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A TOPOLOGICAL RESOLUTION THEOREM

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The eventual goal of this paper is to prove a manifold analogue of Hironaka's resolution theorem for algebraic varieties [H]. Specifically, we show that every compact PL manifold, M, carries an extra structure (A structure) which implies the existence of a smooth manifold $\widetilde{M}$, together with a degree one map (with $\mathbf{Z}/2\mathbf{Z}$ coefficients) $\pi : \widetilde{M} \to M$. In fact, π is a PL homeomorphism in the complement of a union of smooth submanifolds of the form $W_i \times M_i$. This union is not disjoint, but the M_i intersect each other transversally. Moreover, the map π collapses each $W_i \times M_i$ to M_i in some order. Put another way, every compact PL manifold M admits a smooth framed stratification (every stratum has a product neighborhood) such that after a sequence of " topological blow ups " performed along the closed smooth strata we get a compact smooth manifold $\widetilde{M}$ ($\partial\widetilde{M} = \varnothing$ if $\partial M = \varnothing$) together with a degree one map $\pi : \widetilde{M} \to M$. This theorem is used in [AK] to show that the interior of every compact PL manifold is PL homeomorphic to a real algebraic variety. Another application is a way of defining differential forms on a PL manifold by pushing down the relative forms from the smooth resolution spaces.

The work in this paper is to study a certain type of structure, called a C-manifold structure, on PL manifolds. We want to know when C-manifolds behave nicely. In particular, when is there a classifying space BC and a map $BC \to BPL$ such that a PL manifold M has a C structure if and only if the normal bundle, $\nu_M : M \to BPL$, lifts to BC? This question was answered by Levitt [Le] who showed, roughly, that if one can do the standard glueing construction; can collar boundaries; can do regular neighborhood theory; and can form products with smooth manifolds, then one has a classifying space, BC, with the desired properties. In Section 1 we give a discussion of this result with more details than Levitt's, presumably at the expense of readability. We also discuss C bordism theory and prove a fundamental result: for unoriented C bordism, η_*^C, the usual map $\eta_*^C \to H_*(BC; \mathbf{Z}/2\mathbf{Z})$ is monic.

In the second section we discuss a construction, also due to Levitt, for starting with any category of C-manifolds and getting a new category of C-manifolds. Roughly one does the following. Choose any countable collection of C-manifolds, Σ_i, $i = 1, 2, \ldots$,

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where each Σ_i is a C structure on the n_i-sphere, $n_i \geq 2$. A $C(\Sigma)$-manifold is a space of the form $M_0 \cup \coprod_i (c(\Sigma_i) \times M_i)$ where M_0 is a C-manifold; $c(\Sigma_i)$ is the unreduced cone on Σ_i; and M_i is a smooth manifold. The union is formed using a codimension zero embedding (in C) $\coprod_i \Sigma_i \times M_i \to \partial M_0$. The principle result of this section, which is also due to Levitt, is that $C(\Sigma)$-manifolds will behave nicely. We briefly study the relation between C-manifolds and $C(\Sigma)$-manifolds.

Clearly we can now do the following. Start with smooth manifolds and cone some spheres; take this new category and cone some more spheres; take this category... Levitt [Le] proves that if at each stage one cones the exotic spheres, then one gets BPL for BC.

In Section 3 we study A-manifolds. They are defined inductively also, where now at each stage one cones the exotic spheres which are unoriented boundaries. It is not hard to see that there are lots of exotic A-spheres, so PL/A (= the homotopy theoretical fibre of BA$\to$BPL) is definitely not a point. However, we do get the surprising fact that an A sphere bounds an A disc if and only if it bounds an unoriented A-manifold. This result plus the fact that $\eta_*^A \to H_*(BA; \mathbf{Z}/2\mathbf{Z})$ is monic proves $BA = BPL \times PL/A$. In particular BA$\to$BPL has a section so all PL manifolds have A structures. We conclude Section 3 by calculating the homotopy type of PL/A.

In Section 4 we discuss how an A structure on a PL manifold, M, gives rise to a resolution of M which is unique up to concordance of resolutions. One can then easily show that concordance classes of resolutions of M are classified by $\bigoplus H^*(M; \pi_*(PL/A))$. Since $\pi_*(PL/A)$ is infinite if $* > 8$ (Proposition (3.2)), most M have an infinite number of non-concordant resolutions. In particular, any PL manifold of dimension at least 9 has an infinite number of different structures (up to concordance) as an algebraic variety.

1. Smooth manifolds with singularities.

We wish to build an axiomatic framework to study smooth stratified sets where the underlying space is a PL manifold. There are technical difficulties associated with the statement that a smooth manifold is PL and things get no easier upon passage to stratified sets. Our solution to these problems is inelegant but effective. We have one category, $\mathscr{C}$, to be thought of as smooth stratified sets and another category, $\mathscr{L}$, to be thought of as PL stratified sets which are naturally PL manifolds. In both $\mathscr{C}$ and $\mathscr{L}$ we distinguish certain morphisms which we call *cdz embeddings* (short for codimension zero). Finally we have maps from the objects of $\mathscr{L}$ to the objects of $\mathscr{C}$ which can be thought of as stratified piecewise differentiable homeomorphisms. They are called PC *homeomorphisms* in what follows.

The axioms which we now list merely reflect some of the properties one expects $\mathscr{C}$ and $\mathscr{L}$ to have. To fix notation, hereafter Diff denotes the category of compact smooth manifolds with corners and smooth maps; PL denotes the category of compact PL manifolds and PL maps; and TOP denotes the category of compact topological manifolds and continuous maps. There are natural transformations Diff$\to$TOP and PL$\to$TOP but there is *no* natural transformation Diff$\to$PL (which is the source of our difficulties).

We have tried to give a uniform treatment of the properties of both $\mathscr{C}$ and $\mathscr{L}$ wherever possible. We use $\mathscr{P}$ below to denote a category which in practice is often, $\mathscr{C}$ or $\mathscr{L}$, although two other examples do occur.

Axiom I. — There exists a unique object in $\mathscr{P}$ with a unique map to every other object. We note this object by $\varnothing$. Given an object M, *if there is a map* $M \to \varnothing$, *then* $M = \varnothing$.

To every object M *in $\mathscr{P}$ we assign a second object,* ∂M *and a map* $\iota : \partial M \to M$.

There is a natural transformation $\mathscr{P} \to$TOP, *denoted* $|\ |$. *If* $f_1, f_2 : M \to N$ *are maps in $\mathscr{P}$, then* $f_1 = f_2$ *if and only if* $|f_1| = |f_2|$. *There exists a homeomorphism* $|\partial M| \to \partial |M|$ *such that*

$$\begin{array}{ccc} |\partial M| & \to & \partial|M| \\ & {\scriptstyle|\iota|}\searrow \quad \swarrow{\scriptstyle\iota} & \\ & |M| & \end{array}$$

commutes. An object M *in $\mathscr{P}$ is $\varnothing$ if and only if* $|M|$ *is empty.*

Suppose given a continuous map $f : |M_1| \to |\partial M_2|$ *such that the composite:*

$$|M_1| \xrightarrow{f} |\partial M_2| \xrightarrow{|\iota|} |M_2|$$

is the realization of a $\mathscr{P}$ map $h : M_1 \to M_2$. *Then there exists a $\mathscr{P}$-map* $g : M_1 \to \partial M_2$ *such that* $|g| = f$ *and* $h = \iota \circ g$.

There is at least one object of $\mathscr{P}$ whose realization is a point. Given two such objects there exists a unique map between them. Any object M *with* $|M|$ *homeomorphic to* S^1 *is $\mathscr{P}$-homeomorphic to* ∂N, *where* N *is an object with* $|N|$ *homeomorphic to* D^2.

Remark. — $\partial\partial M = \varnothing$ follows from I. We say that an object M in $\mathscr{P}$ has dimension n if $|M|$ has dimension n. We say M is connected if $|M|$ is.

There is a PL version of Axiom I, Axiom I_{PL} in which the natural transformation takes values in PL and the homeomorphism $|\partial M| \to \partial |M|$ is PL. Note that if $\mathscr{P}$ satisfies Axiom I_{PL} it also satisfies I.

Axiom II. — $\mathscr{P}$ has finite disjoint unions, i.e. given a finite collection of objects M_α *in $\mathscr{P}$ there exists an object* $\amalg M_\alpha$ *and maps* $i_\alpha : M_\alpha \to \amalg M_\alpha$ *such that, given maps* $f_\alpha : M_\alpha \to N$, *there exists a unique map* $f : \amalg M_\alpha \to N$ *with* $f_\alpha = f \circ i_\alpha$.

The natural map $\amalg |M_\alpha| \to |\amalg M_\alpha|$ *is a homeomorphism. There exist maps*

$$\partial I_\alpha : \ \partial M_\alpha \to \partial(\amalg M_\alpha)$$

such that $\iota \circ \partial I_\alpha = i_\alpha \circ \iota$ *(so much is already forced from the above plus Axiom I) and such that the map* $\amalg \partial M_\alpha \to \partial(\amalg M_\alpha)$ *is a $\mathscr{P}$-homeomorphism.*

If M *is an object of $\mathscr{P}$ and if* $|M| = K_1 \amalg K_2$ *then there exist objects* M_1 *and* M_2 *in $\mathscr{P}$ with maps* $M_1 \to M$, $M_2 \to M$ *and homeomorphisms* $|M_1| \to K_1$, $|M_2| \to K_2$ *such that the map* $M_1 \amalg M_2 \to M$ *is a $\mathscr{P}$-homeomorphism and* $K_1 \amalg K_2 \to |M_1| \amalg |M_2| \to |M_1 \amalg M_2| \to |M|$ *is the original equivalence.*

Remark. — By induction on the number of components of $|M|$, any object of $\mathscr{P}$ is the disjoint union of connected objects. Since $M \to \varnothing \amalg M$ and $M \to M \amalg \varnothing$ are $\mathscr{P}$-homeomorphisms we will often form infinite disjoint unions in $\mathscr{P}$ if all but finitely many of the objects are $\varnothing$.

Axiom III_{Diff}. — *There is a product* $\mathscr{P} \times \mathrm{Diff} \to \mathscr{P}$, *i.e. given objects* M *in* $\mathscr{P}$ *and* K *in* Diff *we assign an object* $M \times K$ *in* $\mathscr{P}$. *Given maps* $f : M_1 \to M_2$ *and* $g : K_1 \to K_2$ *we assign a map* $f \times g : M_1 \times K_1 \to M_2 \times K_2$. *We require that:*

$$1_M \times 1_K = 1_{M \times K} \quad \textit{and that} \quad (f_1 \times g_1) \circ (f_2 \times g_2) = (f_1 \circ f_2) \times (g_1 \circ g_2).$$

There exists a natural map $\pi_1 : M \times K \to M$. *We can have no* π_2, *since* K *is not necessarily in* $\mathscr{P}$. *We do however have a natural* $\pi_2 : |M \times K| \to |K|$, *also writing* $|\ \ |$ *for the natural transformation* $\mathrm{Diff} \to \mathrm{TOP}$. *The map* $|M \times K| \xrightarrow{|\pi_1| \times \pi_2} |M| \times |K|$ *is a homeomorphism. The map* $\pi_1 : M \times p \to M$ *is a* $\mathscr{P}$*-homeomorphism. The natural map* $\amalg M_\alpha \times K_\beta \to (\amalg M_\alpha) \times (\amalg K_\beta)$ *is a* $\mathscr{P}$*-homeomorphism.*

There exists a natural $\mathscr{P}$*-homeomorphism*

$$\eta : (M \times K_1) \times K_2 \to M \times (K_1 \times K_2).$$

The product in TOP *has a natural transformation* $\eta_{TOP} : (K_1 \times K_2) \times K_3 \to K_1 \times (K_2 \times K_3)$. *We require that the diagrams*

$$\begin{array}{ccccc}
 & & (M \times K_1) \times (K_2 \times K_3) & & \\
 & \overset{\eta}{\nearrow} & & \overset{\eta}{\searrow} & \\
((M \times K_1) \times K_2 \times K_3) & & & & M \times (K_1 \times (K_2 \times K_3)) \\
 & \underset{\eta \times 1}{\searrow} & & \underset{1 \times \eta_{TOP}}{\nearrow} & \\
 & (M \times (K_1 \times K_2)) \times K_3 & \xrightarrow{\eta} & M \times ((K_1 \times K_2) \times K_3) &
\end{array}$$

$$\begin{array}{ccc}
(M \times K_1) \times K_2 & \xrightarrow{\eta} & M \times (K_1 \times K_2) \\
 & \underset{\pi_1}{\searrow} \quad \underset{1_M \times \pi_1}{\swarrow} & \\
 & M \times K_1 &
\end{array}$$

$$\begin{array}{ccc}
|(M \times K_1) \times K_2| & \xrightarrow{|\eta|} & |M \times (K_1 \times K_2)| \\
\Big\downarrow {\scriptstyle |\pi_1| \times \pi_2} & & \Big\downarrow {\scriptstyle \pi_2} \\
|M \times K_1| \times |K_2| & \xrightarrow{\pi_2 \times 1} & |K_1 \times K_2|
\end{array}$$

commute.

Remark. — Although no formal use will be made of coherence theory, articles like MacLane [M] will explain why these diagrams tend to occur.

There is also an Axiom $\mathrm{III}_{\mathrm{PL}}$.

Now in our category $\mathscr{P}$ we have special types of morphisms, called *cdz embeddings* in what follows.

Axiom IV_{Diff}. — *The cdz embeddings have the following properties:*

1) *Any $\mathscr{P}$-homeomorphism is a cdz embedding.*
2) *If the diagram*

$$\begin{array}{ccc} & \overset{g}{\nearrow} & \mathrm{P} \\ \mathrm{N} & & \uparrow h \\ & \underset{f}{\searrow} & \mathrm{M} \end{array}$$

commutes, and if h is a cdz embedding, then f is cdz embedding if and only if g is.

3) *If the diagram*

$$\begin{array}{cccc} & \overset{g}{\nearrow} & \partial\mathrm{P} \xrightarrow{\iota} & \mathrm{P} \\ \mathrm{N} & & & \uparrow h \\ & \underset{f}{\searrow} & \partial\mathrm{M} \xrightarrow{\iota} & \mathrm{M} \end{array}$$

commutes, and if h is a cdz embedding, then f is a cdz embedding if and only if g is.

4) *The disjoint union of cdz embeddings is a cdz embedding.*

5_{Diff}) *If $f:\mathrm{N}\to\mathrm{M}$ is a cdz embedding and if $g:\mathrm{V}\to\mathrm{W}$ is a codimension zero embedding in* Diff, *then $f\times g:\mathrm{N}\times\mathrm{V}\to\mathrm{M}\times\mathrm{W}$ is a cdz embedding.*

6) *From Axiom I we know that the maps $\partial\mathrm{N}\times\mathrm{W}\to\mathrm{N}\times\mathrm{W}$ and $\mathrm{N}\times\partial\mathrm{W}\to\mathrm{N}\times\mathrm{W}$ both factor through $\partial(\mathrm{N}\times\mathrm{W})$. The maps $\partial\mathrm{N}\times\mathrm{W}\to\partial(\mathrm{N}\times\mathrm{W})$ and $\mathrm{N}\times\partial\mathrm{W}\to\partial(\mathrm{N}\times\mathrm{W})$ are both cdz embeddings.*
7) *If $e:\mathrm{N}\to\mathrm{M}$ is a cdz embedding, then $|e|:|\mathrm{N}|\to|\mathrm{M}|$ is a cdz embedding.*
8) *If $e:\mathrm{N}\to\mathrm{M}$ is a cdz embedding, and if $g:|\mathrm{P}|\to|\mathrm{N}|$ is a continuous map such that $|e|\circ g$ is the realization of a $\mathscr{P}$-map, then g is the realization of a $\mathscr{P}$-map.*

Similarly there is also an Axiom $\mathrm{IV}_{\mathrm{PL}}$.

Axiom V. — *Given cdz embeddings $e:\mathrm{N}\to\partial\mathrm{M}_i$, $i=1, 2$, there exists an object* M *and cdz embeddings $u_i:\mathrm{M}_i\to\mathrm{M}$, $i=1, 2$, such that*

$$\begin{array}{ccc} |\mathrm{N}| & \xrightarrow{|\iota\circ e_1|} & |\mathrm{M}_1| \\ {\scriptstyle |\iota\circ e_2|}\downarrow & & \downarrow{\scriptstyle |u_1|} \\ |\mathrm{M}_2| & \xrightarrow[|u_2|]{} & |\mathrm{M}| \end{array}$$

is a pushout in TOP.

If we are given cdz embeddings $f_i : M_i \to P$, $i = 1, 2$ *such that* $e_1 \circ \iota \circ f_1 = e_2 \circ \iota \circ f_2$, *then we can find* M, u_1 *and* u_2 *as above so that the map* $|M| \to |P|$ *is the realization of a cdz embedding.*

Terminology. — We say that M is the result of *glueing* M_1 *and* M_2 *along* N.

Remark. — From I and IV it follows that $\partial(N \times W)$ is the result of glueing $\partial N \times W$ to $N \times \partial W$ along $\partial M \times \partial W$.

Remark. — Glueing behaves well with respect to disjoint unions and products.

Definitions. — We say that a cdz embedding $e : N \to M$ is *complemented* if there exists objects A and X in $\mathscr{P}$ and cdz embeddings $A \to \partial N$, $A \to \partial X$, $X \to M$ such that the resulting square

$$\begin{array}{ccc} |A| & \longrightarrow & |N| \\ \big\downarrow & & \big\downarrow{\scriptstyle |e|} \\ |X| & \longrightarrow & |M| \end{array}$$

is a pushout in TOP.

We require

Axiom VI. — *A cdz embedding* $e : N \to M$ *is complemented if there exists an object* Q *in* $\mathscr{P}$ *and a cdz embedding* $Q \to \partial M$ *such that, in* $|M|$, $|Q| = |N| \cap |\partial M|$.

Remark. — Any cdz embedding is complemented if $\partial M = \varnothing$.

Remark. — Since $\partial\partial M_i = \varnothing$, the cdz embeddings e_i in Axiom V are complemented. Let

$$\begin{array}{ccc} A_i & \longrightarrow & N \\ \big\downarrow & & \big\downarrow{\scriptstyle e_i} \\ X_i & \longrightarrow & \partial M_i \end{array}$$

display the glueing. Then $A_i \to \partial N$ is a cdz embedding whose realization is a homeomorphism. It follows from I and IV that $A_i \to \partial N$ is a $\mathscr{P}$-homeomorphism. It also follows from I and IV that the maps $X_i \to \partial M_i \to M_i \to M$ factor through ∂M so that the maps $X_i \to \partial M$ are cdz embeddings. By I

$$\begin{array}{ccc} |\partial N| & \longrightarrow & |X_1| \\ \big\downarrow & & \big\downarrow \\ |X_2| & \longrightarrow & |\partial M| \end{array}$$

is a pushout, so ∂M is obtained by glueing X_1 to X_2 along ∂N.

Axiom VII. — *Let* $f: N \to \partial M$ *be a cdz embedding. If*

$$\begin{array}{ccc} \partial N & \xrightarrow{\iota} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{g} & \partial M \end{array}$$

is a pushout exhibiting f *as complemented let us suppose given* $\hat{e}: \partial N \times [0, 1] \to W$ *a cdz embedding such that* $\partial N \overset{\pi_1^{-1}}{\to} \partial N \times 0 \to \partial N \times [0, 1] \overset{\hat{e}}{\to} W$ *is* h *and such that* $(|\partial N| \times (0, 1]) \cap |\partial W| = \varnothing$. *Then we can find a cdz embedding* $e: N \times [0, 1] \to M$ *such that*

a) $N \overset{\pi_1^{-1}}{\to} N \times 0 \to N \times [0, 1] \overset{e}{\to} M$ *is* $\iota \circ f$

b) *the square*

$$\begin{array}{ccc} \partial N \times [0, 1] & \xrightarrow{\hat{e}} & W \\ {\scriptstyle \iota \times 1}\downarrow & & \downarrow{\scriptstyle \iota \circ g} \\ N \times [0, 1] & \xrightarrow{e} & M \end{array}$$

commutes, and

c) $((\text{Interior } |N|) \times (0, 1]) \cap |\partial M| = \varnothing$.

Exercise 1. — Any cdz embedding $f: N \to \partial M$ extends to a cdz embedding $N \times [0, 1] \to M$. In particular, boundaries are collared.

Axiom VIII. — *Suppose* I_{PL} *is satisfied. Also suppose given a cdz embedding* $N \to \partial M$ *and a* PL *embedding* $Q \to |M|$ *such that* $Q \cap |\partial M| = |N|$. *Then we can find an object* P *and cdz embeddings* $P \to M$ *and* $N \to \partial P$ *such that*

$$\begin{array}{ccccc} & \nearrow & \partial M & \searrow & \\ N & & & & M \\ & \searrow & \partial P \longrightarrow P & \nearrow & \end{array}$$

commutes and such that $|P| = H_1(Q)$ *where* $H: |M| \times [0, 1] \to |M|$ *is a* PL *isotopy of* $1_{|M|}$ rel $\partial |M|$.

Remark. — If $|N| \subset \partial |M|$ is a regular neighborhood for a subcomplex $X \subset \partial |M|$, and if $Y \subset |M|$ is a subcomplex with $Y \cap \partial |M| = X$, then we can find an object P as above such that $(|P|, |N|)$ is a regular neighbourhood in $(|M|, \partial |M|)$ for (Y_1, X), where (Y_1, X) is isotopic rel X to (Y, X).

These eight axioms on categories will suffice for our needs. The reader should be convinced that Diff satisfies axioms I_{TOP}, II, III_{Diff}, IV_{Diff}, V_{Diff}, VI and VII; and that PL satisfies axioms I_{PL}, II, III_{PL}, IV_{PL}, V_{PL}, VI, VII, and VIII.

We next discuss the PC homeomorphisms. We assume given a category $\mathscr{C}$ satisfying I_{TOP}, II, III_{Diff}, IV_{Diff}, V_{Diff}, VI and VII and a category $\mathscr{L}$ satisfying I_{PL}, II, III_{PL}, IV_{PL}, V_{PL}, VI, VII, and VIII.

We say that $\mathscr{C}$ and $\mathscr{L}$ *admit* PC *homeomorphisms* provided we can do the following. Both $\mathscr{C}$ and $\mathscr{L}$ have natural transformations to TOP. Give $M \in \text{ob}\,\mathscr{C}$ and $X \in \text{ob}\,\mathscr{L}$ we must have a subset, PC(X, M), of Homeo(|X|, |M|), where these subsets satisfy the axioms below.

We adopt the notation "$f: X \to M$ is a PC *homeomorphism*" to mean that f, which is a map from $|X|$ to $|M|$, is in PC(X, M). For uniformity of notation we write $|f|: |X| \to |M|$. Our axioms are

Axiom PC 1. — Given any PC *homeomorphism* $f: X \to M$; *any* $\mathscr{L}$ *homeomorphism* $h: Y \to X$; *and any* $\mathscr{C}$ *homeomorphism* $g: M \to N$, *then* $g \circ f$ *and* $f \circ h$ *are* PC *homeomorphisms.*

Given a PC *homeomorphism* $f: X \to M$ *there exists a* PC *homeomorphism* $\partial f: \partial X \to \partial M$ *such that*

$$\begin{array}{ccc} \partial X & \xrightarrow{\partial f} & \partial M \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ X & \xrightarrow{f} & M \end{array}$$

commutes.

Axiom PC 2. — Suppose given cdz embeddings $e_1: X \to Y$ *and* $e_2: N \to M$; *a* PC *homeomorphism* $f: Y \to M$ *and a continuous map* $g: |X| \to |N|$ *such that* $|e_2| \circ g = |f| \circ |e_1|$. *Then* $g \in \text{PC}(X, N)$.

Suppose given two glueings

$$\begin{array}{ccc} Z & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} N & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M \end{array}$$

the first in $\mathscr{L}$ *and the second in* $\mathscr{C}$. *If we are given* PC *homeomorphisms* $f_i: X_i \to M_i$, $i = 1, 2$, *which agree on* Z, *then the pushed-out map* $|X| \to |M|$ *is in* PC(X, M).

Axiom PC 3. — Given a PC *homeomorphism* $f: X \to M$ *and a piecewise differentiable homeomorphism* $g: K \to W$ *there exists a* PC *homeomorphism, denoted* $f \times g: X \times K \to M \times W$, *such that* $|f \times g|$ *is the composite*

$$|X \times K| \xrightarrow{|\pi_1| \times \pi_2} |X| \times |K| \xrightarrow{|f| \times |g|} |M| \times |W| \xrightarrow{(|\pi_1| \times \pi_2)^{-1}} |M \times W|.$$

Axiom PC 4. — For each object, M, *in* $\mathscr{C}$ *and each* PC *homeomorphism* $f: X \to \partial M$, *there exists an object* Y *in* $\mathscr{L}$; *an* $\mathscr{L}$ *homeomorphism* $h: X \to \partial Y$; *and a* PC *homeomorphism* $g: Y \to M$ *such that* $\iota \circ f = g \circ \iota \circ h$.

Axiom PC 5. — *Suppose given* PC *homeomorphisms* $f_0 : X_0 \to M$ *and* $f_1 : X_1 \to M$. *Suppose also given an* $\mathscr{L}$ *homeomorphism* $g : \partial X_0 \to \partial X_1$ *and a* PC *homeomorphism*

$$H : \partial X_0 \times I \to \partial M \times I$$

such that $\partial H = \partial f_0 \amalg \partial f_1 \circ g$. *Then there exists an* $\mathscr{L}$ *homeomorphism* $G : X_0 \to X_1$ *and a* PC *homeomorphism* $\hat{H} : X_0 \times I \to M \times I$ *such that* $\partial G = g$ *and* $\partial \hat{H}$ *is* H *on* $\partial X_0 \times I$; f_0 *on* $X_0 \times 0$; *and* $f_1 \circ G$ *on* $X_0 \times 1$.

Axiom PC 6. — *Suppose given a cdz embedding* $f : X \to Y$ *in* $\mathscr{L}$ *and a* PC *homeomorphism* $h : Y \to M$. *Suppose further that we have cdz embeddings* $e_1 : Z \to \partial X$ *and* $e_2 : Z \to \partial Y$ *such that* $\iota \circ e_2 = f \circ \iota \circ e_1$; *a* PC *homeomorphism* $g : Z \to W$ *and a cdz embedding* $\hat{e}_2 : W \to \partial M$ *such that* $\hat{e}_2 \circ g = \partial h \circ e_2$. *Then we can find an object* N *in* $\mathscr{C}$; *cdz embeddings* $\hat{e}_1 : W \to \partial N$ *and* $\hat{f} : N \to M$; *and a* PC *homeomorphism* $\hat{h} : X \to N$ *such that*

$$\begin{array}{ccccc} W & \xrightarrow{\hat{e}_1} & \partial N & & \\ {\scriptstyle \hat{e}_2}\downarrow & & & \searrow^{\iota} & \\ & & & & N, \\ & & & \swarrow_{\hat{f}} & \\ \partial M & \xrightarrow{\iota} & M & & \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{e_1} & \partial X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \partial \hat{h}} \\ W & \xrightarrow{\hat{e}_1} & \partial N \end{array}, \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{\hat{h}} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ Y & \xrightarrow{h} & M \end{array}$$

all commute.

Given $\mathscr{C}$, $\mathscr{L}$ and PC homeomorphisms as above, we can define the category of C-manifolds. A C-*manifold* is a triple (X, M, f) when $X \in \text{ob}\,\mathscr{L}$, $M \in \text{ob}\,\mathscr{C}$, and $f : X \to M$ is a PC homeomorphism. A map of (X_1, M_1, f_1) to (X_2, M_2, f_2) is a pair of maps $h : X_1 \to X_2$ in $\mathscr{L}$ and $g : M_1 \to M_2$ in $\mathscr{C}$ such that $|g| \circ |f_1| = |f_2| \circ |h|$. A cdz embedding is a map (h, g) where both h and g are cdz embeddings. The obvious definitions of composite and identity clearly makes the collection of C-manifolds and maps into a category.

Since $\mathscr{C} = \text{Diff}$, $\mathscr{L} = \text{PL}$ with the piecewise differentiable homeomorphisms being the PC homeomorphisms satisfies all our requirements ([Hi], [H-M], [Mu] and [R-S]), we get the category, *Tri*, of triangulated smooth manifolds.

Note that the category of C-manifolds satisfies Axioms I_{PL}, II, III_{Tri}, IV, V, and VI, when we leave the reader to make the obvious definitions. The necessary proofs are tedious but not hard.

Axioms VIII and PC 6 combine to give a regular neighborhood theory for C-manifolds. Given a C-manifold, M, and a simplicial complex, K, with a map $f : K \to |M|$, we say that K, or, properly (K, f), is a *subcomplex* of M if f is a PL embedding. We say that a pair of complexes (K, L), with a map $f : K \to |M|$, is a *proper pair* if f is a PL embedding; $f(L) \subset |\partial M|$; and $f(K) \cap |\partial M| = f(L)$.

We say that a cdz embedding $e : N \to M$ is a *regular neighborhood* for (K, f) if, in $|M|$, $|N|$ is a regular neighborhood in the usual sense. Given C-manifolds W and N and cdz embeddings $e : N \to M$, $e_1 : W \to \partial N$, and $e_2 : W \to \partial M$ such that

$e\circ\iota\circ e_1=\iota\circ e_2$ we say that (N, W) is a *regular neighborhood* for the proper pair (K, L) if $(|N|, |W|)\subset(|M|, \partial|M|)$ is a regular neighborhood for (K, L) in the usual sense. We have

Proposition **(1.1)**. — *Let* (K, L) *be a proper pair in* M. *Suppose given a regular neighborhood* $e_2: W\rightarrow\partial M$ *for* L. *Then there is a* PL *isotopy of* $|M|$ rel $\partial|M|$, *denoted* H_t, *and a* C*-manifold* M *with cdz embeddings* $e: N\rightarrow M$ *and* $e_1: W\rightarrow\partial N$ *such that*

a) $H_0=1_{|M|}$
b) $e\circ\iota\circ e_1=\iota\circ e_2$
c) (N, W) *is a regular neighborhood for the proper pair* $(H_1(K), L)$.

The proof is left to the reader. The reason for the isotopy in Axiom VIII, and hence in Proposition (1.1) will become apparent in Section 2.

The usual definitions in manifold theory all work for C-manifolds. Glueing guarantees that bordism is an equivalence relation, so we have the unoriented bordism group of C-manifolds, η_*^C. We say that a C-manifold, M, is oriented if we have chosen an orientation for $|M|$. Oriented bordism is defined and we denote the groups by Ω_*^C.

A C structure on a PL manifold K is a C-manifold M and a PL homeomorphism $\alpha: K\rightarrow|M|$. Given a codimension zero sub-manifold, L, of ∂K and a C structure (N, β) on L, we say that the C structure (M, α) *extends* (N, β) if there is a cdz embedding $e: N\rightarrow\partial M$ such that

$$\begin{array}{ccc} L\subset\partial K & \rightarrow & K \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\alpha} \\ |N| & \xrightarrow{|\iota\circ e|} & |M| \end{array}$$

commutes. Two C structures on K, say (M_0, α_0), (M_1, α_1), are concordant if there is a C structure on $K\times I$ extending the obvious one on $K\times 0\amalg K\times 1$.

If both (M_0, α_0) and (M_1, α_1) extend a structure (N, β) on L, we say that they are *concordant rel* L if we can find a C structure on $K\times I$ extending the following one. Since $1_{[0,1]}: [0, 1]\rightarrow[0, 1]$ is a Tri manifold, products with [0, 1] make sense. We have a C-manifold $N\times[0, 1]$ which, via $\beta\times 1$, is a structure on $L\times[0, 1]$. In PL, $K\times 0\cup L\times[0, 1]\cup K\times 1$ is a codimension zero submanifold of $\partial(K\times I)$. By the glueing axiom, the C-manifolds M_0, $L\times[0, 1]$, and M_1 can be glued to get a C-manifold P and an obvious PL homeomorphism $\gamma: K\times 0\cup L\times[0, 1]\cup K\times 1\rightarrow|P|$. This is the structure we insist that our concordance extend.

Exercise 2. — Prove that concordance rel L is an equivalence relation.

We need one last axiom.

Countability axiom. — Given a C structure on a codimension zero submanifold $L \subset \partial K$, the collection of concordance classes of C structures on K rel L, denoted $\mathscr{S}(K, \text{rel } L)$, is a countable set.

We can now state our first main theorem.

Theorem I. — *We suppose given a category of* C-*manifolds for which the countability axiom holds. Then we have a* CW *complex* BC *with the following properties.*

a) *To every* C-*manifold* M *there is associated a unique homotopy class of maps* $\nu_M : |M| \to BC$.

b) *There is a map* $BC \to BPL$ *such that, given a* C *structure* (N, β) *on* L, L *a codimension zero submanifold of* ∂K, K *a* PL *manifold, there is a one to one correspondence between concordance classes* rel L *of* C *structures on* K, *and homotopy classes of lifts,* rel L *of the diagram*

$$\begin{array}{ccccc} L & \xrightarrow{\beta} & |N| & \xrightarrow{\nu_N} & BC \\ \downarrow & & & & \downarrow \\ K & & \xrightarrow{\quad\nu_K\quad} & & BPL \end{array}$$

c) *There is a product* $BC \times BO \to BC$ *such that* $BC \to BC \times BO \to BC$ *is the identity and such that*

$$\begin{array}{ccc} BC \times BO & \longrightarrow & BC \\ \downarrow & & \downarrow \\ BPL \times BO & \longrightarrow & BPL \end{array}$$

weakly homotopy commutes.

(Following Adams [A] we say that two maps $f, g : X \to Y$ are *weakly homotopic* if, given any finite simplicial complex, T, and map $\alpha : T \to X$, $f \circ \alpha$ and $g \circ \alpha$ are homotopic.)

The proof of Theorem I is well-known to the experts and Levitt [L] can be profitably consulted for a more intuitive version of the same proof. In outline, we follow Wall [W] and consider stable C-manifold thickenings. This gives a representable functor with representing space BC. Part a) is then easy and part c) follows from the lemma that $B\,\text{Tri} \cong BO$. Then we prove a Cairns-Hirsch theorem from which b) follows.

Definition. — Given a finite simplicial complex, T, an n-dimensional C *thickening* of T is an n-dimensional C-manifold M and a simple homotopy equivalence $\alpha : T \to |M|$.

Two thickenings (M_0, α_0) and (M_1, α_1) are equivalent if we can find an $(n+1)$-dimensional thickening (W, β) of T and a cdz embedding $E: M_0 \amalg M_1 \to \partial W$ such that

$$T \amalg T \xrightarrow{\alpha_0 \amalg \alpha_1} |M_0| \amalg |M_1| \to |M_0 \amalg M_1| \xrightarrow{E} |\partial W| \xrightarrow{|\iota|} |W| \text{ and } T \amalg T \xrightarrow{\beta \amalg \beta} |W|$$

are homotopic. Let $H_n(T)$ denote the equivalence classes of n-dimensional thickenings of T.

Exercise 3. — Prove that equivalence of thickenings is an equivalence relation and that $H_n(T)$ is a countable set.

Given an n-dimensional thickening (M, α) of T we obtain an $(n+1)$-dimensional thickening $(M \times I, \alpha \times 1)$ of T. It is easily checked that this extends to a set map $\sigma: H_n(T) \to H_{n+1}(T)$. Let $H(T)$ denote the direct limit and let $\sigma_\infty: H_n(T) \to H(T)$ be the natural map.

To prove that $H(\ )$ is a representable functor we must first define $f^*: H(T_2) \to H(T_1)$ for a map $f: T_1 \to T_2$. To begin we define a slight variant. For each n, define $\hat{f}: H_n(T_2) \to$ (subsets of $H_n(T_1)$) as follows. If $u_i \in H_n(T_i)$, $i = 1, 2$, we say that $u_1 \in \hat{f}(u_2)$ if we can find representatives (M_i, α_i) for u_i, $i = 1, 2$, and a cdz embedding $e: M_1 \to M_2$ such that

$$\begin{array}{ccc} T_1 & \xrightarrow{f} & T_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ |M_1| & \xrightarrow{|e|} & |M_2| \end{array}$$

homotopy commutes. If $\hat{f}(u_2)$ always has precisely one element, we let

$$f^*: H_n(T_2) \to H_n(T_1)$$

denote the underlying map of sets.

Clearly $\hat{f}$ depends only on the homotopy class of f. Moreover, $\sigma\hat{f}(u) \subset \hat{f}(\sigma(u))$ and, if $g: T_2 \to T_3$, $\bigcup_{v \in \hat{g}} \hat{f}(v) \subset \widehat{(g \circ f)}(u)$. We also have

Proposition **(1.2).** — *If* $n > 2 \dim T_1 + 2$, *then* $\hat{f}(u)$ *contains precisely one element for each* $u \in H_n(T_2)$.

Proof. — Given (M_2, α_2), a representative for u, the composite $T_1 \xrightarrow{f} T_2 \xrightarrow{\alpha_2} |M_2|$ is homotopic to a PL embedding. By Proposition (1.1) we can find (M_1, α_1) and a cdz embedding $e: M_1 \to M_2$ such that

$$\begin{array}{ccc} T_1 & \xrightarrow{f} & T_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ |M_1| & \xrightarrow{|e|} & |M_2| \end{array}$$

homotopy commutes, and $|M_1|$ is a regular neighborhood for T_1. In particular, α_1 is a simple homotopy equivalence. This shows that $\hat{f}(u)$ is non-empty. A relative version of the same argument shows that $\hat{f}(u)$ has only one element. ■

Hence we get a map $f^*: H(T_2) \to H(T_1)$ which depends only on the homotopy class of f and such that $(g \circ f)^* = f^* \circ g^*$ and $1_T^* = 1_{H(T)}$. Hence $H(\ )$ is a homotopy functor.

To prove that $H(\ )$ is representable we use Brown's theorem [B]. Hence we must show that H satisfies a wedge axiom, a Mayer-Vietoris axiom, and a countability axiom. We proceed in reverse order.

That $H(S^r)$ is countable follows easily from the countability axiom and well-known results about PL.

For the Mayer-Vietoris axiom let us be given finite simplicial complexes, S, T_1, T_2 and simplicial maps $f_i: S \to T_i$ $i = 1, 2$. Let Z denote the double mapping cylinder. We have simplicial maps $h_i: T_i \to Z$, $i = 1, 2$, such that $h_1 \circ f_1$ is homotopic to $h_2 \circ f_2$. We want to show that, given $u_i \in H(T_i)$, $i = 1, 2$, such that $f_1^* u_1 = f_2^* u_2$ in $H(S)$, then there exists $v \in H(Z)$ such that $h_i^* v = u_i$, $i = 1, 2$.

By well-known arguments, we can find, for some n, n-thickenings (M_i, α_i) representing u_i, $i = 1, 2$; and thickenings (N_i, β_i) of S with cdz embeddings $e_i: N_i \to M_i$ such that $|e_i| \circ \beta_i \sim \alpha_i \circ f_i$. Moreover (N_1, β_1) is equivalent to (N_2, β_2).

If (W, b) is an equivalence between N_1 and N_2, let P be $(M_1 \times [0, 1]) \amalg (M_2 \times [0, 1])$ glued to W along $N_1 \amalg N_2$ where

$$N_1 \amalg N_2 \xrightarrow{e_1 \amalg e_2} M_1 \amalg M_2 \to (M_1 \times 0) \amalg (M_2 \times 0)$$
$$\longrightarrow \partial((M_1 \times [0, 1]) \amalg (M_2 \times [0, 1]))$$

is the required cdz embedding. It is easy to check that there is a simple homotopy equivalence $\gamma: Z \to |P|$ such that $\sigma(M_i, \alpha_i) \in \hat{f}_i(P, \gamma)$ $i = 1, 2$. The result follows.

For the wedge axiom we need to prove that $H_n(\text{pt})$ is a one element set, $n \geq 0$, and that any orientable n-thickening of S^1 is the boundary of an $(n+1)$-thickening of D^2, $n \geq 1$. The results are obvious for $n = 0$ and $n = 1$ respectively from Axiom 1.

If (M, α) is an n-thickening of a point, p, notice that $\alpha: p \to |M|$ can, by a homotopy, be assumed to factor through $|\partial M|$. Use (1.1) to find N and a cdz embedding $e: N \to \partial M$ with $|N|$ a regular neighborhood for $\alpha(p)$. Let $\beta: p \to |N|$ denote α restricted to $|N|$. Then $\sigma(N, \beta)$ is seen to be equivalent to (M, α) using the collaring Axiom (VII). Hence $H_{n-1}(p) \to H_n(p)$ is onto and our result follows. The same idea works for oriented thickenings of S^1.

To prove the wedge axiom we proceed as follows. We use the same notation as for the Mayer-Vietoris axiom only now we further assume that S is a point.

We want to show that $h_1^* \times h_2^*: H(Z) \to H(T_1) \times H(T_2)$ is a set isomorphism. Since $H(S)$ has only one element, the Mayer-Vietoris axiom shows that $h_1^* \times h_2^*$ is onto. In fact, if $n > 2\max(\dim T_1, \dim T_2) + 3$, any element in $H_n(Z)$ is equivalent to one of the form we constructed in our proof of the Mayer-Vietoris axiom. To see this, first observe that by (1.2), $H_{n+1}(Z) \to H_n(Z)$ is onto.

Let (M, α) be an $(n-1)$-thickening of Z. By general position and (1.1) we can find thickenings (M_i, α_i) of T_i, $i=1, 2$ and a cdz embedding $e_1 \amalg e_2 : M_1 \amalg M_2 \to M$ such that $|e_i| \circ \alpha_i = \alpha \circ h_i$, $i=1, 2$. By general position and (1.1) we can find an $(n+1)$-thickening (W, β) of S and a cdz embedding $E : W \to M \times [-1, 0]$ such that

$$\begin{array}{ccccc} M_1 \amalg M_2 & \longrightarrow & \partial W & \longrightarrow & W \\ \downarrow{\scriptstyle e_1 \amalg e_2} & & & & \downarrow{\scriptstyle E} \\ M & \to & M \times 0 & \to & M \times [-1, 0] \end{array}$$

commutes. We also have a cdz embedding

$$(e_1 \times 1) \amalg (e_2 \times 1) : (M_1 \times [0, 1]) \amalg (M_2 \times [0, 1]) \to M \times [0, 1].$$

We can glue $M \times [0, 1]$ to $M \times [-1, 0]$ along $M \times 0$ to get $M \times [-1, 1]$. We can glue W to $(M_1 \times [0, 1]) \amalg (M_2 \times [0, 1])$ along $(M_1 \times 0) \amalg (M_2 \times 0)$ to get P and we can actually choose P so that we have a cdz embedding $P \to M \times [-1, 1]$. This shows that $\sigma(M, \alpha)$ is equivalent to one of our special thickenings. Since $H_{n-1}(Z) \to H_n(Z)$ is onto, any n-thickening is equivalent to one of our special thickenings.

Now suppose we have (P, γ) and $(\overline{P}, \overline{\gamma})$, two special n-thickenings of Z. If M_1 is equivalent to $\overline{M}_1$, and if M_2 is equivalent to $\overline{M}_2$, via (W_1, β_1), (W_2, β_2) respectively, we can glue $P \amalg \overline{P}$ to $W_1 \amalg W_2$ along $M_1 \amalg M_2 \amalg \overline{M}_1 \amalg \overline{M}_2$ to get a C-manifold Q. It turns out that Q is a thickening of $T_1 \vee T_2 \vee S^1$. Use regular neighborhood theory (1.1) and general position to find an orientable n-thickening of S^1, (R, δ), and a cdz embedding $R \to Q$. Let D denote an $(n+1)$-thickening of the disc that R bounds. Glue $Q \times [0, 1]$ to D along R $(R \to Q \times 1)$ to get V. There is a simple homotopy equivalence $\lambda : Z \to |V|$ and it is easy to check that there are cdz embeddings $e : P \to \partial V$ and $\overline{e} : \overline{P} \to \partial V$ such that $|\iota \circ e| \circ \gamma \sim \lambda \sim |\iota \circ \overline{e}| \circ \overline{\gamma}$. Hence $\sigma(\overline{P}, \overline{\gamma})$ is equivalent to $\sigma(P, \gamma)$ and so (P, γ) is equivalent to $(\overline{P}, \overline{\gamma})$ since $H_n(Z) \to H_{n+1}(Z)$ is monic by (1.2). This proves the wedge axiom.

Hence Brown's theorem [B] applies and we have a CW complex, BC, and a natural transformation $H(T) \cong [T, BC]$. Given a C-manifold, M, and the map $1_{|M|} : |M| \to |M|$ we have a thickening of $|M|$, and hence a map $\nu_M : |M| \to BC$ well-defined up to homotopy.

Adams' [A] discussion of natural transformations is valid in our case also. Any natural transformation of representable theories, whose representing spaces have countable homotopy type, is induced by a map of representing spaces. The catch is that the map is only well-defined up to weak homotopy.

To any C-manifold M we associate the homotopy class of the map $\nu_{|M|} : |M| \to BPL$, the PL normal bundle of $|M|$. This defines a natural transformation $H(T) \to k^0_{PL}(T)$, and hence a map $BC \to BPL$ unique up to weak homotopy type. Without further mention we assume that the map has been made into a fibration.

Given the category Tri, by the above, we get a classifying space B Tri. To any Tri manifold M, we associate the smooth normal bundle map $\nu_M : M \to BO$. This defines a map $B\,\mathrm{Tri} \to BO$. Now using PC 5 the reader can show that if (X, M, f) and (Y, M, g) are C-manifolds, then they are concordant as structures on $|X|$. Since BO classifies smooth thickenings (Wall [W]) one can use the above remark, plus its relative version, to prove that $B\,\mathrm{Tri} \to BO$ is a homotopy equivalence.

The space $BC \times BO$ classifies pairs consisting of a C thickening, M, of T and a Tri thickening, W, of T. Then $M \times W$ is a C thickening of $T \times T$ and we use the diagonal map to induce a thickening of T itself. This defines a natural transformation $BC \times BO \to BC$ such that the map $BC \to BC \times BO \to BC$ is weakly homotopic to the identity. But such a map is a homotopy equivalence, so we can choose the map $BC \times BO \to BC$ so that $BC \to BC \times BO \to BC$ is actually homotopic to the identity. The diagram

$$\begin{array}{ccc} BC \times BO & \longrightarrow & BC \\ \downarrow & & \downarrow \\ BPL \times BO & \longrightarrow & BPL \end{array}$$

weakly homotopy commutes as the reader may check.

For part b) of Theorem I, let K be a PL manifold; L a codimension zero submanifold of ∂K; and (N, β) a C structure on L. Part b) of Theorem I is well-known to be equivalent to the

Cairns-Hirsch theorem. — *With notation as above, and with* C *structure* $(N \times I, \beta \times 1)$ *on* $L \times I$, *the natural map*

$$\mathcal{S}(K, \mathrm{rel}\, L) \to \mathcal{S}(K \times I, \mathrm{rel}\, L \times I)$$

is an isomorphism. $\mathcal{S}(K, \mathrm{rel}\, L)$ *is the set of concordance classes of* C *structures on* K, rel L. *(See the countability axiom.)*

Proof. — We first show that the map is onto. We have a C-manifold M; a cdz embedding $e : N \times I \to \partial M$; and a PL homeomorphism $\alpha : K \times I \to |M|$ such that

$$\begin{array}{ccccc} L \times I & \longrightarrow & (\partial K) \times I & \longrightarrow & K \times I \\ {\scriptstyle \beta \times 1} \downarrow & & & & \downarrow {\scriptstyle \alpha} \\ |N \times I| & \xrightarrow{|e|} & |\partial M| & \xrightarrow{|\iota|} & |M| \end{array}$$

commutes.

There is an embedding $L \to L \times 0 \to \partial(L \times I)$ and $L \times I$ is a codimension zero submanifold of $\partial(K \times I)$. In fact, there is a PL manifold X and cdz embeddings $\partial(L \times I) \to \partial X$, $X \to \partial(K \times I)$ such that

$$\begin{array}{ccc} \partial(L \times I) & \longrightarrow & L \times I \\ \downarrow & & \downarrow \\ X & \longrightarrow & \partial(K \times I) \end{array}$$

is a glueing in PL.

The following illustration may prove helpful. Let K be a square (fig. 1) with L being one side. Then $K \times I$ is a cube (fig. 2) and $L \times I$ is the shaded side facing the viewer. The two unshaded sides and the three hidden sides make up X.

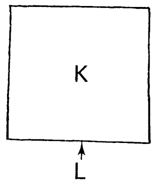

FIG. 1

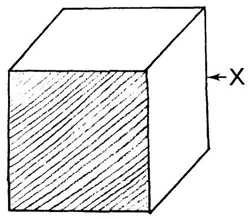

FIG. 2

There is an embedding of K in X so that (K, L) is a proper pair in $(X, \partial X)$. The manifold X has a C structure defined as follows. The cdz embedding e is complemented and we let

$$\begin{array}{ccc} \partial(N \times I) & \longrightarrow & N \times I \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ W & \longrightarrow & \partial M \end{array}$$

be a glueing exhibiting the complement. Then there is a PL homeomorphism $\gamma : X \to W$. Now the thickening (N, β) of L is a regular neighborhood for L, so by Proposition (1.1) we can find a C manifold V; cdz embeddings $\varepsilon : N \to \partial V$ and $V \to W$; and a PL isotopy H_t of $|W|$ rel $|\partial W|$ with the property that (V, N) is a regular neighborhood for the proper pair (K, L) in $(W, \partial W)$. But the inclusion $K \to |V|$ can be homotoped through embeddings rel L until it is a PL homeomorphism, since K is a codimension zero submanifold of its regular neighborhood $|V|$. Let $\lambda : K \to |V|$ denote such a PL homeomorphism. It is clear that

$$\begin{array}{ccc} L & \xrightarrow{\beta} & |N| \\ \downarrow & & \downarrow{\scriptstyle |\varepsilon|} \\ \partial K & \xrightarrow{\partial\lambda} & |\partial V| \end{array}$$

commutes, and it is easy to check that $(V \times I, \lambda \times 1)$ is equivalent rel $(L \times I)$ to (M, α) using the relative existence of collars, Axiom VII.

That the map $\mathscr{S}(K, \mathrm{rel}\, L) \to \mathscr{S}(K \times I, \mathrm{rel}\, L \times I)$ is $1-1$ follows by a similar argument. Details are omitted. ∎

One corollary of theorem I is

Corollary. — The group $\pi_r(PL/C)$ *is in* $1-1$ *correspondence with concordance classes of* C *structures on* S^r.

Remark. — Connected sum, which can be defined using our C-regular neighborhood theory, gives a group structure on $\mathscr{S}(S^r)$. This group structure is the same as the one coming from the correspondence with $\pi_r(PL/C)$ if $p \times S^r$ is the C structure on S^r corresponding to $0 \in \pi_r(PL/C)$.

There is a map $\eta_*^C \to H_*(BC; \mathbf{Z}/2\mathbf{Z})$ defined by sending M^n to

$$\nu_{M*}([M]) \in H_n(BC; \mathbf{Z}/2\mathbf{Z})$$

where $[M] \in H_n(|M|; \mathbf{Z}/2\mathbf{Z})$ is the fundamental class. It is easy to see that this map is well-defined. If BSC is defined as the pull-back

$$\begin{array}{ccc} BSC & \longrightarrow & BC \\ \downarrow & & \downarrow \\ BSPL & \longrightarrow & BPL \end{array}$$

then we get a similar map $\Omega_*^C \to H_*(BSC; Z)$.

Theorem II. — The map $\eta_*^C \to H_*(BC; \mathbf{Z}/2\mathbf{Z})$ *is monic.*

Remark. — One can also prove $\Omega_*^C \otimes \mathbf{Z}_{(2)} \to H_*(BSC; \mathbf{Z}_{(2)})$ is split monic.

Proof. — Let $X_n \subset BC$ be the inverse image of an $(n-1)$-skeleton for BPL under our map $BC \to BPL$. Then the universal bundle over BPL has a unique representative as an n-dimensional bundle when pulled back to X_n. If γ_n denotes this bundle, let $T(\gamma_n)$ denote its Thom space. We get a natural map $\Sigma T(\gamma_n) \to T(\gamma_{n+1})$, so we get a spectrum, which we denote by MC. We also get a map of spectra $MC \to MPL$.

We have a map $\eta_*^C \to \pi_*(MC)$ defined as follows. For a given C-manifold M, we can embed $|M^n|$ in S^{n+k}, k large. Then the map $|M| \to BC$ factors through X_{n+k} and so we get a map of spectra $T(\nu_M) \to MC$ where $T(\nu_M)$ is the Thom spectrum associated to the PL normal bundle for $|M|$. Since $T(\nu_M)$ has a reduction, a C structure on M^n defines an element in $\pi_n(MC)$. It is not hard to see that this element only depends on the unoriented bordism class of M, and hence we get a map $\eta_*^C \to \pi_*(MC)$. It is easy to see that this map is a homomorphism.

Lashof [LA] has given us another way to think about $\pi_*(MC)$. He shows that $\pi_*(MC)$ is the group of bordism classes of PL manifolds K and lifts of $\nu_K : K \to BPL$ up to BC. But by Theorem I b) this is the same as η_*^C. Moreover the map described one paragraph above gives the isomorphism.

(One can define MSC similarly and prove that $\Omega_*^C \to \pi_* MSC$ is an isomorphism.)

The reader should check that the map

$$\eta_*^C \to \pi_*(MC) \xrightarrow{\text{Hurewicz}} H_*(MC; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\text{Thom}} H_*(BC; \mathbf{Z}/2\mathbf{Z})$$

is the map we previously defined.

The following result is useful and easy to prove. Let X be a countable CW complex and let $f_1, f_2 : X \to BPL$ be weakly homotopic maps. Then the two Thom spectra constructed from f_1 and f_2, say MF_1 and MF_2, are equivalent by an equivalence h such that

$$\begin{array}{ccc} \pi_* MF_1 & \xrightarrow{h_*} & \pi_* MF_2 \\ & \searrow \quad \swarrow & \\ & \pi_* MPL & \end{array} \quad \text{and} \quad \begin{array}{ccc} H_*(MF_1; \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{h_*} & H_*(MF_2; \mathbf{Z}/2\mathbf{Z}) \\ & {\scriptstyle\text{Thom}}\searrow \quad \swarrow{\scriptstyle\text{Thom}} & \\ & H_*(X; \mathbf{Z}/2\mathbf{Z}) & \end{array}$$

commute. The proof is to observe that there are finite subcomplexes, X_i, of X with $X_0 \subset X_1 \subset \ldots \subset X$ and $X = \bigcup_{i=0}^{\infty} X_i$. We can construct a spectrum based on this decomposition of X and map it to both MF_1 and MF_2 by equivalences. Details are omitted.

This means that, since the square in Theorem I c) weakly homotopy commutes, we get a map of spectra $MC \wedge MO \to MC$ such that

$$\begin{array}{ccc} H_*(MC \wedge MO) & \rightarrow & H_*(MC) \\ \downarrow & & \downarrow \\ H_*(BC \times BO) & \longrightarrow & H_*(BC) \end{array}$$

commutes.

Then the map $MC = MC \wedge S^0 \to MC \wedge MO \to MC$ is an equivalence since

$$\begin{array}{ccccc} H_*(MC) & \longrightarrow & H_*(MC \wedge MO) & \longrightarrow & H_*(MC) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_*(BC) & \longrightarrow & H_*(BC \times BO) & \longrightarrow & H_*(BC) \end{array}$$

commutes, and the bottom composite is the identity.

Remark to the experts. — Via $\lim^1$ arguments one can show that MC is an MO-module spectrum and that MSC is an MSO-module spectrum.

But by work of Thom [Th] (and see [Ta]) there is a map of the Eilenberg-MacLane spectrum $H\mathbf{Z}_2$ to MO so we get that $\pi_* MC \to H_*(MC; \mathbf{Z}/2\mathbf{Z}) \to MO_*(MC) \to \pi_* MC$ is an isomorphism. Hence $\eta_*^C \to H_*(MC; \mathbf{Z}/2\mathbf{Z})$ is monic. ∎

Remark. — The proof that $\Omega_*^C \otimes \mathbf{Z}_{(2)} \to H_*(BSC; \mathbf{Z}_{(2)})$ is split monic is in [Ta]. Also one could prove that MC and $MSC_{(2)}$ are products of Eilenberg-MacLane spectra. Origins of Theorem II can be traced back to [B-L-P].

2. New structures from old.

In section one we built an elaborate theory but provided few examples. The case $\mathscr{C} = \text{Diff}$, $\mathscr{L} = \text{PL}$, and PC being piecewise differentiable is our one example so far. Here we wish to give a general process to construct a whole host of examples.

We start with a category of C-manifolds. We also assume given a countable set, Σ, of C structures on spheres. Hence an element in our set is an object Σ_r in $\mathscr{C}$, an object Γ_r in $\mathscr{L}$, a PC homeomorphism $u_r : \Gamma_r \to \Sigma_r$ and a PL homeomorphism $v_r : S^{n_r} \to |\Gamma_r|$. We then introduce the category of C-stratified sets with Σ—like singularities.

Specifically, we have a category $\mathscr{C}(\Sigma)$, a category $\mathscr{L}(\Sigma)$, and a class of $PC(\Sigma)$ homeomorphisms between the objects of $\mathscr{L}(\Sigma)$ and $\mathscr{C}(\Sigma)$. To describe them let us first fix an indexing set, $\mathscr{I}$, for Σ.

Then an object in $\mathscr{C}(\Sigma)$ consists of an object in $\mathscr{C}$, denoted L_0; objects in Diff, L_r, one for each $r \in \mathscr{I}$ and a cdz embedding $\amalg \beta_r : \amalg (\Sigma_r \times L_r) \to \partial L_0$. Notice this is possible only if all but finitely many of the L_r are empty. We denote this object by (L_0, L_r, β_r).

A map in $\mathscr{C}(\Sigma)$ between $(L_0^{(1)}, L_r^{(1)}, \beta_r^{(1)})$ and $(L_0^{(2)}, L_r^{(2)}, \beta_r^{(2)})$, is a map $f_0 : L_0^{(1)} \to L_0^{(2)}$ in $\mathscr{C}$, and smooth maps $f_r : L_r^{(1)} \to L_r^{(2)}$ such that

$$\begin{array}{ccc} \Sigma_r \times L_r^{(1)} & \xrightarrow{1 \times f_r} & \Sigma_r \times L_r^{(2)} \\ {\scriptstyle \beta_r^{(1)}}\downarrow & & \downarrow{\scriptstyle \beta_r^{(2)}} \\ \partial L_0^{(1)} & & \partial L_0^{(2)} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ L_0^{(1)} & \xrightarrow{\quad f_0 \quad} & L_0^{(2)} \end{array}$$

commutes for each $r \in \mathscr{I}$. Composition is defined by $(f_0, f_r) \circ (g_0, g_r) = (f_0 \circ g_0, f_r \circ g_r)$.

The category $\mathscr{L}(\Sigma)$ is defined analogously with objects (X_0, X_r, b_r) and a map (f_0, f_r) between $(X_0^{(1)}, X_r^{(1)}, b_r^{(1)})$ and $(X_0^{(2)}, X_r^{(2)}, b_r^{(2)})$ is a map $f_0 : X_0^{(1)} \to X_0^{(2)}$ in $\mathscr{L}$, and PL maps $f_r : X_r^{(1)} \to X_r^{(2)}$ such that

$$\begin{array}{ccc} \Gamma_r \times X_r^{(1)} & \xrightarrow{1 \times f_r} & \Gamma_r \times X_r^{(2)} \\ {\scriptstyle b_r^{(1)}}\downarrow & & \downarrow{\scriptstyle b_r^{(2)}} \\ \partial X_0^{(1)} & & \partial X_0^{(2)} \\ \downarrow & & \downarrow \\ X_0^{(1)} & \xrightarrow{\quad f_0 \quad} & X_0^{(2)} \end{array}$$

commutes for each $r \in \mathscr{I}$. We still require $\amalg b_r : \amalg (\Gamma_r \times X_r) \to \partial X_0$ to be a cdz embedding.

Since $\mathscr{C}(\Sigma)$ and $\mathscr{L}(\Sigma)$ are defined so similarly we shall mostly discuss $\mathscr{C}(\Sigma)$ and leave to the reader to produce the corresponding results for $\mathscr{L}(\Sigma)$.

The natural transformation $\mathscr{C}(\Sigma)$ to TOP is given as follows. There is a natural map $\rho : |\amalg (\Sigma_r \times L_r)| \to \amalg (c|\Sigma_r| \times |L_r|)$ defined as the composite

$$|\amalg (\Sigma_r \times L_r)| \to \amalg |(\Sigma_r \times L_r)| \xrightarrow{\amalg (|\pi_1| \times \pi_2)} \amalg (|\Sigma_r| \times |L_r|) \xrightarrow{\amalg (j \times 1)} \amalg (c|\Sigma_r| \times |L_r|)$$

where $j : |\Sigma_r| \to c|\Sigma_r|$ is the map which sends $|\Sigma_r|$ to the boundary of $c|\Sigma_r|$, the unreduced cone on $|\Sigma_r|$, in the usual fashion. Then, given an object $L = (L_0, L_r, \beta_r)$ in $\mathscr{C}(\Sigma)$, $|L|$ is a choice of pushout for

$$\begin{array}{ccc} |\amalg (\Sigma_r \times L_r)| & \xrightarrow{|\iota \circ (\amalg \beta_r)|} & |L_0| \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow \\ \amalg (c|\Sigma_r| \times |L_r| & \longrightarrow & |L| \end{array}$$

i.e. $|L| = |L_0| \cup \amalg (c|\Sigma_r| \times |L_r|)$.

The map $|\iota\circ(\amalg\beta_r)|$ is a cdz embedding of the topological manifold $|\amalg(\Sigma_r\times L_r)|$ into the boundary of the topological manifold $|L|$. The map ρ is also a cdz embedding into the boundary since $c|\Sigma_r|$ is just an (n_r+1)-disc. Hence $|L|$ is a topological manifold.

Given a map $f=(f_0, f_r)$

$$\begin{array}{ccc} |\amalg(\Sigma_r\times L_r^{(1)})| & \xrightarrow{|(\amalg 1\times f_r)|} & |\amalg(\Sigma_r\times L_r^{(2)})| \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\rho} \\ \amalg(c|\Sigma_r|\times|L_r^{(1)}|) & \xrightarrow{\amalg(1\times|f_r|)} & \amalg(c|\Sigma_r|\times|L_r^{(2)}|) \end{array}$$

and

$$\begin{array}{ccc} |\amalg(\Sigma_r\times L_r^{(1)})| & \xrightarrow{|\iota\circ(\amalg\beta_r^{(1)})|} & |L_0^{(1)}| \\ \Big\downarrow{\scriptstyle|\amalg(1\times f_r)|} & & \Big\downarrow{\scriptstyle|f_0|} \\ |\amalg(\Sigma_r\times L_r^{(2)})| & \xrightarrow{|\iota\circ(\amalg\beta_r^{(2)})|} & |L_0^{(2)}| \end{array}$$

both commute. Hence we get a unique map $|f|:|L^{(1)}|\to|L^{(2)}|$ making the map of squares into a commutative cube.

It is easily seen that this gives a natural transformation as does the similar definition for $\mathscr{L}(\Sigma)$, which now lands in PL.

A PC homeomorphism $f: X\to L$ is a homeomorphism $|X|\to|L|$ which is the pushout of a PC homeomorphism $f_0: X_0\to L_0$ and piecewise differentiable homeomorphisms $f_r: X_r\to L_r$. A collection (f_0, f_r) will give a map on the realizations if

$$\begin{array}{ccc} \Gamma_r\times X_r & \xrightarrow{u_r\times f_r} & \Sigma_r\times L_r \\ \Big\downarrow{\scriptstyle b_r} & & \Big\downarrow{\scriptstyle\beta_r} \\ \partial X_0 & \xrightarrow{\partial f_0} & \partial L_0 \end{array}$$

commutes for each $r\in\mathscr{I}$. Hence we require that a PC homeomorphism $f: X\to L$ be one with $|f|$ given by realizing a collection (f_0, f_r) as above.

Given $L=(L_0, L_r, \beta_r)$ we define ∂L as follows. Let

$$\begin{array}{ccc} \amalg(\Sigma_r\times\partial L_r) & \xrightarrow{\amalg(1\times\iota)} & \amalg(\Sigma_r\times L_r) \\ \Big\downarrow{\scriptstyle \iota\circ(\amalg\gamma_r)} & & \Big\downarrow{\scriptstyle \amalg\beta_r} \\ X & \xrightarrow{\quad\psi\quad} & \partial L_0 \end{array}$$

be a square exhibiting $\amalg\beta_r$ as a complemented cdz embedding. Then $\partial L=(X, \partial L_r, \gamma_r)$. The map $\partial L\to L$ is given by $\iota\circ\psi: X\to L_0$ and $\iota: \partial L_r\to L_r$.

In order to get $\mathscr{C}(\Sigma)$ to satisfy all of the axioms, we must require that $n_r>0$ for all $r\in\mathscr{I}$. This requirement will remain in force for the rest of the paper.

We have a natural transformation $\mathscr{C}\to\mathscr{C}(\Sigma)$ given by sending L to $(L, \varnothing, \beta_r)$ where $\beta_r: \Sigma_r\times\varnothing\to\partial L$ is the map $\Sigma_r\times\varnothing=\varnothing\to\partial L$. Since $n_r>0$, all points and circles in $\mathscr{C}(\Sigma)$ are in the image of this natural transformation, so our requirements on them are fulfilled. We leave to the reader the task of finishing the proof of

Proposition **(2.1)**. — *$\mathscr{C}(\Sigma)$ satisfies* I; *$\mathscr{L}(\Sigma)$ satisfies* I_{PL}.

Now if $L_\alpha=(L_0^{(\alpha)}, L_r^{(\alpha)}, \beta_r^{(\alpha)})$ is a finite collection of objects in $\mathscr{C}(\Sigma)$, let $\amalg L_\alpha$ be $(\amalg L_0^{(\alpha)}, \amalg L_r^{(\alpha)}, \amalg\beta_r^{(\alpha)})$. Define disjoint union similarly in $\mathscr{L}(\Sigma)$. The reader can check

Proposition **(2.2)**. — *$\mathscr{C}(\Sigma)$ and $\mathscr{L}(\Sigma)$ satisfy* II.

If we are given an object in $\mathscr{C}(\Sigma)$, say $(L_0, L_r, \beta_r)=L$ and a smooth manifold, M, then let $L\times M$ be given as follows. The objects are $L_1\times M$ and $L_r\times M$. The cdz embedding $\amalg(L_r\times M)\to\partial(L_0\times M)$ is the composite

$$\amalg(L_r\times M)\to(\amalg L_r)\times M\xrightarrow{(\amalg\beta_r)\times 1}\partial L_0\times M\to\partial(L_0\times M).$$

If (f_0, f_r) is a map in $\mathscr{C}(\Sigma)$ and g is a smooth map, $(f_0, f_r)\times g=(f_0\times g, f_r\times g)$.

We define $\pi_1: (L_0\times M, L_r\times M, \hat{\beta}_r)\to(L_0, L_r, \beta_r)$ by $L_0\times M\to L_0$ via π_1 and $L_r\times M\to L_r$ via π_1. The maps $|L_0\times M|\xrightarrow{\pi_2}|M|$ and $\amalg(c|\Sigma_r|\times|L_r\times M|)\to|M|$ agree on $|\amalg(\Sigma_r\times L_r)|$ so we get a pushout map $\pi_2: |L\times M|\to|M|$. With similar definitions in $\mathscr{L}(\Sigma)$ the reader can show

Proposition **(2.3)**. — *$\mathscr{C}(\Sigma)$ satisfies* III_{Diff} *and $\mathscr{L}(\Sigma)$ satisfies* III_{PL}.

It is easy to see that (f_0, f_r) is a $\mathscr{C}(\Sigma)$ homeomorphism if and only if f_0 is a $\mathscr{C}$ homeomorphism and f_r is a diffeomorphism for each $r\in\mathscr{I}$. For cdz embeddings we have

Definition. — A map (f_0, f_r) is a cdz embedding in $\mathscr{C}(\Sigma)$ if $f_0 : L_0 \to M_0$ is a cdz embedding in $\mathscr{C}$; $f_r : L_r \to M_r$ is a cdz embedding in Diff; and, in $|M_0|$

$$|\Sigma_r \times M_r| \cap |L_0| = |\Sigma_r \times L_r|.$$

With this definition it is not hard to prove

Proposition (**2.4**). — *$\mathscr{C}(\Sigma)$ satisfies* IV_{Diff}; *$\mathscr{L}(\Sigma)$ satisfies* IV_{PL}.

We also have

Proposition (**2.5**). — *$\mathscr{C}(\Sigma)$ and $\mathscr{L}(\Sigma)$ satisfy* V.

The proof is a tedious check that the following definition is correct. Given $e_0^{(i)} : N_0 \to M_0^{(i)}$ and $e_r^{(i)} : N_r \to M_r^{(i)}$ such that $(e_0^{(i)}, e_r^{(i)})$ is a cdz embedding $e_i : N \to M$, $i = 1, 2$, let M_0 be $M_0^{(1)}$ glued to $M_0^{(2)}$ along N_0, and let M_r be $M_r^{(1)}$ glued to $M_r^{(2)}$ along N_r. The map $\amalg \beta_r : \amalg(\Sigma_r \times M_r) \to \partial M_0$ is the restriction to the boundary of the maps

$$\amalg(\Sigma_r \times M_r^{(1)}) \to \partial M_0^{(1)} \to M_0^{(1)} \quad \text{and} \quad \amalg(\Sigma_r \times M_r^{(2)}) \to \partial M_0^{(2)} \to M_0^{(2)}.$$

It must be a cdz embedding by IV 3).

Next we show

Proposition (**2.6**). — *$\mathscr{C}(\Sigma)$ and $\mathscr{L}(\Sigma)$ satisfy* VI.

Proof. — First show that under the hypothesis our cdz embedding (f_0, f_r) has the property that $f_r : N_r \to M_r$ and $f_0 : N_0 \to M_0$ are complemented. This involves some elementary set theory, properties of smooth (or PL) manifolds, and Axioms I, IV, V and VI for $\mathscr{C}$ (or $\mathscr{L}$).

Let

$$\begin{array}{ccc} B_r & \longrightarrow & N_r \\ \downarrow & & \downarrow \\ X_r & \longrightarrow & M_r \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B_0 & \longrightarrow & N_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & M_0 \end{array}$$

display the complements. Use elementary set theory to get maps $|\amalg(\Sigma_r \times X_r)| \to |\partial X_0|$ and $|\amalg(\Sigma_r \times B_r)| \to |\partial B_0|$ giving a commutative cube

$$\begin{array}{ccc} |\amalg(\Sigma_r \times B_r)| & \longrightarrow & |\amalg(\Sigma_r \times N_r)| \\ \downarrow & & \downarrow \\ |\amalg(\Sigma_r \times X_r)| & \longrightarrow & |\amalg(\Sigma_r \times M_r)| \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} |B_0| & \longrightarrow & |N_0| \\ \downarrow & & \downarrow \\ |X_0| & \longrightarrow & |M_0| \end{array}$$

Use Axiom IV to show that there are cdz embeddings $\amalg(\Sigma_r \times X_r) \to \partial X_0$ and $\amalg(\Sigma_r \times B_r) \to \partial B_0$ realizing the above maps. Check that the maps $(B_0, B_r) \to \partial N$; $(B_0, B_r) \to \partial(X_0, X_r)$; and $(X_0, X_r) \to (M_0, M_r)$ are cdz embeddings. ∎

The following is not hard to show.

Proposition **(2.7)**. — *$\mathscr{C}(\Sigma)$ and $\mathscr{L}(\Sigma)$ satisfy* VII.

Proof (very sketchy). — Fix notation as in the statement of Axiom VII. Let $M = (M_0, M_r, \beta_r)$ and $N = (N_0, N_r, \alpha_r)$. Since the axiom is true in Diff and PL we can get the embedding defined correctly on the $N_r \times [0, 1]$. This then gives a collar on $|\partial N_0| \cap |\partial M_0|$ union $\amalg(N_r \times [0, 1])$, which we extend to $\partial N_0 \times [0, 1]$ using Axiom VII. It is then easy to check that the map $(N_0 \times [0, 1], N_r \times [0, 1]) \to (M_0, M_r)$ is a cdz embedding. ∎

We also have

Proposition **(2.8)**. — *$\mathscr{L}(\Sigma)$ satisfies* VIII.

Proof. — Inside $|M|$ we have submanifolds (cone point)$\times|M_r|$. Along $|N|$, ∂Q is already transverse to (cone point)$\times|M_r|$, so by an isotopy rel $|\partial M|$ we can assume that ∂Q is transverse to (cone point)$\times|M_r|$. By a further isotopy rel $|\partial M|$ we can in fact assume $\partial Q \cap (c|\Sigma_r| \times |M_r|) = c|\Sigma_r| \times Q_r$ for some PL manifolds Q_r. Let Q_0 denote the closure of $Q - (\amalg(c|\Sigma_r| \times |M_r|))$.

Use Axiom VIII to extend the $\mathscr{L}$ structure N_0 glued to $\amalg(\Sigma_r \times Q_r)$ along $\amalg(\Sigma \times \partial N_r)$, on ∂Q_0 to all of Q_0. ∎

We now proceed to the PC(Σ) homeomorphisms.

Notation. — A collection of maps (f_0, f_r), where $f_0 : X_0 \to M_0$, $f_r : X_r \to M_r$ is called a PC(Σ) *homeomorphism* provided f_0 is a PC homeomorphism; the f_r are piecewise differentiable homeomorphisms; and

$$\begin{array}{ccc} \Sigma_r \times X_r & \xrightarrow{1 \times f_r} & \Sigma_r \times M_r \\ \downarrow & & \downarrow \\ \partial X_0 & \xrightarrow{\partial f_0} & \partial M_0 \end{array}$$

commutes, where for notational convenience we replace u_r by 1.

Axioms PC(Σ) 1, PC(Σ) 2, and PC(Σ) 3 are easily seen to be satisfied.

To see that PC(Σ) 4 is satisfied we first fix our notation. Let

$$(f_0, f_r) : (X_0, X_r, \gamma_r) \to \partial(M_0, M_r, \beta_r)$$

be our PC(Σ)-homeomorphism. Hence we have piecewise differentiable homeomorphisms $f_r : X_r \to \partial M_r$ which we can extend to piecewise differentiable homeomorphisms $g_r : Y_r \to M_r$ such that

$$\begin{array}{ccc} X_r & \xrightarrow{f_r} & \partial M_r \\ {\scriptstyle h_r}\downarrow & & \downarrow \\ Y_r & \xrightarrow{g_r} & M_r \end{array}$$

commutes, where $h_r : X_r \to \partial Y_r$ is a PL homeomorphism. Let

$$\begin{array}{ccc} \amalg \Sigma_r \times \partial M_r & \longrightarrow & \amalg \Sigma_r \times M_r \\ \downarrow & & \downarrow \\ W & \longrightarrow & \partial M_0 \end{array}$$

display the complemented cdz embedding $\amalg \beta_r$. Then $f_0 : X_0 \to W$ is a PC homeomorphism. In $\mathscr{L}$, form the glueing

$$\begin{array}{ccc} \amalg (\Sigma_r \times X_r) & \longrightarrow & \amalg (\Sigma_r \times Y_r) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Q \end{array}$$

and note that the pushout map $|Q| \to |\partial M_0|$ is a PC homeomorphism. Hence we can find an object Y_0 in $\mathscr{L}$; an $\mathscr{L}$ homeomorphism $\hat{h}_0 : Q \to \partial Y_0$; and a PC homeomorphism $g_0 : Y_0 \to M_0$.

The composite $\amalg (\Sigma_r \times Y_r) \to Q \to \partial Y_0$ is a cdz embedding so we have an object, Y, and (g_0, g_r) is a PC(Σ) homeomorphism $Y \to M$. The boundary of Y is (X_0, X_r, γ_r) where the map $X \to Y$ is given by $X_0 \to Q \to \partial Y_0 \to Y_0$ and $X_r \to \partial Y_r \to Y_r$. ∎

The verification of PC(Σ) 5 is notationally more complex but the same idea works. First fix things up on the $X_r \to M_r$ portion using the properties of Diff and PL and then note that we can apply PC 5 to fix up the $X_0 - M_0$ portion. Axioms PC(Σ) 6 follows from a similar line of argument. Details are left to the reader.

Hence we have the category of $\mathscr{C}(\Sigma)$-*manifolds*; namely A $\mathscr{C}(\Sigma)$-manifold is a triple (X, M, f) with $X \in \text{ob}\, \mathscr{L}(\Sigma)$, $M \in \text{ob}\, \mathscr{C}(\Sigma)$ and $f: X \to M$ is a PC(Σ)-homeomorphism, and the maps between $\mathscr{C}(\Sigma)$-manifolds defined in the usual way. It is not hard to prove the countability axiom and so we have a classifying space $BC(\Sigma)$; a map $BC(\Sigma) \to BPL$; and bordism theories $\eta_*^{C(\Sigma)}$ and $\Omega_*^{C(\Sigma)}$ with a monic map

$$\eta_*^{C(\Sigma)} \to H_*(BC(\Sigma); \mathbf{Z}/2\mathbf{Z}).$$

The natural transformation $\mathscr{C} \to \mathscr{C}(\Sigma)$ that we defined extends to a natural transformation of the category of C-manifolds to the category of $C(\Sigma)$-manifolds. The relation between these two categories is given by

Theorem III. — There are exact sequences

$$\dots \longrightarrow \eta_*^{C} \xrightarrow{i_*} \eta_*^{C(\Sigma)} \xrightarrow{\tau} \bigoplus_{r \in \mathscr{I}} \eta_{*-(n_r+1)} \xrightarrow{r} \eta_{*-1}^{C} \longrightarrow \dots$$

$$\dots \longrightarrow \Omega_*^{C} \xrightarrow{i_*} \Omega_*^{C(\Sigma)} \xrightarrow{\tau} \bigoplus_{r \in \mathscr{I}} \Omega_{*-(n_r+1)} \xrightarrow{r} \Omega_{*-1}^{C} \longrightarrow \dots$$

where η_*, Ω_* denote the smooth unoriented, oriented bordism groups.

The maps are given as follows. The map i_* is just the map induced by the natural transformation of C-manifolds to $C(\Sigma)$-manifolds. The map τ is defined by

$$\tau(M_0, M_r, \beta_r) = (\dots, [M_r], \dots) \in \bigoplus_{r \in \mathscr{I}} \eta_{*-(n_r+1)}.$$

If (M, M_r, β_r) is orientable so are all the M_r, so the same formula defines both τ's. The map r is defined in both cases by the formula $r(\dots, [M_r], \dots) = \sum_{r \in \mathscr{I}} [\Sigma_r \times M]$. The sum is finite. We leave it for the reader to check that our maps are well-defined on bordism classes and that our sequences are exact.

The following result has some amusing consequences. Let $I(C)$ denote the kernel of the map $H^*(BPL; \mathbf{Z}/2\mathbf{Z}) \to H^*(BC; \mathbf{Z}/2\mathbf{Z})$. It is easy to check that this kernel is independent of which map $BC \to BPL$ we choose in the unique weak homotopy class. We have similarly defined ideals $I(C(\Sigma))$ and $I(o)$ in $H^*(BPL; \mathbf{Z}/2\mathbf{Z})$.

Theorem IV. — $I(C) . I(o) \subset I(C(\Sigma))$.

Proof. — If $x \in H^*(BPL; \mathbf{Z}/2\mathbf{Z})$ does not vanish in $H^*(BC(\Sigma))$, then we can find a $C(\Sigma)$-manifold M such that, under the map $\nu_M : |M| \to BC(\Sigma)$, x pulls back non-zero (in general the dimension of M is much larger than that of x). Hence we need only show that $y.z$ pulls back to zero in every $C(\Sigma)$-manifold when $y \in I(C)$, $z \in I(o)$.

But $|M| = |M_0| \cup \coprod (c|\Sigma_r| \times |M_r|)$. The class y restricts to zero on $|M_0|$ and the class z restricts to zero in $\coprod (c|\Sigma_r| \times |M_r|)$. Hence $y.z$ vanishes on $|M|$. ■

We also have the following useful result.

Theorem V. — Let a be the minimum dimension of any element of Σ. By hypothesis $a \geq 1$. Then $\pi_r(C(\Sigma)/C) = 0$ for $0 \leq r \leq a-1$.

Proof. — We have an exact sequence:

$$\pi_{r+1}(\mathrm{PL/C}) \to \pi_{r+1}(\mathrm{PL/C}(\Sigma)) \to \pi_r(\mathrm{C}(\Sigma/\mathrm{C}) \to \pi_r(\mathrm{PL/C}) \to \pi_r(\mathrm{PL/C}(\Sigma)).$$

This takes just a minute to see. Since

$$\begin{array}{ccc} \mathrm{BC} & \to & \mathrm{BC}(\Sigma) \\ & \searrow \quad \swarrow & \\ & \mathrm{BPL} & \end{array}$$

weakly homotopy commutes we do not have an obvious map $\mathrm{PL/C} \to \mathrm{PL/C}(\Sigma)$. But if $\mathrm{K} \subset \mathrm{PL/C}$ is any finite complex we do get a map $\mathrm{K} \to \mathrm{PL/C}(\Sigma)$. Since PL/C is a countable complex, the homotopy extension theorem implies that we have a map $\mathrm{PL/C} \to \mathrm{PL/C}(\Sigma)$ and a diagram

$$\begin{array}{ccccc} \mathrm{PL/C} & \longrightarrow & \mathrm{BC} & \longrightarrow & \mathrm{BPL} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PL/C}(\Sigma) & \longrightarrow & \mathrm{BC}(\Sigma) & \longrightarrow & \mathrm{BPL} \end{array}$$

such that the squares weakly homotopy commute. This gives our exact sequence.

But now a moment's pause convinces one that if M is a $\mathrm{C}(\Sigma)$-manifold of dimension $\leq a$, then M is actually a C-manifold because if $\mathrm{M}_r \neq \varnothing$ for any r, $\dim \mathrm{M} = ((\dim \Sigma_r) + 1) + \dim \mathrm{M}_r \geq a + 1$. ■

Now that we have our general setup we can iterate the construction. Start with Tri and cone some exotic spheres to get BC_1; cone some C_1 exotic spheres to get BC_2; and continue. Clearly we can get lots of theories this way. Levitt [Le] showed that, if at each stage the exotic spheres one coned contained precisely one representative from each non-trivial concordance class of exotic spheres, the limit space $\mathrm{BC}_1 \to \mathrm{BC}_2 \to$ was BPL. We wish to consider this construction in general.

We are to be given a countable sequence of categories $\mathrm{C}_0 = \mathrm{Tri}, \mathrm{C}_1, \mathrm{C}_2, \ldots$ where $\mathrm{C}_{k+1} = \mathrm{C}_k(\Sigma)$ for some choice of exotic spheres Σ. There is our natural transformation $\lambda : \mathrm{C}_k \to \mathrm{C}_{k+1}$, which we recall has the property that any C_{k+1} map $f : \lambda(\mathrm{M}_1) \to \lambda(\mathrm{M}_2)$ is of the form $\lambda(g)$ for a (unique) C_k map $g : \mathrm{M}_1 \to \mathrm{M}_2$. Define a limit category C as follows. An object in C is just an object in C_k for some k. Given two objects M_0 and M_1 in C suppose $\mathrm{M}_0 \in \mathrm{ob}\, \mathrm{C}_{k_0}$ and $\mathrm{M}_1 \in \mathrm{ob}\, \mathrm{C}_{k_1}$. A morphism in C, $f : \mathrm{M}_0 \to \mathrm{M}_1$ is a morphism in C_k from $\lambda^{k-k_1}(\mathrm{M}_0) \to \lambda^{k-k_0}(\mathrm{M}_1)$ where $k = \max(k_0, k_1)$ and λ^0 is the identity. To compose two morphisms, apply λ iterated until all three objects are in one C_k and then compose there. It is easy to check that we have a category.

It is tedious, but basically not hard to construct natural transformations $\mathrm{C}_k \to \mathrm{C}$ and $\mathrm{C} \to \mathrm{PL}$ such that $\mathrm{C}_k \to \mathrm{C} \to \mathrm{PL}$ is our natural transformation. Moreover one can prove that C satisfies Axioms I_{PL}, II, $\mathrm{III}_{\mathrm{Tri}}$, $\mathrm{IV}_{\mathrm{Tri}}$, V, VI, and VII as well as Proposition (1.1).

Hence we get a classifying space BC and maps $\mathrm{BC}_k \to \mathrm{BC}$ and $\mathrm{BC} \to \mathrm{BPL}$. The composite $\begin{array}{ccc} BC_k & \to & BC_{k+1} \\ & \searrow \swarrow & \\ & BC & \end{array}$ weakly homotopy commutes. The reader can easily show that, for any finite simplicial complex, T, $\lim_{k\to\infty} [T, BC_k] = [T, BC]$. In particular $H_*(BC; G) = \lim_{k\to\infty} H_*(BC_k; G)$; $\pi_* BC = \lim_{k\to\infty} \pi_*(BC_k)$; etc. We have a C bordism. The map $\eta_*^C \to H_*(BC; \mathbf{Z}/2\mathbf{Z})$ is monic, and $\eta_*^C = \lim_{k\to\infty} \eta_*^{C_k}$. We also have the result that $\pi_r(PL/C)$ is the set of concordance classes of C structures on S^r and that $\pi_r(PL/C) = \lim_{k\to\infty} \pi_r(PL/C_k)$.

Theorem IV can be used to prove a nice result. Levitt [Le] has proved that every compact PL manifold has a C structure for a certain limit category described above. We have a similar result in Section 3. However, we have the following

Proposition (**2.9**). — *Let* C_r *be any category constructed from* Tri *by r-iterates of our construction. Then there is a* PL *manifold* $M^{8(r+1)}$ *such that no manifold which is unoriented* PL *bordant to* M *has a* C_r *structure. The manifold* M *is independent of* C_r.

Proof. — By calculations of Brumfiel, Madsen, and Milgram [B-M-M], there is a unique element in $H^8(BPL; \mathbf{Z}/2\mathbf{Z})$ which goes to zero in $H^8(BO; \mathbf{Z}/2\mathbf{Z})$. This element, x, generates a polynomial algebra and, by Theorem IV, x^{r+1} must vanish on any C_r-manifold. But results of Browder, Liulevicius, and Peterson [B-L-P] imply that there is an $8(r+1)$ dimensional manifold, M, such that $\langle x^{r+1}, [M]\rangle$ is a non-zero characteristic number. ■

3. A-structures and PL manifolds

In this section we discuss in some detail the following special case of our general construction in Section 2. Let $A_0 = \text{Tri}$. Define A_{k+1} inductively from A_k by $A_{k+1} = A_k(\Sigma)$ where Σ is defined by: Σ contains one representative, Σ_r, from each non-trivial concordance class of A_k structures on a sphere provided that Σ_r is an unoriented A_k boundary.

By Theorem V and induction $\pi_r(PL/A_k) = 0$ for $0 \leq r \leq 6$. Hence Σ satisfies the hypothesis that $\dim \Sigma_r > 0$ for all $r \in \mathscr{I}$, and since Σ is a subset of $\bigoplus \pi_*(PL/A_k)$, Σ is countable by induction.

We let A denote the limit category. The following result is crucial to our analysis.

Proposition (**3.1**). — *The natural map* $\pi_r(PL/A) \to \eta_r^A$ *is monic.*

Proof. — The map just assigns to the A structure, M, on S^r the underlying unoriented bordism class of M. There is a similar map $\pi_r(PL/A_k) \to \eta_r^{A_k}$ for each k and the first map is just the limit over k of the latter maps.

Consider an element N in the kernel of the map $\pi_r(\mathrm{PL}/\mathrm{A}_k) \to \eta_r^{\mathrm{A}_k}$. Then we are permitted to cone an exotic sphere concordant to N in A_{k+1} manifold theory. Hence N is in the kernel of $\pi_r(\mathrm{PL}/\mathrm{A}_k) \to \pi_r(\mathrm{PL}/\mathrm{A}_{k+1})$.

In fact, one can prove that

$$\ker(\pi_r(\mathrm{PL}/\mathrm{A}_k) \to \eta_r^{\mathrm{A}_k}) = \ker(\pi_r(\mathrm{PL}/\mathrm{A}_k) \to \pi_r(\mathrm{PL}/\mathrm{A}_{k+1})).$$

An easy corollary of Theorem III is that $\eta_*^{\mathrm{A}_k} \to \eta_*^{\mathrm{A}_{k+1}}$ is monic. Hence $\eta_*^{\mathrm{A}_k} \to \eta_*^{\mathrm{A}}$ is monic. The proposition follows by a simple limit argument. ∎

An easy corollary of Proposition (3.1) is that $\pi_r(\mathrm{PL}/\mathrm{A})$ is a $\mathbf{Z}/2\mathbf{Z}$-vector space. It is also clear that

$$\begin{array}{ccc} \pi_r(\mathrm{PL}/\mathrm{A}) & \longrightarrow & \eta_r^{\mathrm{A}} \\ \downarrow & & \downarrow \\ \mathrm{H}_r(\mathrm{PL}/\mathrm{A}; \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & \mathrm{H}_r(\mathrm{BA}; \mathbf{Z}/2\mathbf{Z}) \end{array}$$

commutes, where the maps going clockwise have been described already and where $\pi_r(\mathrm{PL}/\mathrm{A}) \to \mathrm{H}_r(\mathrm{PL}/\mathrm{A}; \mathbf{Z}/2\mathbf{Z})$ is just the Hurewicz map followed by mod 2 reduction and the bottom horizontial map is induced by the map $\mathrm{PL}/\mathrm{A} \to \mathrm{BA}$.

But Proposition (3.1) and Theorem II show that the map

$$\pi_r(\mathrm{PL}/\mathrm{A}) \to \mathrm{H}_r(\mathrm{BA}; \mathbf{Z}/2\mathbf{Z})$$

is monic: hence split monic!

We digress briefly. Let E be a product of Eilenberg-MacLane spaces: so E is determined by $\pi_*\mathrm{E}$. If X is any space, then given any homomorphism $\lambda : \mathrm{H}_*(\mathrm{X}; \mathbf{Z}) \to \pi_*\mathrm{E}$ we can find a map $f : \mathrm{X} \to \mathrm{E}$ such that

$$\begin{array}{ccc} \mathrm{H}_*(\mathrm{X}; \mathbf{Z}) & \longrightarrow & \mathrm{H}_*(\mathrm{E}; \mathbf{Z}) \\ & {}_{\lambda}\searrow \quad \nearrow_{\text{Hurewicz}} & \\ & \pi_*\mathrm{E} & \end{array}$$

commutes. We now return to our analysis of the map $\mathrm{PL}/\mathrm{A} \to \mathrm{BA}$.

If E is a product of Eilenberg-MacLane spaces such that we have an isomorphism $\varphi : \pi_*(\mathrm{PL}/\mathrm{A}) \to \pi_*\mathrm{E}$, let $\lambda : \mathrm{H}_*(\mathrm{BA}; \mathrm{Z}) \to \pi_*\mathrm{E}$ be the composite

$$\mathrm{H}_*(\mathrm{BA}; \mathbf{Z}) \longrightarrow \mathrm{H}_*(\mathrm{BA}; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\text{splitting}} \pi_*(\mathrm{PL}/\mathrm{A}) \xrightarrow{\varphi} \pi_*\mathrm{E}.$$

Construct $f : \mathrm{BA} \to \mathrm{E}$ as above.

It is easy to show that $\pi_*(\mathrm{PL}/\mathrm{A}) \to \pi_*(\mathrm{BA}) \to \pi_*\mathrm{E}$ is just φ. Hence PL/A is a product of Eilenberg-MacLane spaces and we have a map $\mathrm{BA} \to \mathrm{PL}/\mathrm{A}$ splitting the inclusion of a fibre $\mathrm{PL}/\mathrm{A} \to \mathrm{BA}$. The map $\mathrm{BA} \to \mathrm{BPL} \times \mathrm{PL}/\mathrm{A}$ is easily seen to be a homotopy equivalence. In particular we have

Theorem VI. — *Every compact* PL *manifold has an* A *structure.*

Remark. — One can even show that any paracompact PL manifold has an A structure by writing it as a union of compact PL manifolds. We refrain.

For amusement we finish this section with a complete analysis of PL/A. All that remains is to compute $\pi_*(\mathrm{PL}/\mathrm{A})$ and we have

Proposition **(3.2)**. — *Since* $\pi_r(\mathrm{PL}/\mathrm{A})$ *is a* $\mathbf{Z}/2\mathbf{Z}$*-vector space it is determined by its dimension* δ_r. *We have*

$$\delta_r = \begin{cases} 0 & 0 \leq r \leq 7 \\ 26 & r = 8 \\ \textit{infinite but countable} & r \geq 9 \end{cases}$$

Proof. — That $\delta_r = 0$ for $r \leq 6$ follows as we saw from Theorem V. For $r = 7$, any exotic A_k sphere must come from A_0 and hence is coned in A_1. Therefore $\pi_7(\mathrm{PL}/\mathrm{A}_1) = 0$, so Theorem V and induction show that $\pi_7(\mathrm{PL}/\mathrm{A}_k) = 0$ for $k \geq 1$.

Now suppose that $\pi_r(\mathrm{PL}/\mathrm{A}_k)$ has an element, Σ, whose image in $\Omega_r^{\mathrm{A}_k}$ has infinite order. Then, in $\pi_{r+1}(\mathrm{PL}/\mathrm{A}_{k+1})$ we have an element whose image in $\Omega_{r+1}^{\mathrm{A}_{k+1}}$ has infinite order and we further have that the image of $\pi_{r+1}(\mathrm{PL}/\mathrm{A}_{k+1}) \to \eta_{r+1}^{\mathrm{A}_{k+1}}$ is an infinite dimensional subpace.

To prove this observe that if Σ does not bound in $\eta_r^{\mathrm{A}_k}$, then 2Σ does, so we can assume that Σ bounds in $\eta_r^{\mathrm{A}_k}$ but has infinite order in $\Omega_r^{\mathrm{A}_k}$.

Now in the passage to A_{k+1} we can assume that we cone $n\Sigma$ for all $n \neq 0$. Consider a set of integers $\{r_1, r_2, \ldots\}$ such that all but finitely many of the r_n are zero and such that $\sum_{n=1}^{\infty} n r_r = 0$. Then, if $r = \sum_{n=1}^{\infty} |r_n|$, we have that $\amalg r_n(n\Sigma)$ bounds an A_k manifold, W, PL homeomorphic to an r-holed sphere. Hence the A_{k+1} manifold $(\mathrm{W}, \mathrm{P}_n, \beta_n)$, where P_n is a set of points of cardinality r, is PL homeomorphic to S^{r+1}. See figure 3.

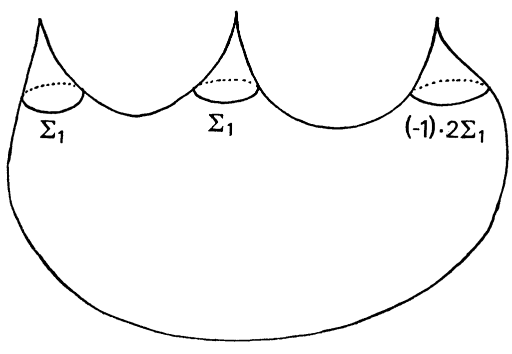

Fig. 3

In the exact sequence of Theorem III, $\tau(W, P_n, \beta_n)$ is $(r_1, r_2, \ldots) \bmod 2$ or $(r_1, r_2, \ldots)$. In either case it is easy to choose the r_n's to prove the result.

Now in dimension 8 we can still do the same construction. Since

$$\pi_7(PL/A_0) = \mathbf{Z}/28\mathbf{Z}$$

there are 27 non-trivial concordance classes of exotic 7-spheres. We can form an 8-sphere by choosing positive integers $r_1, r_2, \ldots, r_{27}$ such that $\sum_{n=1}^{27} nr_n \equiv 0 \ (28)$. Hence we may choose $r_2, \ldots, r_{27}$ arbitrarily and then r_1 is forced. All this arithmetic is mod 28. If some $r_n \neq 0$, the resulting sphere has infinite order in $\Omega_8^{A_k}$. Moreover, the image of $\pi_8(PL/A_1) \to \eta_8^{A_1}$ has dimension at least 26. Since any 8-dimensional A_1 sphere has only point singularities, the image has dimension precisely 26. Since $\pi_8(PL/A) \to \eta_8^A$ and $\eta_8^{A_1} \to \eta_8^A$ are monic, $\pi_8(PL/A)$ has dimension at least 26.

But since $\pi_r(PL/A_k) = 0$ for $0 \leq r \leq 7$ and $k \geq 1$, by Theorem V we see that $\pi_8(PL/A)$ has dimension exactly 26.

Since $\pi_8(PL/A_1)$ has elements of infinite order in $\Omega_8^{A_1}$, our earlier remarks show that the dimension of $\pi_r(PL/A)$ is infinite for $r > 8$. ∎

4. Remarks on algebraic varieties

From the work of Akbulut and King [A-K] it follows that the interior of any compact PL manifold with A structure is PL homeomorphic to a real algebraic variety. If we start with a smooth manifold Nash [N] and Tognoli [To] showed that the variety could be chosen non-singular. Clearly this is not possible for a non-smooth manifold, but one gets good control on the singularity set. The variety one associates to an A_k manifold has the property that the singular set (of the singular set (... (of the variety))) $(k+1$ times) is empty.

The relationship between all varieties homeomorphic to PL manifolds and those coming from A structures is less clear. In some sense, A structures correspond to " as non-singular as possible " varieties, since, for example, we have permitted no cones over the honest sphere, an operation which introduces non-essential singular points. A conjecture which would make the above precise is

Conjecture I. — If a PL *manifold is* PL *homeomorphic to a variety* V *for which*

$$\mathrm{Sing}(\mathrm{Sing}(\ldots(\mathrm{Sing}(V)))) = \varnothing$$

for $k+1$ *repetitions of* Sing, *then the* PL *manifold has an* A_k *structure.*

Not as good but perhaps easier would be to show that the manifold $M^{8(k+1)}$ of (2.9) is not PL homeomorphic to a variety V with $\mathrm{Sing}^{k+1}(V) = \varnothing$.

Since all PL manifolds are varieties one looks around for other natural objects which might be varieties. Integral homology manifolds are an obvious class and one is tempted to try and use work of Maunder and Martin [M-M] to push the result through by an attack similar to the above.

One fails because the Cairns-Hirsch theorem is false. Put another way, there are homology cobordism bundles which are trivial but are not products. Still the result remains an attractive

Conjecture II. — Any (triangulated) $\mathbf{Z}_{(2)}$ *homology manifold is* PL *homeomorphic to a real algebraic variety.*

Hopefully there will be other uses for Theorem VI. In particular, any compact PL manifold M has a resolution to a smooth manifold as follows. Let X_k be an A_k manifold (M_0, M_r, β_r) with $M=|X_k|=|M_0|\cup\amalg|c\Sigma_r\times M_r|$. Choose A_{k-1} manifolds W_r with $\partial W_r=\Sigma_r$ and let X_{k-1} be the A_{k-1} manifold given by the glueing, in A_{k-1} of

$$\begin{array}{ccc} \amalg(\Sigma_r\times M_r) & \longrightarrow & M_0 \\ \downarrow & & \downarrow \\ \amalg(W_r\times M_r) & \longrightarrow & X_{k-1} \end{array}$$

i.e. $|X_{k-1}|=|M_0|\cup\amalg|W_r\times M_r|$. There is a PL map $\pi:|X_{k-1}|\to|X_k|$. The map π is a PL homeomorphism on $|M_0|$ and on $|\amalg W_r\times M_r|$ has the form

$$|\amalg(W_r\times M_r)|\to\amalg(|W_r|\times|M_r|)\xrightarrow{\rho_r\times 1}\amalg(c|\Sigma_r|\times|M_r|)\to|X_k|,$$

where $\rho_r:|W_r|\to c|\Sigma_r|$ is a fixed PL map extending the identity on the boundaries. In fact by [A-K] if one wishes one can modify the W_i's so that a spine of W_i is the union of transversally intersecting closed (empty boundary) A_{k-1} submanifolds $\bigcup_j L_{ij}$ with π collapsing each $L_{ij}\times M_i$ onto M_i. Clearly we can iterate this process to get a resolution sequence

$$\widetilde{M}=|X_0|\xrightarrow{\pi}|X_1|\xrightarrow{\pi}\dots\xrightarrow{\pi}|X_k|=M.$$

Since $X_0\in\mathrm{Tri}$, $|X_0|$ is a smooth manifold. Furthermore it is clear that $\pi:\widetilde{M}\to M$ collapses each $W_i\times M_i$ onto M_i in some order.

Now we can make a choice of the W_r and ρ_r once and for all. There is some choice involved in a pushout which got us X_{k-1}. It is easy to prove that any two choices of X_{k-1} are concordant (even in the smooth case you may or may not have corners as you choose so you can do no better than concordance). Hence the resolution is well-defined up to concordance once some universal choices have been made.

Notice that even if the PL manifold M is orientable, it is quite likely that the smooth manifold $\widetilde{M}$ will not be. Notice also that if $\widetilde{M}$ does happen to be orientable then the map $\widetilde{M}\to M$ will have degree one. Since Brumfiel [Br] has shown that there are oriented PL manifolds with no degree one map from a smooth manifold, there is no hope of a general oriented resolution theorem. One can of course use the work in the first two sections to construct a space BSA and a map $BSA \to BSPL$ such that an oriented PL manifold M has an oriented resolution if and only if $\nu_M : M \to BSPL$ lifts to BSA, but the above remarks show that honest obstructions exist to making the lift.

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