Int}\,\overset{\frown}{y_m c}$ must contain the point $h^{-1}(d'_m)$ of a compact leaf of $\mathscr{F}''$ by Lemma 3. This is also a contradiction. Thus $h(c) = d'_m$ holds. This implies that c and d'_m lie on a compact leaf L of $\mathscr{F}'$ by Lemma 4.

However, since c is a point of a compact leaf of the boundary of a plus Reeb component, L has a contracting holonomy in the negative direction of S^1 in this side, and, on the other hand, since d'_m is a point of a compact leaf of the boundary of a minus Reeb component, L has a contracting holonomy in the direction of S^1 in the same side. This is a contradiction. Therefore there exists no discontinuous point of h on $\widehat{\sigma c}$ and

$$h\,|\,(\widehat{\sigma c}-\{\sigma\}) : \widehat{\sigma c}-\{\sigma\} \to \widehat{h(c)\sigma}-\{\sigma\}$$

is a C^∞ diffeomorphism. The point $h(c)$ must belong to a compact leaf of $\bar{\mathscr{F}}''$ by Lemma 3. Thus, making use of the same argument as above, we have

$$h(c) = c'.$$

This completes the proof of Lemma 5.

Proposition 3. — (i) *Let* $\hat{L}$ *be a compact leaf of* $\bar{\mathscr{F}}'$ *of* $\partial(S^1\times D^2)$ *and let* L *be the leaf of* $\mathscr{F}'$ *containing* $\hat{L}$. *Then* L *is compact and an annulus in case* $a=1$ *and an annulus or a Möbius band in case* $a=2$ *such that* $\partial L = L\cap\partial(S^1\times D^2)$ *consists of two compact leaves of* $\bar{\mathscr{F}}'$ *in case* $a=1$ *and of one or two compact leaves of* $\bar{\mathscr{F}}'$ *in case* $a=2$, *where a is the longitudinal number of* $\bar{\mathscr{F}}'$.

(ii) *For a plus Reeb component* $\bar{\mathscr{F}}_R^{(+)}$ *in* $\bar{\mathscr{F}}'$, *there exists a plus half Reeb component* $\mathscr{F}_{R/2}^{(+)}$ *in* $\mathscr{F}'$ *such that* $\bar{\mathscr{F}}_R^{(+)}$ *is the restriction of* $\mathscr{F}_{R/2}^{(+)}$ *to* $|\bar{\mathscr{F}}_R^{(+)}|$, *where it may happen that the compact leaf of* $\mathscr{F}_{R/2}^{(+)}$ *forms a Möbius band in* $\bar{\mathscr{F}}'$ *identified by a free* $\mathbf{Z}_2$ *action in case* $a=2$. *Let* $A=\bigcup_x A^{(x)}$ *be as in Section* 2, *then* $\{A^{(x)}\cap L';\ L'\in\mathscr{F}_{R/2}^{(+)}\}$ *consists of concentric half circles.*

Proof. — There is a natural isomorphism from $\bar{\mathscr{F}}'$ to $\bar{\mathscr{F}}''$ of ∂A and the compact leaves corresponding by this isomorphism are the boundary of an annulus which is the restriction of a leaf of $\mathscr{F}'$ to $S^1\times D^2-\mathrm{Int}\,A$. Thus the first part of (i) is an immediate consequence of Lemma 4. For a compact leaf $\hat{L}$ of the boundary of $|\bar{\mathscr{F}}_R^{(+)}|$, there exists a compact leaf L containing $\hat{L}$ as above. According to Lemma 5, ∂L consists of the two compact leaves of $\bar{\mathscr{F}}_R^{(+)}$ in case $a=1$ and of one or two compact leaves in case $a=2$. Thus the second part of (i) is proved.

Now we prove (ii). Let L be the compact leaf of $\mathscr{F}'$ containing a compact leaf of $\partial|\bar{\mathscr{F}}_R^{(+)}|$. Assume L is annular. Let R denote the closure of a connected component of $S^1\times D^2-L$ which contains $\mathrm{Int}|\bar{\mathscr{F}}_R^{(+)}|$. Since, as was shown in the proof of Lemma 5, $R\cap A^{(x)}$ consists of concentric half circles, $\mathscr{F}'|R$ is a plus half Reeb component. Thus Proposition 3 is proved. In case L is a Möbius band, the same arguments hold by considering the double covering of $S^1\times D^2$.

4. TS components

First we prove the following lemma.

Lemma 6. — *Let* $\hat{L}$ *be a compact leaf of* $\bar{\mathscr{F}}'$ *which is a boundary of a minus Reeb component* $\bar{\mathscr{F}}_R^{(-)}$ *or a slope component* $\bar{\mathscr{F}}_S$ *and let* L *be the compact leaf of* $\mathscr{F}'$ *containing* $\hat{L}$ (Proposition 3, (i)). *Let* B *denote the closure of a connected component of* $S^1 \times \partial D^2 - \partial L$ *which contains* $\mathrm{Int}|\bar{\mathscr{F}}_R^{(-)}|$ *or* $\mathrm{Int}|\bar{\mathscr{F}}_S|$, *where* $|\bar{\mathscr{F}}_R^{(-)}|$, $|\bar{\mathscr{F}}_S|$ *denote underlying submanifolds. Then* $\hat{L}' = \partial L - \hat{L}$ *is a compact leaf of* $\bar{\mathscr{F}}'$ *which is a boundary of a minus Reeb component or a slope component contained in* B, *unless* $\partial L = \hat{L}$.

Proof. — First assume that the longitudinal number a is 1. Suppose that there exists a family of compact leaves $\{\hat{L}_i'\}$ of $\bar{\mathscr{F}}'|B$ which accumulates to $\hat{L}'$. Then, by Proposition 3, (i), we have a family of compact leaves $\{L_i\}$ of $\mathscr{F}'$ such that

$$\partial L_i = \hat{L}_i \cup \hat{L}_i' = L_i \cap (S^1 \times \partial D^2).$$

Thus $\{\hat{L}_i\}$ accumulates to $\hat{L}$ which contradicts the assumption on $\hat{L}$. Thus $\hat{L}'$ is a boundary of a plus or minus Reeb component, or of a slope component. But $\hat{L}'$ cannot be a boundary of a plus Reeb component by Proposition 3, (ii). In case $a=2$, the same arguments hold by considering the double covering of $S^1 \times D^2$. Note that it may happen that $\partial L = \hat{L}$ in this case. Thus this lemma is proved.

In the following $\bar{\mathscr{F}}_i$ denotes a minus Reeb component or a slope component contained in $\bar{\mathscr{F}}'$. $\bar{\mathscr{F}}_1$ and $\bar{\mathscr{F}}_2$ are called to be *connected by a compact leaf* L, denoted by $\bar{\mathscr{F}}_1 \underset{L}{\sim} \bar{\mathscr{F}}_2$, if there exists a compact annular leaf (resp. a Möbius band in case $a=2$) L of $\mathscr{F}'$ with $\partial L = \bar{L} \cup \bar{L}'$ (resp. $\partial L = \bar{L}$) such that $\bar{L} \subset |\bar{\mathscr{F}}_1|$, $\bar{L}' \subset |\bar{\mathscr{F}}_2|$ (resp. $\bar{L} \subset |\bar{\mathscr{F}}_1|$, $\bar{L} \subset |\bar{\mathscr{F}}_2|$) and that $\mathrm{Int}|\bar{\mathscr{F}}_1|$ and $\mathrm{Int}|\bar{\mathscr{F}}_2|$ are contained in the same connected component of $S^1 \times \partial D^2 - \bar{L} - \bar{L}'$ (resp. $S^1 \times \partial D^2 - \bar{L}$) (Fig. 10).

Further, $\bar{\mathscr{F}}_0$ and $\bar{\mathscr{F}}_m$ are called to be *connected* if there exists a sequence $\bar{\mathscr{F}}_1, \bar{\mathscr{F}}_2, \ldots, \bar{\mathscr{F}}_{m-1}$ such that

$$\bar{\mathscr{F}}_i \underset{L_i}{\sim} \bar{\mathscr{F}}_{i+1} \quad (i=0, 1, \ldots, m-1)$$

for some compact leaves L_i $(i=0, 1, \ldots, m-1)$ of $\mathscr{F}'$.

By Lemma 6, q copies of the minus Reeb components in $\bar{\mathscr{F}}'$ are divided into connected components.

Lemma 7. — *Let* $\mathscr{C} = \{\bar{\mathscr{F}}_j^{(-)}; j=1, 2, \ldots, m\}$ *be a connected component of* q *copies of the minus Reeb component in* $\bar{\mathscr{F}}'$. *Then there exist two slope components* $\bar{\mathscr{F}}_S^{(1)}$, $\bar{\mathscr{F}}_S^{(2)}$ *in* $\bar{\mathscr{F}}^{(1)}$ *such that* $\{\bar{\mathscr{F}}_j^{(-)}; j=1, 2, \ldots, m\} \cup \{\bar{\mathscr{F}}_S^{(1)}, \bar{\mathscr{F}}_S^{(2)}\}$ *is a connected component of the set of* q *copies of the minus Reeb component and slope components in* $\bar{\mathscr{F}}'$. *Further,* $\bar{\mathscr{F}}_S^{(1)}$ *and* $\bar{\mathscr{F}}_S^{(2)}$ *are connected by a compact leaf.*

Proof. — First we assume that the longitudinal number a is 1. Let

$$\{\bar{\mathscr{F}}_j^{(-)}; j=1, 2, \ldots, m, m+1, \ldots, m'\} \cup \{\bar{\mathscr{F}}_S^{(\delta)}; \delta \in \Delta\}$$

be a connected component of the set of q copies of the minus Reeb component and slope components in $\bar{\mathscr{F}}'$ containing $\mathscr{C}$. Let L be an arbitrary compact leaf of $\mathscr{F}'$, then

$L\cap A^{(x)}$ is a simple curve in $A^{(x)}$ and $L\cap A^{(x)}$ divides $A^{(x)}$ into two connected components, say $A_1^{(x)}$ and $A_2^{(x)}$. Since the Euler number $\chi(A_i^{(x)})$ is equal to 1 $(i=1, 2)$, we have

$$A_i^{(x)}\cap\hat{\Sigma}^{(+)}\neq\varnothing \qquad (i=1, 2).$$

Since $\bar{\mathscr{F}}'$ contains only a finite number of plus Reeb components, this observation shows that Δ is a finite set, say $\Delta=\{\delta_i; i=1, 2, \ldots, r\}$.

Let L_k $(k=0, 1, \ldots, r+m'-1)$ be compact leaves which connect

$$\{\bar{\mathscr{F}}_j^{(-)}; j=1, 2, \ldots, m'\}\cup\{\bar{\mathscr{F}}_S^{(\delta_i)}; i=1, 2, \ldots, r\}$$

and let $Q^{(x)}$ denote the closure of a connected component of $A^{(x)}-\bigcup_{k=0}^{r+m'-1} L_k$ intersecting $\mathrm{Int}\,|\bar{\mathscr{F}}_j^{(-)}|$ and $\mathrm{Int}\,|\bar{\mathscr{F}}_S^{(\delta_i)}|$. Denote by $\hat{Q}^{(x)}$ the double of $Q^{(x)}$ obtained by pasting $Q^{(x)}\cap\partial A^{(x)}$. Then the Euler number $\chi(\hat{Q}^{(x)})$ is equal to $2-(m'+r)$. On the other hand, a C^∞ vector field on $\hat{Q}^{(x)}$ introduced naturally by $\{\ell_\lambda^{(x)}\}_{\lambda\in A(x)}$ is tangent to $\partial\hat{Q}^{(x)}$ and has exactly m' singular points of index -1. Thus we have $r=2$.

Further compact leaves having contracting holonomy in the negative direction of S^1 are only contained in $\bar{\mathscr{F}}_S^{(\delta_1)}$ and $\bar{\mathscr{F}}_S^{(\delta_2)}$. In order to be connected by a compact leaf, compact leaves in $\bar{\mathscr{F}}_j^{(-)}$ or in $\bar{\mathscr{F}}_S^{(\delta_i)}$ should have the same holonomy. Therefore $\bar{\mathscr{F}}_S^{(\delta_1)}$ and $\bar{\mathscr{F}}_S^{(\delta_2)}$ should be connected by a compact leaf which implies that $m=m'$.

In case $a=2$, the same arguments hold by considering the double covering of $S^1\times D^2$. Thus this lemma is proved.

Let $\mathscr{C}$ be as in Lemma 7, then, by Lemma 7, there exists slope components $\bar{\mathscr{F}}_S^{(1)}$, $\bar{\mathscr{F}}_S^{(2)}$ and compact leaves L_k $(k=0, 1, 2, \ldots, m+1)$ of $\mathscr{F}'$ such that

$$\bar{\mathscr{F}}_S^{(1)} \underset{L_1}{\frown} \bar{\mathscr{F}}_1^{(-)}, \quad \bar{\mathscr{F}}_i^{(-)} \underset{L_{i+1}}{\frown} \bar{\mathscr{F}}_{i+1}^{(-)} \quad (i=1, 2, \ldots, m-1),$$

$$\bar{\mathscr{F}}_m^{(-)} \underset{L_{m+1}}{\frown} \bar{\mathscr{F}}_S^{(2)}, \quad \bar{\mathscr{F}}_S^{(2)} \underset{L_0}{\frown} \bar{\mathscr{F}}_S^{(1)}.$$

Let $Q(\mathscr{C})$ or simply Q denote the closure of a connected component of $S^1\times D^2-\bigcup_{i=0}^{m+1} L_i$ containing $\mathrm{Int}\,|\bar{\mathscr{F}}_j^{(-)}|$ $(j=1, 2, \ldots, m)$. The C^∞ foliation $\mathscr{F}'|Q$ of codimension one which is the restriction of $\mathscr{F}'$ to $Q(\mathscr{C})$ is called a TS *component of type m* with respect to $\mathscr{C}$ and denoted by TS_m. We denote by $|TS_m|$ the underlying submanifold Q of TS_m.

Proposition 4. — There exists a C^∞ *foliation* $\mathscr{F}'$ *of codimension one of* $S^1\times D^2$ *transverse to the plus Reeb component such that* $\mathscr{F}'$ *contains a* TS *component of type m.*

Proof. — Figure 5 and Figure 10 show the existence of a TS component of type 1. Similary a TS component of type m exists for any $m\geqq 1$.

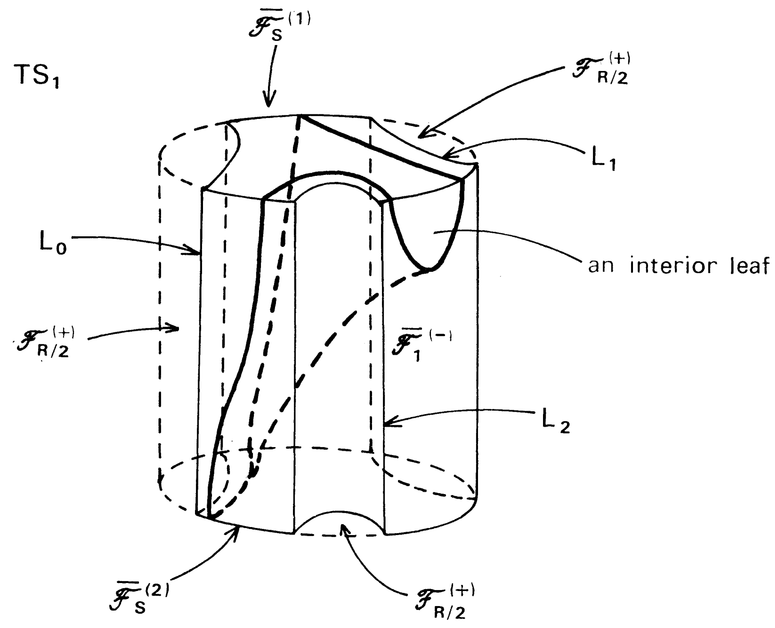

FIG. 10

The following proposition is an immediate consequence of Lemma 7.

Proposition 5. — For a minus Reeb component $\overline{\mathscr{F}}_R^{(-)}$ *in* $\overline{\mathscr{F}}'$, *there exists a* TS *component* TS_m *of type m such that* $|TS_m| \supset |\overline{\mathscr{F}}_R^{(-)}|$.

Proposition 6. — The TS *component* TS_m *of type m with respect to*

$$\mathscr{C} = \{\overline{\mathscr{F}}_j^{(-)}; j = 1, 2, \ldots, m\}$$

has the following properties:

(i) *The underlying submanifold* $|TS_m|$ *of* $S^1 \times D^2$ *is a compact connected 3-dimensional* C^∞ *manifold with corner, where the corner consists of the boundaries of compact leaves* L_k $(k = 0, 1, \ldots, m+1)$ *and is* C^∞ *diffeomorphic to* $S^1 \times D^2$ *by straightening the corner. The set* $|TS_m| \cap (S^1 \times \partial D^2)$ *consists of minus Reeb components* $\overline{\mathscr{F}}_j^{(-)}$ $(j = 1, 2, \ldots, m)$ *and slope components* $\overline{\mathscr{F}}_S^{(i)}$ $(i = 1, 2)$.

(ii) *The intersection* $|TS_m| \cap (\{x\} \times D^2)$ $(x \in S^1)$ *is a polygon with* $2(m+2)$ *vertices (resp. a polygon with* $4m+4$ *vertices or two disjoint polygons with* $2(m+2)$ *vertices) if the longitudinal number a is* 1 *(resp.* 2*)*.

(iii) *The compact leaves in* TS_m *are exactly* L_k $(k = 0, 1, \ldots, m+1)$. *They are annular in case* $a = 1$ *and one of them may be a Möbius band in case* $|TS_m| \cap (\{x\} \times D^2)$ *is a polygon with* $4m+4$ *vertices. The compact leaf* L_k *has a contracting holonomy in the positive (resp. negative) direction of* S^1 *if* $k = 1, 2, \ldots, m+1$ *(resp.* $k = 0$*)*.

(iv) *Every non-compact leaf* L *of* TS_m *meets* Int $|\overline{\mathscr{F}}_j^{(-)}|$ $(j=1, 2, \ldots, m)$ *and* Int $|\overline{\mathscr{F}}_S^{(1)}|$, Int $|\overline{\mathscr{F}}_S^{(2)}|$.

(v) *Let* H *denote the subset of* $\mathbf{R}^2$ *defined by*

$$\mathrm{H}=[0, 2m+1]\times\mathbf{R}$$
$$-\left\{(x,y)\,;y-c_k>\frac{1}{(x-k)(k+1-x)},\, k<x<k+1,\, k=1, 3, \ldots, 2m-1\right\},$$

where c_k *is a constant*; *then every non-compact leaf in* TS_m *is* C^∞ *diffeomorphic to* H (Fig. 11).

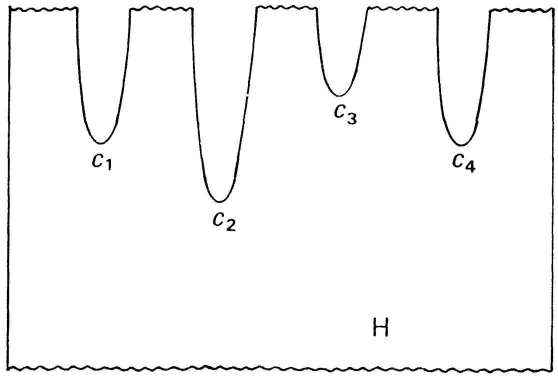

Fig. 11

Proof. — Properties (i), (ii), (iii) are obvious. So we prove (iv), (v) here. A non-compact leaf L of $\mathscr{F}'\,|\,\mathrm{Q}$ meets $\mathrm{A}^{(x)}$ for some $x\in\mathrm{S}^1$. Then $\mathrm{L}\cap\mathrm{A}^{(x)}$ contains a simple curve $\ell_\lambda^{(x)}$ of $\{\ell_\lambda^{(x)}\}_{\lambda\in\Lambda(x)}$. Let (x,y) be an end point of $\ell_\lambda^{(x)}$, then (x,y) belongs to the interior of one of $|\overline{\mathscr{F}}_j^{(-)}|$ $(j=1, 2, \ldots, m)$ or $|\overline{\mathscr{F}}_S^{(i)}|$ $(i=1, 2)$ by the identification of $\partial\mathrm{A}$ and $\mathrm{S}^1\times\partial\mathrm{D}^2$. Assume that $(x,y)\in|\overline{\mathscr{F}}_j^{(-)}|$, then it is easy to see that $\mathrm{L}\cap|\overline{\mathscr{F}}_j^{(-)}|$ contains a point (x',y') and (x'',y'') which lie near to L_j and L_{j+1} respectively. Since $\ell^{(x')}(y')\subset\mathrm{L}$, $\ell^{(x'')}(y'')\subset\mathrm{L}$, we have $\mathrm{L}\cap|\overline{\mathscr{F}}_{j-1}^{(-)}|\neq\varnothing$, $\mathrm{L}\cap|\overline{\mathscr{F}}_{j+1}^{(-)}|\neq\varnothing$. By iterating this process for $|\overline{\mathscr{F}}_j^{(-)}|$ and $|\overline{\mathscr{F}}_S^{(i)}|$, (iv) is proved.

Let $\overline{\mathrm{L}}'$ be a non-compact leaf of $\overline{\mathscr{F}}_S^{(1)}$ and let $\overline{\mathrm{L}}'\subset\mathrm{L}'$ $(\mathrm{L}'\in\mathscr{F}')$. For $(x,y)\in\overline{\mathrm{L}}'$ $(x\in\mathrm{S}^1, y\in\partial\mathrm{D}^2)$, making use of the identification $\mathrm{S}^1\times\mathrm{D}^2=\mathrm{A}$, we consider a simple curve $\ell^{(x)}(y)$ in $\mathrm{A}^{(x)}$. If (x,y) is near to L_1, then $[\ell^{(x)}(y)]$ is a point of $|\overline{\mathscr{F}}_1^{(-)}|$ because $\ell^{(x)}(y)$ lies near to L_1. Let (x,y_1) denote a point of $|\overline{\mathscr{F}}_1^{(-)}|$ which is symmetric to $[\ell^{(x)}(y)]$ with respect to $\hat{\Sigma}^{(-)}\cap|\overline{\mathscr{F}}_1^{(-)}|$, then it is obvious that $(x,y_1)\in\mathrm{L}'$. Thus $\ell^{(x)}(y_1)\subset\mathrm{L}'$. Therefore, in general, for a point $(x,y)\in\overline{\mathrm{L}}'$, there exists a sequence $\ell^{(x)}(y_0)$, $\ell^{(x)}(y_1)$, $\ldots$, $\ell^{(x)}(y_s)$ of simple curves in $\mathrm{A}^{(x)}$ $(s\leqq m)$ such that

1) $(x,y_0)=(x,y)\in|\overline{\mathscr{F}}_S^{(1)}|$, $[\ell^{(x)}(y_s)]\in|\overline{\mathscr{F}}_S^{(2)}|$,
2) $\ell^{(x)}(y_k)\subset\mathrm{L}'$ $(k=0, 1, \ldots, s)$,
3) $[\ell^{(x)}(y_k)]$ and (x,y_{k+1}) are points of $|\overline{\mathscr{F}}_{i_k}^{(-)}|$ which are symmetric with respect to $\hat{\Sigma}^{(-)}\cap|\overline{\mathscr{F}}_{i_k}^{(-)}|$.

Further if (x, y) is sufficiently near to L_0, then $s = 1$ and the above sequence consists of a simple curve $\ell^{(x)}(y)$ such that $[\ell^{(x)}(y)] \in |\mathscr{F}_S^{(2)}|$.

By these observations, we can define a C^∞ diffeomorphism f from H onto $L' \cap A$ such that f maps $H \cap ([0, 2m+1] \times \{u\})$ onto $\bigcup_{k=0}^{s} \ell^{(x)}(y_k)$. Obviously L′ is diffeomorphic to H. Thus (v) is proved.

Let TS_m be a TS component of type m with respect to $\mathscr{C}$ as above, $|\widetilde{TS_m}|$ the universal covering of $|TS_m|$ and $\tilde{\pi} : |\widetilde{TS_m}| \to |TS_m|$ the projection. For the natural projection $p_1 : |TS_m| \to S^1$ which is the restriction of the projection to the first factor $S^1 \times D^2 \to S^1$, there exist the covering map $\tilde{\pi}'$ and the natural projection $\tilde{p}$ satisfying the following commutative diagram:

$$\begin{array}{ccc} |\widetilde{TS_m}| & \xrightarrow{\tilde{\pi}} & |TS_m| \\ \Big\downarrow{\scriptstyle \tilde{p}} & & \Big\downarrow{\scriptstyle p_1} \\ \mathbf{R} & \xrightarrow{\tilde{\pi}'} & S^1. \end{array}$$

Denote by $\tilde{\mathscr{F}}'$ the C^∞ foliation of codimension one of $|\widetilde{TS_m}|$ defined by $\{\tilde{\pi}^{-1}(L') ; L' \in TS_m\}$. Let $h^{(1)}$ (resp. $h^{(2)}$) be a C^∞ diffeomorphism from the open interval $]0, 1[$ onto a connected component of $\tilde{\pi}^{-1}(\mathrm{Int}\, |\mathscr{F}_S^{(1)}| \cap (\{x\} \times \partial D^2))$ (resp. $\tilde{\pi}^{-1}(\mathrm{Int}\, |\mathscr{F}_S^{(2)}| \cap (\{x\} \times \partial D^2)))$, then $\{h^{(1)}(t)\}$ $(0 < t < 1)$ is an index set for leaves of $\tilde{\mathscr{F}}' | \tilde{\pi}^{-1}(|\mathscr{F}_S^{(1)}|)$. The leaf $\tilde{L}_t$ of $\tilde{\mathscr{F}}'$ containing $h^{(1)}(t)$ intersects $\tilde{\pi}^{-1}(\hat{\Sigma}^{(-)} \cap |\mathscr{F}_j^{(-)}|)$ (resp. $\tilde{\pi}^{-1}(|\mathscr{F}_S^{(2)}| \cap h^{(2)}(]0, 1[)))$ at one point, say $f_j(t)$ (resp. $h^{(2)}(\bar{f}(t))$) for $0 < t < 1$. Then it is easy to see that

$$\tilde{p} \circ f_j : \;]0, 1[\to \mathbf{R} \qquad (j = 1, 2, \ldots, m)$$

$$\bar{f} : \;]0, 1[\to]0, 1[$$

are C^∞ diffeomorphism. The maps f_j $(j = 1, 2, \ldots, m)$ and $\bar{f}$ are called *lag functions* for TS_m. The lag functions depend on the choice of $\{x\}$ and $h^{(i)}$ $(i = 1, 2)$.

Now we define a standard TS component of type m. Let P_{2m+4} denote the regular polygon of $2m + 4$ vertices and let $\hat{Q}(m)$ be a compact connected orientable 3-dimensional C^∞ manifold with corner obtained from $P_{2m+4} \times I$ by identifying $P_{2m+4} \times \{0\}$ and $P_{2m+4} \times \{1\}$ after twisting of b times, where b is an integer. The boundary $\partial\hat{Q}(m)$ consists of $2m+4$ annuli, say $(S^1 \times I)_i$ $(i = 0, 1, \ldots, 2m+3)$, whose boundaries are corners of $\hat{Q}(m)$.

By the turbulization of $\mathrm{Int}\, \hat{Q}(m)$ in neighborhoods of $m+1$ annuli $(S^1 \times I)_{2i}$ $(i = 1, \ldots, m+1)$ along $(S^1 \times I)_{2i}$ in the direction of S^1 for $i = 1, 2, \ldots, m+1$ and in the negative direction of S^1 for $i = 0$, a C^∞ foliation $\hat{\mathscr{F}}(m)$ of codimension one of $\hat{Q}(m)$ is constructed. Compact leaves in $\hat{\mathscr{F}}(m)$ are $(S^1 \times I)_{2i}$ $(i = 0, 1, \ldots, m+1)$ and

$\hat{\mathscr{F}}(m) | (S^1 \times I)_{2i+1}$ is the minus Reeb component if $i = 1, 2, \ldots, m$ and the slope component if $i = 0, m+1$. The foliation $\hat{\mathscr{F}}(m)$ is called the *standard* TS *component of type m* and denoted by $\widehat{TS}_m$. Clearly the lag functions f_j of $\widehat{TS}_m$ satisfy

$$f_1 = f_2 = \ldots = f_m.$$

5. Classification theorems for foliations transverse to the Reeb component

Theorem 1. — Let $\mathscr{F}'$ be a C^∞ foliation of codimension one transverse to the plus Reeb component $\mathscr{F}_R^{(+)}$ of $S^1 \times D^2$: $\mathscr{F}' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$. Then the following conditions hold:

(i) *Let $[L_{comp}] = a\alpha + b\beta$ be the homology class of $H_1(S^1 \times \partial D^2; \mathbf{Z})$ represented by a compact leaf L_{comp} in $\bar{\mathscr{F}}' = \mathscr{F}' | (S^1 \times \partial D^2)$, where $\alpha = [S^1 \times \{*\}]$ with the given orientation, $\beta = [\{*\} \times \partial D^2]$ and $a \geqq 0$. Then we have $a = 1$ or 2 (the number a is the longitudinal number of $\bar{\mathscr{F}}'$).*

(ii) *$\mathscr{F}'$ consists of p copies of the plus half Reeb component, s_m copies of the* TS *component of type m for $m = 1, 2, \ldots, u$, and a finite number of foliated* I*-bundles over $S^1 \times I$ in case $a = 1$ and over $S^1 \times I$ or the Möbius band in case $a = 2$ such that*

$$p - \sum_{m=1}^{u} m s_m = 2 \quad \textit{if} \quad a = 1,$$

$$p - \sum_{m=1}^{u} m s_m = 1 \quad \textit{if} \quad a = 2,$$

and that the foliated I*-bundles are trivial* I*-bundles in case $a = 1$.*

Proof. — (i) and a part of (ii) concerning plus and minus half Reeb components and TS components are direct consequences of Proposition 2 and Proposition 5.

Let $(\mathscr{F}_{R/2}^{(+)})_i$ $(i = 1, 2, \ldots, p)$ and $(TS_m)_j$ $(j = 1, 2, \ldots, s_m)$ be the plus Reeb components and the TS components of type m in $\mathscr{F}'$ respectively, and let

$$M = S^1 \times D^2 - \left(\bigcup_{i=1}^{p} \operatorname{Int} |(\mathscr{F}_{R/2}^{(+)})_i|\right) - \left(\bigcup_{m=1}^{u} \left(\bigcup_{j=1}^{s_m} \operatorname{Int} |(TS_m)_j|\right)\right).$$

Let C be a connected component of M, then, by Proposition 3, (ii) and Proposition 5, the family of simple curves formed by the intersection of $A^{(x)}$ and leaves of $\mathscr{F}' | C$ are transverse to $\partial A^{(x)}$. This implies that $\mathscr{F}' | C$ is a foliated I-bundle isomorphic to $\bar{\mathscr{F}}'_C \times I$ or $\tilde{\bar{\mathscr{F}}}'_C \times I/\mathbf{Z}_2$, where $\bar{\mathscr{F}}'_C$ denotes the restriction of $\bar{\mathscr{F}}' = \mathscr{F}' | \partial A$ to one of the connected components of $C \cap (S^1 \times \partial D^2)$ and $\tilde{\bar{\mathscr{F}}}'_C$ denotes its double covering. Since $\bar{\mathscr{F}}'_C$ is a foliated I-bundle over S^1 such that this bundle is trivial if $a = 1$ and is trivial or a Möbius band if $a = 2$. Thus this theorem is proved.

In order to state the classification theorem for $t_1(\mathscr{F}_R^{(+)})$, we introduce the concept of TS diagram. TS *diagrams* consist of finite number of smooth simple arcs $\hat{\ell}$

$(i=1, 2, \ldots, r)$ in the 2-disc D^2 and symbols $+$, $-$, $\times$ and $||$ on $2r$ arc intervals of ∂D^2 divided by $\partial\hat{\ell}_i$ $(i=1, 2, \ldots, r)$ satisfying the following conditions (Fig. 12, 14):

(i) $\hat{\ell}_i$ $(i=1, 2, \ldots, r)$ are mutually disjoint smooth simple arcs in D^2 intersecting ∂D^2 transversely such that

$$\hat{\ell}_i \cap \partial D^2 = \partial \hat{\ell}_i \quad (i=1, 2, \ldots, r).$$

(ii) Let N_i $(i=1, 2, \ldots, r+1)$ denote the closures of connected components of $D^2 - \bigcup_{i=1}^{r} \hat{\ell}_i$. Then the symbols are given as follows:

(a) if $N_i \cap \partial D^2$ consists of one connected component, then the symbol $(+)$ is given on this arc interval;

(b) if $N_i \cap \partial D^2$ consists of two connected components, then the symbol $||$ is given on each arc interval of $N_i \cap \partial D^2$;

(c) if the number k of connected components of $N_i \cap \partial D^2$ is $\geqq 3$, then the symbol $(-)$ is given for $k-2$ arc intervals of $N_i \cap \partial D^2$ and the symbol $\times$ is given for the rest two arc intervals. Further two arc intervals with symbol $\times$ are contained in a connected component of

$$\partial D^2 - [(k-2) \text{ arc intervals with symbol } (-)].$$

(iii) Let p and q denote the numbers of arc intervals having the symbol $(+)$ and $(-)$ respectively. Then $p-q=2$.

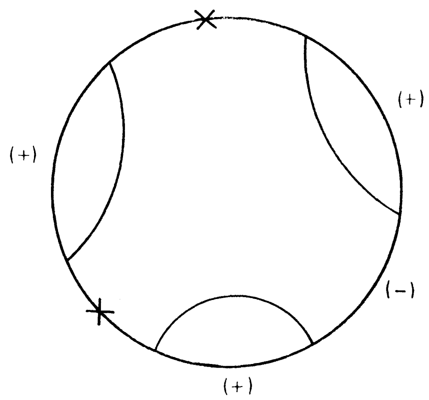

FIG. 12

The TS diagram of Figure 12 corresponds to $\mathscr{F}'$ illustrated in Figure 5.

Two TS diagrams are *isomorphic* if and only if there exists a C^∞ diffeomorphism of D^2 preserving simple arcs $\{\hat{\ell}_i\}$ and symbols. Let $A = \bigcup_x A^{(x)}$ be as in Section 2, then the TS diagram illustrates $A^{(x)} \cap L'$ $(L' \in \mathscr{F}')$ (see Fig. 5).

The following theorem is an immediate consequence of Theorem 1 and the definition of TS diagrams.

Theorem 2. — Let $\mathscr{F}' \in t_1(\mathrm{S}^1 \times \mathrm{D}^2, \mathscr{F}_{\mathrm{R}}^{(+)})$. *Then a pair* (a, b) *of integers and an isomorphism class of* TS *diagrams satisfying the following conditions correspond uniquely to* $\mathscr{F}'$.

(i) $a = 1$ *or* 2. *The homology class* $[\mathrm{L}_{\mathrm{comp}}]$ *of* $\mathrm{H}_1(\mathrm{S}^1 \times \partial\mathrm{D}^2; \mathbf{Z})$ *represented by a compact leaf* $\mathrm{L}_{\mathrm{comp}}$ *in* $\overline{\mathscr{F}}' = \mathscr{F}' | \mathrm{S}^1 \times \partial\mathrm{D}^2$ *is* $a\alpha + b\beta$. *In case* $a = 1$, $\mathscr{F}'$ *is transversely orientable. In case* $a = 2$, $\mathscr{F}'$ *is transversely non-orientable. The* TS *diagram should be invariant under the action of order* 2 *if* $a = 2$.

(ii) *Arc intervals of* $\partial\mathrm{D}^2$ *with symbols* $+$, $-$, $\times$ *and* $||$ *represent the plus, the minus Reeb components, the slope components and foliated* I*-bundles over* S^1 *(i.e. union of slope components and compact leaves) contained in* $\overline{\mathscr{F}}'$ *respectively.*

(iii) *In case* $a = 1$ *(resp.* $a = 2$*),* N_i *(resp. a pair of* N_i *which is invariant under the action of order* 2*) represents the plus half Reeb component or the* TS *component of type m or a foliated* I*-bundle over* $\mathrm{S}^1 \times \mathrm{I}$ *(resp.* $\mathrm{S}^1 \times \mathrm{I}$ *or the Möbius band) if* $\mathrm{N}_i \cap \partial\mathrm{D}^2$ *consists of one or* $m + 2$ *or* 2 *connected components respectively.*

(iv) *Each simple arc* $\hat{\ell}_i$ *represents a compact leaf diffeomorphic to* $\mathrm{S}^1 \times \mathrm{I}$ *in case* $a = 1$ *and to* $\mathrm{S}^1 \times \mathrm{I}$ *or the Möbius band in case* $a = 2$.

In the following we consider the topological classification for $t_1(\mathrm{S}^1 \times \mathrm{D}^2, \mathscr{F}_{\mathrm{R}}^{(+)})$.

Lemma 8. — Any TS *component* TS_m *of type m is topologically isomorphic to the standard* TS *component* $\widehat{\mathrm{TS}_m}$ *of type m if the longitudinal number is* 1.

Proof. — Let $\mathrm{L}_0, \mathrm{L}_1, \ldots, \mathrm{L}_{m+1}$ (resp. $\hat{\mathrm{L}}_0, \hat{\mathrm{L}}_1, \ldots, \hat{\mathrm{L}}_{m+1}$) be compact leaves of TS_m (resp. $\widehat{\mathrm{TS}_m}$). We fix C^∞ diffeomorphisms

$$\eta_i : \mathrm{S}^1 \times \mathrm{I} \to \mathrm{L}_i \text{ (resp. } \hat{\eta}_i : \mathrm{S}^1 \times \mathrm{I} \to \hat{\mathrm{L}}_i) \qquad (i = 0, 1, \ldots, m+1).$$

Then it is easy to see that there exists a collar c (resp. $\hat{c}$) of $\bigcup_{i=0}^{m+1} \mathrm{L}_i$ (resp. $\bigcup_{i=0}^{m+1} \hat{\mathrm{L}}_i$) in $|\mathrm{TS}_m|$ (resp. $|\widehat{\mathrm{TS}_m}|$):

$$c : \Big(\bigcup_{i=0}^{m+1} \mathrm{L}_i\Big) \times [0, 1] \to |\mathrm{TS}_m|, \quad c(z, 0) = z \quad \Big(z \in \bigcup_{i=0}^{m+1} \mathrm{L}_i\Big)$$

$$\text{(resp. } \hat{c} : \Big(\bigcup_{i=0}^{m+1} \hat{\mathrm{L}}_i\Big) \times [0, 1] \to |\widehat{\mathrm{TS}_m}|, \quad \hat{c}(z, 0) = z \quad \Big(z \in \bigcup_{i=0}^{m+1} \hat{\mathrm{L}}_i\Big))$$

such that

(a) $c(\bigcup_{i=0}^{m+1} \mathrm{L}_i \times \{1\})$ (resp. $\hat{c}(\bigcup_{i=0}^{m+1} \hat{\mathrm{L}}_i \times \{1\})$) is transverse to TS_m (resp. $\widehat{\mathrm{TS}_m}$),

(b) $c | (\{z\} \times [0, 1])$ (resp. $\hat{c} | (\{z\} \times [0, 1])$) is transverse to TS_m (resp. $\widehat{\mathrm{TS}_m}$),

(c) $c(\eta_i(\{x\} \times \mathrm{I}), t)$ (resp. $\hat{c}(\hat{\eta}_i(\{x\} \times \mathrm{I}), t)$) is contained in a leaf of TS_m (resp. $\widehat{\mathrm{TS}_m}$), where $t \in [0, 1]$.

Since, by Proposition 6, the restriction of TS_m (resp. $\widehat{TS_m}$) to

$$Q' = |TS_m| - c\Big(\big(\bigcup_{i=0}^{m+1} L_i\big) \times [0, 1[\Big) \quad (\text{resp. } \hat{Q}' = |\widehat{TS_m}| - \hat{c}\Big(\big(\bigcup_{i=0}^{m+1} \hat{L}_i\big) \times [0, 1[\Big))$$

is a C^∞ foliation of codimension one whose leaves are the regular polygon P_{2m+4} of $2m+4$ vertices, it follows from the Reeb stability theorem [5] that $TS_m | Q'$ (resp. $\widehat{TS_m} | \hat{Q}'$) is a product foliation. Thus, as is easily verified, there exists a C^∞ diffeomorphism

$$h_{Q'}: \ Q' \to \hat{Q}'$$

such that

(i) $h_{Q'}$ preserves the foliations $TS_m | Q'$ and $\widehat{TS_m} | \hat{Q}'$,
(ii) $h_{Q'}(L_i \times \{1\}) = \hat{L}_i \times \{1\}$.

Further, letting $h_{Q'}(z, 1) = (z', 1)$ $(z \in L_i)$, we define surjective homeomorphisms

$$h_i: \ c(L_i \times [0, 1]) \to \hat{c}(\hat{L}_i \times [0, 1]), \qquad i = 0, 1, \ldots, m+1,$$

by

$$h_i(z, t) = (z', \xi_z(t)),$$

where $\xi_z: [0, 1] \to [0, 1]$ is a surjective homeomorphism depending continuously on z. By a suitable choice of ξ_z, the homeomorphisms h_i preserves the foliations $TS_m | c(L_i \times [0, 1])$ and $\widehat{TS_m} | \hat{c}(\hat{L}_i \times [0, 1])$.

Then the homeomorphism

$$h: \ |TS_m| \to |\widehat{TS_m}|$$

defined by $h | Q' = h_{Q'}$ and $h | c(L_i \times I) = h_i$ $(i = 0, 1, \ldots, m+1)$ is a surjective homeomorphism preserving foliations TS_m and $\widehat{TS_m}$. Thus this lemma is proved.

The following theorem is an immediate consequence of Theorem 2 and Lemma 8.

Theorem 3. — *Let* $\mathscr{F}_1'$, $\mathscr{F}_2' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$. *Suppose that* $\mathscr{F}_1'$ *and* $\mathscr{F}_2'$ *satisfy the following conditions*:

(i) $\mathscr{F}_1'$ *and* $\mathscr{F}_2'$ *have the same longitudinal number*;
(ii) *there exists an isomorphism between their* TS *diagrams, say* f_0;
(iii) *for foliated* I-*bundles in* $\mathscr{F}_1'$ *and in* $\mathscr{F}_2'$ *corresponding by* f_0, *there exists an isomorphism between them compatible with* f_0.

Then $\mathscr{F}_1'$ *and* $\mathscr{F}_2'$ *are topologically isomorphic.*

Let us consider $\mathscr{F}_1' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ consisting of 3 copies of the half Reeb component and one TS component of type 1 (Fig. 12). We represent $\mathscr{F}_1'$ by illustrating $\mathscr{F}_1' \cap A^{(x)}$ and $\mathscr{F}_1' | (S^1 \times \partial D^2)$ by dotted curves in Figure 13.

Let $\mathscr{F}_1''$ be an element of $t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ represented by real curves illustrating $\mathscr{F}_1'' \cap A^{(x)}$ and $\mathscr{F}_1'' | (S^1 \times \partial D^2)$ in Figure 13. The foliation $\mathscr{F}_1''$ consists of 2 copies of the half Reeb component and a foliated I-bundle over $S^1 \times I$, and is transverse to $\mathscr{F}_1'$:

$$\mathscr{F}_1'' \pitchfork \mathscr{F}_1'.$$

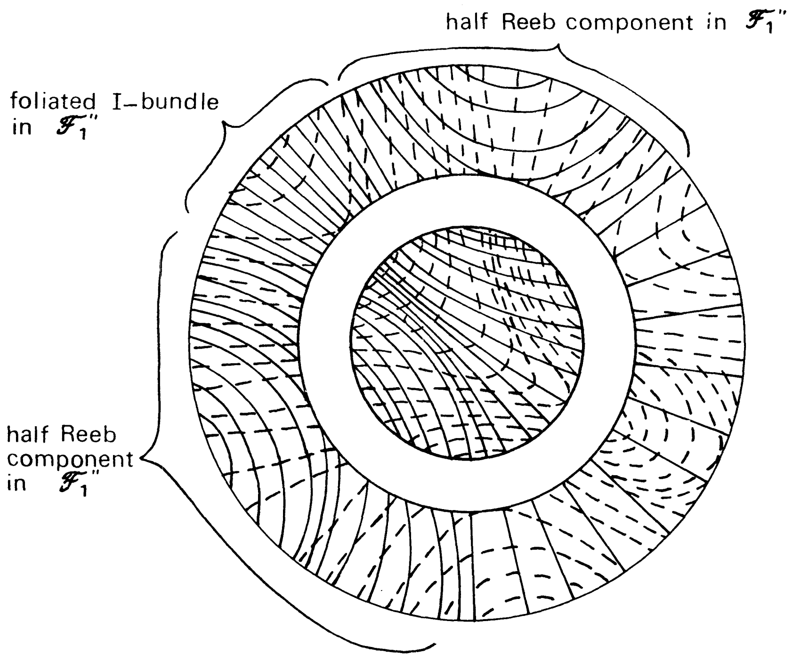

FIG. 13

For $\mathscr{F}' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ consisting of m copies of the half Reeb component and one TS component of type $m-1$, we can construct $\mathscr{F}'' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ such that $\mathscr{F}'' \pitchfork \mathscr{F}'$ by similar methods. It seems to us that, for any $\mathscr{F}' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$, there exists always $\mathscr{F}'' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ such that $\mathscr{F}' \pitchfork \mathscr{F}''$. However, in general, $\mathscr{F}''$ is not unique, because $\mathscr{F}_1''$ above is also transverse to $\mathscr{F}_1''' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ consisting of 2 copies of the half Reeb component (Fig. 4).

6. Foliations transverse to the Reeb foliation of S^3

Let $S^3 = (S_1^1 \times D_1^2) \bigcup_h (D_2^2 \times S_2^1)$ be the decomposition of the 3-sphere into the union of two solid tori, where $h : S_1^1 \times \partial D_1^2 \to \partial D_2^2 \times S_2^1$ is given by $h(x, y) = (x, y)$.

Let $\mathscr{F}_R$ denote the Reeb foliation of S^3. We fix orientations on S_1^1 and S_2^1 so that $'\mathscr{F}_R = \mathscr{F}_R | (S_1^1 \times D_1^2)$ and $''\mathscr{F}_R = \mathscr{F}_R | (D_2^2 \times S_2^1)$ are the plus Reeb components.

Let $\mathscr{F}'$ be a C^∞ foliation of codimension one of S^3 transverse to $\mathscr{F}_R$. Then we have

$$\mathscr{F}' | (S_1^1 \times D_1^2),\ \mathscr{F}' | (D_2^2 \times S_2^1) \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)}).$$

Since $\mathscr{F}'$ is transversely orientable, their longitudinal numbers are both 1. The restrictions $\mathscr{F}' | (S_1^1 \times D_1^2)$ and $\mathscr{F}' | (D_2^2 \times S_2^1)$ must be isomorphic by h on their boundaries. Thus it is obvious that the homology class $[L_{\text{comp}}]$ of $H_1(S_1^1 \times \partial D_1^2; \mathbf{Z})$ represented by a compact leaf L_{comp} of $\tilde{\mathscr{F}}' = \mathscr{F}' \mid \partial(S^1_1 \times D^2_1)$ having the orientation compatible with the orientations of S^1_1 and S^1_2 is $\alpha + \beta$, where $\alpha = [S^1_1 \times \{*\}]$, $\beta = [\{**\} \times S^1_2]$ $(\{*\}, \{**\} \in \partial D^2)$ with given orientations (Theorem 1, (i)).

Conversely from two elements $\mathscr{F}'_1, \mathscr{F}'_2 \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ with the longitudinal number 1 such that $\mathscr{F}'_1 \mid \partial(S^1 \times D^2)$ is isomorphic to $\mathscr{F}'_2 \mid \partial(S^1 \times D^2)$ by the map $(x, y) \to (y, x)$, we obtain an $\mathscr{F}' \in t_1(S^3, \mathscr{F}_R)$ by identifying their boundaries. Thus the following theorem holds:

Theorem 4. — Let $\mathscr{F}'$ be a C^∞ foliation of codimension one transverse to the Reeb foliation $\mathscr{F}_R$ of S^3: $\mathscr{F}' \in t_1(S^3, \mathscr{F}_R)$. *Then $\mathscr{F}'$ is obtained from two foliations $\hat{\mathscr{F}}'_1, \hat{\mathscr{F}}'_2 \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ with longitudinal number* 1 *such that $\hat{\mathscr{F}}'_1 \mid \partial(S^1 \times D^2)$ is isomorphic to $\hat{\mathscr{F}}'_2 \mid \partial(S^1 \times D^2)$ by the map $(x, y) \to (y, x)$ identifying their boundaries.*

Let D^2_i $(i = 0, 1, \ldots, m+1)$ be 2-discs imbedded disjointly in the 2-sphere S^2 and let $C = (S^2 - \bigcup_{i=0}^{m+1} \operatorname{Int} D^2_i) \times S^1$. We fix an orientation on S^1. A *full* TS *component of type* $(m; r)$, denoted by $\overline{TS}_{(m;\, r)}$, is a C^∞ foliation $\mathscr{F}_C$ of codimension one of C having the following properties:

(i) Compact leaves of $\mathscr{F}_C$ are $\partial D^2_i \times S^1$ $(i = 0, 1, \ldots, m+1)$.

(ii) Interior leaves of $\mathscr{F}_C$ are transverse to $\{x\} \times S^1$ $(x \in S^2 - \bigcup_{i=0}^{m+1} D^2_i)$.

(iii) $\mathscr{F}_C$ has a contracting holonomy in the negative direction of S^1 on $\partial D^2_i \times S^1$ $(i = 0, 1, \ldots, r)$ and in the positive direction of S^1 on $\partial D^2_i \times S^1$ $(i = r+1, r+2, \ldots, m+1)$, where $0 \leq r \leq m$. We note that a full TS component contains compact leaves having contracting holonomy in different directions of S^1.

Example A. — Let us consider two copies of

$$\hat{\mathscr{F}}' \in t_1(S^1 \times D^2, \mathscr{F}_R^{(+)}) \quad \text{with} \quad [L_{comp}] = \alpha + \beta.$$

We may suppose that h gives an isomorphism $\mathscr{F}_R^{(+)} \mid (S^1 \times \partial D^2) \to \mathscr{F}_R^{(+)} \mid (S^1 \times \partial D^2)$. Thus the C^∞ foliation $\mathscr{F}'$ obtained from two copies of $\hat{\mathscr{F}}'$ identifying their boundaries by h is an element of $t_1(S^3, \mathscr{F}_R)$. If $\hat{\mathscr{F}}'$ consists of p copies of the plus half Reeb component, s_m copies of the TS component of type m $(m = 1, 2, \ldots, u)$, then $\mathscr{F}'$ consists of p copies of the Reeb component and s_m copies of the full TS component of type $(m; 0)$.

Example B (Koichi Yano). — Let $\hat{\mathscr{F}}'_1, \hat{\mathscr{F}}'_2$ be elements of $t_1(S^1 \times D^2, \mathscr{F}_R^{(+)})$ with $[L_{comp}] = \alpha + \beta$ such that their TS diagrams are given by Figure 14, *(a)*, *(b)* respectively and that h gives an isomorphism $\hat{\mathscr{F}}'_1 \mid (S^1 \times \partial D^2) \to \hat{\mathscr{F}}'_2 \mid (S^1 \times \partial D^2)$ (thus, the symbol || in their TS diagrams represent slope components). The C^∞ foliation $\mathscr{F}'$ obtained from $\hat{\mathscr{F}}'_1$ and $\hat{\mathscr{F}}'_2$ identifying their boundaries by h is an element of $t_1(S^3, \mathscr{F}_R)$ consisting of 7 copies of the Reeb component, a full TS component of type (1; 1) and a full TS component of type (3; 1).

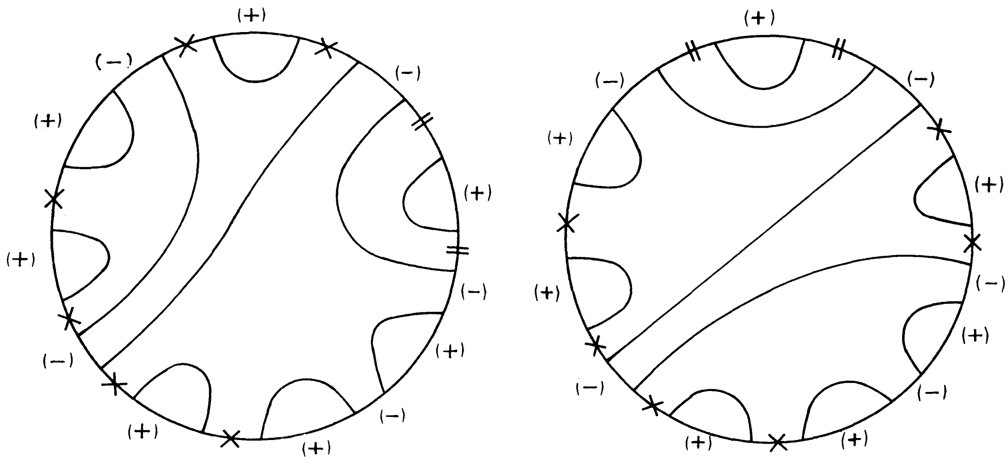

Fig. 14

Theorem 5. — *Let* $\mathscr{F}' \in t_1(S^3, \mathscr{F}_R)$, *then* $\mathscr{F}'$ *consists of a finite number of Reeb components, full* TS *components and foliated* I*-bundles over* $S^1 \times S^1$. *Furthermore, let* ℓ_R, etc. (*resp.* ℓ_{TS}) *denote a closed curve in a Reeb component* (*resp. a full* TS *component*) *in* $\mathscr{F}'$ *homotopic to the longitude and transverse to* $\mathscr{F}'$, *then* ℓ_R *and* ℓ_{TS} *are both unknotted and the linking number of* ℓ_R *and* ℓ'_R *is* ± 1.

Proof. — Let $\hat{\mathscr{F}}'_1$, $\hat{\mathscr{F}}'_2$ be as in Theorem 4: $\mathscr{F}' = \hat{\mathscr{F}}'_1 \cup \hat{\mathscr{F}}'_2$. First assume that the number of compact leaves contained in $\hat{\mathscr{F}}'_1$ (thus also in $\hat{\mathscr{F}}'_2$) is finite. This is equivalent to the assumption that foliated I-bundles in $\hat{\mathscr{F}}'_1$ and in $\hat{\mathscr{F}}'_2$ contain only a finite number of compact leaves. For a compact leaf L_1 in $\hat{\mathscr{F}}'_1$, $L_1 \cap (S^1 \times \partial D^2)$ consists of two compact leaves in $\hat{\mathscr{F}}'_2 \mid (S^1 \times \partial D^2)$, say $\bar{L}_1$, $\bar{\bar{L}}_1$. Then there exists a unique compact leaf in $\hat{\mathscr{F}}'_2$, say L_2 (resp. L'_2), which contains $\bar{L}_1$ (resp. $\bar{\bar{L}}_1$). Thus the set of compact leaves in $\hat{\mathscr{F}}'_1$ and $\hat{\mathscr{F}}'_2$ forms a finite number of compact leaves in $\mathscr{F}'$ which are diffeomorphic to $S^1 \times S^1$. By Proposition 2, (iii), the numbers of plus half Reeb components in $\hat{\mathscr{F}}'_1$ and in $\hat{\mathscr{F}}'_2$ are the same and they form the same number of Reeb components in $\mathscr{F}'$. Further, as is easily verified, TS components and foliated I-bundles in $\hat{\mathscr{F}}'_1$ and in $\hat{\mathscr{F}}'_2$ form a finite number of full TS components and foliated I-bundles over $S^1 \times S^1$ with finite compact leaves in $\mathscr{F}'$ (Examples A, B). Since $[L_{comp}] = \alpha + \beta$, it follows from the construction as above that ℓ_R and ℓ_{TS} are unknotted and the linking number of ℓ_R and ℓ'_R is ± 1. Thus the theorem is proved in this case.

Now suppose that $\hat{\mathscr{F}}'_1$ (thus also $\hat{\mathscr{F}}'_2$) contains foliated I-bundles with infinite numbers of compact leaves. We denote A, $A^{(x)}$ and $\{\ell_\lambda^{(x)}\}_{\lambda \in \Lambda(x)}$ of Section 3 for $\hat{\mathscr{F}}'_1$ (resp. $\hat{\mathscr{F}}'_2$) by A_1, $A_1^{(x)}$ and $\{'\ell_\lambda^{(x)}\}_{\lambda \in \Lambda'(x)}$ (resp. A_2, $A_2^{(x)}$ and $\{''\ell_\lambda^{(x)}\}_{\lambda \in \Lambda''(x)}$). Since $[L_{comp}] = \alpha + \beta$, the diffeomorphism h induces naturally a diffeomorphism

$$\bar{h} : \partial A_1^{(x_1)} \to \partial A_2^{(x_2)}$$

such that $\bar{h}(y) \in h(\bar{L})$ for $y \in \bar{L}$. Let $\hat{S}$ be the 2-sphere obtained from $A_1^{(x_1)}$ and $A_2^{(x_2)}$ identifying their boundaries by $\bar{h}$. Then we may consider that

$$\mathscr{L} = \{'\ell_\lambda^{(x_1)}\}_{\lambda \in \Lambda'(x_1)} \cup \{''\ell_\lambda^{(x_2)}\}_{\lambda \in \Lambda''(x_2)}$$

is a family of integral curves of a C^∞ vector field Y on $\hat{S}$ which is non-singular except $\partial A_1^{(x_1)} \cap (\hat{\Sigma}^{(+)} \cup \hat{\Sigma}^{(-)})$. Here $\mathscr{L}$ represents the leaves of $\mathscr{F}'$. If there exists a sequence of an infinite number of compact leaves $L_1^{(1)}, L_2^{(2)}, L_1^{(3)}, L_2^{(4)}, \ldots, L_1^{(2s-1)}, L_2^{(2s)}, \ldots$ such that

$$L_1^{(2s-1)} \in \hat{\mathscr{F}}_1', \quad L_2^{(2s)} \in \hat{\mathscr{F}}_2',$$

$$L_1^{(2s-1)} \cap L_2^{(2s)} \neq \varnothing, \quad L_2^{(2s)} \cap L_1^{(2s+1)} \neq \varnothing \quad (s = 1, 2, 3, \ldots),$$

and that the union $(\bigcup_{s=1}^{\infty} L_1^{(2s-1)}) \cup (\bigcup_{s=1}^{\infty} L_2^{(2s)})$ is a non-compact leaf of $\mathscr{F}'$, then, letting $L_1^{(2s-1)} \cap A_1^{(x_1)} = \hat{\ell}^{(2s-1)}$, $L_2^{(2s)} \cap A_2^{(x_2)} = \hat{\ell}^{(2s)}$, the union $\hat{\ell} = \bigcup_{s=1}^{\infty} \hat{\ell}^{(s)}$ is a non-compact integral curve of Y. By the Poincaré-Bendixson theorem, the ω-limit set of $\hat{\ell}$ is a circle, say ω. We denote by L_ω the leaf of $\mathscr{F}'$ containing ω and by $\bar{L}_\omega$ a connected component of $L_\omega \cap \partial A_1$. Then, it is obvious that $\bar{L}_\omega$ cannot be a non-compact leaf (i.e. an interior leaf of a slope component) of $\hat{\mathscr{F}}_1' | \partial A_1$. This implies that L_ω is a compact leaf of $\mathscr{F}'$ diffeomorphic to $S^1 \times S^1$, say $L_\omega = \omega \times S^1$. Let $\bar{L}^{(2s-1)}$ (resp. $\bar{L}^{(2s)}$) denote a connected component of $L_1^{(2s-1)} \cap \partial A_1$ (resp. $L_2^{(2s)} \cap \partial A_2$), then there exists a sequence of compact leaves $\bar{L}^{(s_1)}, \bar{L}^{(s_2)}, \ldots, \bar{L}^{(s_q)}, \ldots$ of $\hat{\mathscr{F}}_1' | \partial A_1 = \hat{\mathscr{F}}_2' | \partial A_2$ such that this sequence converges to $\bar{L}_\omega$. If, for a given integer q', there always exists an integer $q > q'$ such that a slope component of $\hat{\mathscr{F}}_1' | \partial A_1$ (maybe contained in a foliated I-bundle of $\hat{\mathscr{F}}_1' | \partial A_1$) situated between $\bar{L}^{(s_q)}$ and $\bar{L}^{(s_{q+1})}$, then L_ω has contracting holonomy in both $[\omega]$, $[\{*\} \times S^1]$ $(* \in \omega)$. Since $\mathscr{F}'$ is of class C^∞, this contradicts to the Kopell's theorem [2]. Therefore, for a compact leaf $\bar{L}$ of the boundary of a slope component of $\hat{\mathscr{F}}_1' | \partial A_1$, the saturation of $\bar{L}$ in $\mathscr{F}'$ is a compact leaf of $\mathscr{F}'$. Thus, as is easily verified, for a slope component $\bar{\mathscr{F}}_S'$ in $\hat{\mathscr{F}}_1' | \partial A_1$, one of the following occurs:

(i) The saturation of $|\bar{\mathscr{F}}_S'|$ in $\mathscr{F}'$ contains a TS component of $\hat{\mathscr{F}}_1'$ or $\hat{\mathscr{F}}_2'$.

(ii) The saturation of $|\bar{\mathscr{F}}_S'|$ in $\mathscr{F}'$ forms a foliated I-bundle over $S^1 \times S^1$ with two compact leaves.

Further, let $\bar{\mathscr{F}}$ be a subset of $\hat{\mathscr{F}}_1' | \partial A_1$ which satisfies the following:

(a) $\bar{\mathscr{F}}$ consists of compact leaves;
(b) the union of the leaves in $\bar{\mathscr{F}}$ is diffeomorphic to $S^1 \times I$;
(c) the boundary of $|\bar{\mathscr{F}}|$ consists of two compact leaves which belong to the boundaries of slope components.

Then the saturation of $|\bar{\mathscr{F}}|$ in $\mathscr{F}'$ is a foliated I-bundle over $S^1 \times S^1$.

It is obvious that the union of two foliated I-bundles over $S^1 \times S^1$ with a common compact leaf forms a foliated I-bundle over $S^1 \times S^1$. Moreover, let F be the saturation in $\mathscr{F}'$ of a sufficiently small subset of ∂A_1 such that the boundary of F consists of two compact leaves, then we can show that $\mathscr{F}' | F$ is a foliated I-bundle over $S^1 \times S^1$ by constructing a vector field transverse to $\mathscr{F}' | F$.

By the observation above, there exist foliated I-bundles $\mathscr{F}_1', \mathscr{F}_2', \ldots, \mathscr{F}_u'$ over $S^1 \times S^1$ in $\mathscr{F}'$ such that $S^3 - \bigcup_{i=1}^{u} \operatorname{Int} |\mathscr{F}_i'|$ contains only a finite number of compact leaves. So the theorem reduces to the case above. Thus the theorem is proved.

Remark. — The properties of the full TS component in $\mathscr{F}'$ depend mainly on the lag functions of two TS components contained in it. For example, see [10] for the Godbillon-Vey classes of TS components.

7. Foliations of codimension one of S^3 admitting no transverse foliation of codimension one

Let k be a fibred knot in S^3. That is, letting $N(k)$ be a tubular neighborhood of k, $E = S^3 - \operatorname{Int} N(k)$ is a C^∞ fibre bundle over S^1, $\pi : E \to S^1$, with fibre $G - \operatorname{Int} D^2$ where G is a closed surface of genus g and D^2 is a 2-disc imbedded in G. For example, the intersection k of $S^3 = \{(z_1, z_2) ; |z_1|^2 + |z_2|^2 = 1\}$ and $\{(z_1, z_2) ; z_1^p + z_2^q = 0\}$ is a fibred knot [3].

Let $\mathscr{F}$ be a C^∞ foliation of codimension one of S^3 constructed by the spinnable structure having the fibred knot as the axis ([8]). That is, by choosing suitable orientations on S^1 of $S^1 \times D^2$ and on S^1 of $E \to S^1$, $\mathscr{F}$ is the union of the plus Reeb component $\mathscr{F}_R^{(+)}$ of $N(k) = S^1 \times D^2$ and the C^∞ foliation $\mathscr{F}_\pi^{(+)}$ of Proposition 2: $\mathscr{F} = \mathscr{F}_R^{(+)} \cup \mathscr{F}_\pi^{(+)}$. Then we have the following theorem.

Theorem 6. — Let $\mathscr{F}$ be a C^∞ foliation of codimension one of S^3 defined from a fibred knot k as above, where the genus g is $\geqq 1$. Then $\mathscr{F}$ does not admit any transverse C^∞ foliation of codimension one:

$$t_1(S^3, \mathscr{F}) = \varnothing.$$

Proof. — Suppose there exists $\mathscr{F}' \in t_1(S^3, \mathscr{F})$. Let α and β denote the homology classes of $H_1(\partial E; \mathbf{Z})$ represented by a meridian of $N(k)$ and $\partial(G - \operatorname{Int} D^2)$ with orientations chosen as above respectively. Then, by Proposition 2, a compact leaf L_{comp} of $\mathscr{F}' | \partial E$ represents a homology class $\bar{a}\alpha + \bar{b}\beta$ $(\bar{a} \geqq 0)$, and $\mathscr{F}' | \partial E$ contains p copies of the plus Reeb component and q copies of the minus Reeb component, where

$$\bar{a}(p - q) = 2(1 - 2g).$$

On the other hand, also by Proposition 2, (iii), since $\mathscr{F}'$ is transversely orientable, $\mathscr{F}' | \partial N(k)$ contains p' copies of the plus Reeb component and q' copies of the minus Reeb component with

$$(*) \qquad p' - q' = 2$$

and $\bar{b}=\pm 1$. Since

$$p-q=\pm(p'-q'),$$

it follows that

$$(**) \qquad p-q=-2, \quad \bar{a}=2g-1.$$

$\mathscr{F}'$ has a Reeb component ([4]), say $\mathscr{F}'_R$, $|\mathscr{F}'_R|\subset S^3$. If $|\mathscr{F}'_R|\subset N(k)$, then obviously $|\mathscr{F}'_R|\subset \operatorname{Int} N(k)$. Thus $\mathscr{F}\,|\,|\mathscr{F}'_R|$ consists of compact leaves. On the other hand, since $\mathscr{F}\,|\,|\mathscr{F}'_R|\in t_1(\mathscr{F}'_R)$, $\mathscr{F}\,|\,|\mathscr{F}'_R|$ must contain non-compact leaves by Theorem 1. This contradiction implies that

$$|\mathscr{F}'_R|\not\subset N(k).$$

Similarly we have

$$|\mathscr{F}'_R|\not\subset E.$$

Since $\mathscr{F}'\,|\,N(k)\in t_1(N(k),\mathscr{F}_R^{(+)})$, $\mathscr{F}'\,|\,N(k)$ satisfies the conditions of Theorem 1. Now we consider $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$. If there exists a TS component in

$$\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|),$$

say TS_m, then the longitude of $|TS_m|$ and that of $|\mathscr{F}'_R|$ must coincide. This contradicts that TS_m contains compact leaves having the contracting holonomy in both positive and negative directions (Proposition 6). Thus $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$ does not contain a TS component. Similarly, by the fact that $[L_{comp}]=(2g-1)\alpha\pm\beta$, $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$ does not contain a foliated I-bundle. Therefore, $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$ should consist of half Reeb components.

Since $\mathscr{F}\,|\,|\mathscr{F}'_R|$ is transverse to a Reeb component $\mathscr{F}'_R$, $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ consists of half Reeb components, TS components and foliated I-bundles over $S^1\times I$ by Theorem 1. If $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ contains a TS component, say $TS'_{m'}$, then, since compact leaves of $TS'_{m'}$ have contracting holonomy, they must be subsets of ∂E. However, $\mathscr{F}$ has the contracting holonomy in one direction on $\partial N(k)$ in the side of E. This contradicts the property of TS components about contracting holonomy. Thus $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ cannot contain a TS component. Similarly, by the fact that $[L_{comp}]=(2g-1)\alpha\pm\beta$, $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ cannot contain a foliated I-bundle. Thus $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ must consist of half Reeb components. This shows that $E\cap|\mathscr{F}'_R|$ is diffeomorphic to the disjoint union of a finite number of copies of $S^1\times D^2_+$. Further, since $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$ consists of half Reeb components, $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$ is a half Reeb component, and, thus, $\mathscr{F}'_R\,|\,(N(k)\cap|\mathscr{F}'_R|)$ is also a half Reeb component.

Moreover, since $\mathscr{F}'_R\,|\,(\partial N(k)\cap|\mathscr{F}'_R|)$ is a plus Reeb component in $\mathscr{F}'\,|\,\partial N(k)$, it follows from (*), (**) that $\mathscr{F}'_R\,|\,(\partial E\cap|\mathscr{F}'_R|)$ is a minus Reeb component in $\mathscr{F}'\,|\,\partial E$.

Let $c(\partial E)$ be a sufficiently small collar of ∂E in E and denote $A_E=\overline{E-c(\partial E)}$. Then, for a non-compact leaf L of $\mathscr{F}\,|\,(E\cap|\mathscr{F}'_R|)$, $L\cap A_E$ is a half-disc. Thus the Euler number $\chi(L\cap A_E)=1$.

We may consider that the family of curves $\{L\cap A_E\cap L'; L'\in\mathscr{F}'\}$ is that of integral curves of a C^∞ vector field V_L on $L\cap A_E$ such that V_L is tangent to $L\cap A_E\cap\partial|\mathscr{F}'_R|$ and is non-singular except at one point of $L\cap\partial A_E$, say $\hat{p}$. The singularity of V_L at $\hat{p}$ is of minus type, because $\mathscr{F}'_R\,|\,(\partial E\cap|\mathscr{F}'_R|)$ is a minus Reeb component in $\mathscr{F}'\,|\,\partial E$. Let $D(L\cap A_E)$ denote the double of $L\cap A_E$ obtained from two copies of $L\cap A_E$ by pasting at $L\cap\partial A_E$, then the Euler number of $D(L\cap A_E)$ is -1. On the other hand, it follows from $\chi(L\cap A_E)=1$ that $\chi(D(L\cap A_E))=1$. This is a contradiction. Thus there exists no $\mathscr{F}'$. This completes the proof.

As a corollary of Theorem 6, we have the following theorem.

Theorem 7. — *Let* $\bar{\mathscr{F}}$ *be a* C^∞ *foliation of codimension 2 which is a subfoliation of* $\mathscr{F}$ *of Theorem 6. Then* $\bar{\mathscr{F}}$ *does not admit any transverse* C^∞ *foliation of codimension one*:

$$t_0(S^3, \bar{\mathscr{F}})=\varnothing.$$

Proof. — Clearly $\bar{\mathscr{F}}$ exists (cf. Proposition 1). Suppose $\mathscr{F}'\in t_0(S^3, \bar{\mathscr{F}})$. Then it is obvious that $\mathscr{F}'\in t_1(S^3, \mathscr{F})$. This contradicts the result of Theorem 6.

8. Problems

The following are some problems raised by the results of this paper.

Problem 1. — Classify or characterize C^∞ subfoliations of codimension 2 of the Reeb component $(S^1\times D^2, \mathscr{F}_R^{(+)})$. In case the restriction of the subfoliation to $S^1\times\partial D^2$ consists of two copies of the half Reeb foliation, do they coincide with $\mathscr{F}_R^{(+)}\cap\mathscr{F}'$ $(\mathscr{F}'\in t_1(\mathscr{F}_R^{(+)})$ of Fig. 4)?

Problem 2. — Determine $t_1^m(S^1\times D^2, \mathscr{F}_R^{(+)})$ for $m=2, 3, \ldots$. Does there exist a stability: $t_1^m(S^1\times D^2, \mathscr{F}_R)=t_1^{m+2}(S^1\times D^2, \mathscr{F}_R)=\ldots=t_1^{m+2j}(S^1\times D^2, \mathscr{F}_R)=\ldots$?

Problem 3. — Classify or characterize C^∞ subfoliations of codimension 2 of the Reeb foliation $(S^3, \mathscr{F}_R)$.

Problem 4. — Characterize C^∞ foliations of codimension 2 of S^3 which have superfoliations of codimension one.

Problem 5. — Does there exist a C^∞ foliation $\mathscr{F}$ of codimension one of S^3 such that $t_1(\mathscr{F})\neq\varnothing$ and $t_1(\mathscr{F})\cap t_1(\mathscr{F}')=\varnothing$ for some $\mathscr{F}'\in t_1(\mathscr{F})$.

Problem 6. — Characterize C^∞ foliations contained in the limit of the sequence $\{(S^3, \mathscr{F}_R)\}\subset t_1(S^3, \mathscr{F}_R)\subset\ldots\subset t_1^m(S^3, \mathscr{F}_R)\subset\ldots$. Does there exist a stability for this sequence? Is $t_1^2(S^3, \mathscr{F}_R)=t_1(S^3, \mathscr{F}_R)$?

Problem 7. — Does there exist a C^∞ foliation of codimension one of S^3 such that $t_1(S^3, \mathscr{F})$ (or the limit of the sequence $t_1(S^3, \mathscr{F})\subset\ldots\subset t_1^m(S^3, \mathscr{F})\subset\ldots$) is equal to the set of C^∞ foliations of codimension one which admit transverse C^∞ foliations of codimension one.

Problem 8. — For $(S^3, \mathscr{F}') \in t_1(S^3, \mathscr{F}_R)$, is it true that " the Godbillon-Vey number zero " implies " cobordant to zero "?

Problem 9. — Consider the deformation classes in $t_1(S^3, \mathscr{F}_R)$.

Problem 10. — Find conditions for C^∞ foliated manifolds to admit transverse foliations.

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