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THE WHITEHEAD GROUP OF A POLYNOMIAL EXTENSION

by H. BASS, A. HELLER and R. G. SWAN (1)

§ 1. Introduction.

This paper is a sequel to " K-theory and Stable Algebra " [1], hereafter referred to as [K]. The object is to study the behavior of the functor K^1, constructed in [K, § 12], under polynomial and related extensions. The analogous problem for its companion, K^0, was handled already by Grothendieck (see [2, Prop. 8] or [6, § 9]), and it will be convenient for us first to recall his result.

If A is a ring, $K^0(A)$ is the " Grothendieck group " of finitely generated projective left A-modules (see [K, § 12] or [2, § 4] or § 3 below). Before stating Grothendieck's theorem we need a definition: A is *left regular* if A is left noetherian and if every finitely generated left A-module has finite homological dimension.

Theorem (Grothendieck). — If A *is left regular and t is an indeterminate, then*

$$K^0(A) \to K^0(A[t])$$

is an isomorphism.

Actually, Grothendieck's proof is in an algebro-geometric setting, so it applies here only when A is commutative. We give here a proof of a generalization of it in the above form to graded rings (Theorem 6, § 5). The argument here is inspired by Serre's discussion of the theorem in [6, § 9].

Let T be an infinite cyclic group with generator t, and let $A[T] = A[t, t^{-1}]$ denote the group ring over A. It is immediate from Grothendieck's methods that $K^0(A[t]) \to K^0(A[t, t^{-1}])$ is also an isomorphism. See § 5 for details.

Hilbert's Syzygy Theorem (see, e.g. [5, Theorem 1.5]) says that $A[t]$ is left regular if A is, and it follows that $A[t, t^{-1}]$ is likewise, being a ring of quotients of $A[t]$. Hence an induction on n yields:

Corollary. — Let T *be a free abelian group with basis* $t_1, \ldots, t_n$ *and let* A *be a left regular ring. Then*

$$K^0(A) \to K^0(A[t_1, \ldots, t_n]) \to K^0(A[T])$$

are isomorphisms.

To construct the functor K^1 one considers automorphisms α of projective left A-modules (always finitely generated). If T is an infinite cyclic group, such α can be

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viewed as A[T]-modules in an obvious way, and as such, we can speak of exact sequences of α's. $K^1(A)$ is now the Grothendieck group defined by such α, but with some relations in addition to those given by exact sequences. Namely, if α and β are automorphisms of the same module, we want $\alpha\beta$ and $\alpha\oplus\beta$ to agree in $K^1(A)$. (See § 3 for details.)

If $\alpha\in \mathbf{GL}(n, A)$, we can view α as an automorphism of the free module A^n and hence assign it a class in $K^1(A)$. (Since $K^1(A)$ is abelian it is unimportant that $\mathbf{GL}(n, A)\to \mathrm{Aut}(A^n)$ is an antihomomorphism.) Letting $\mathbf{GL}(A)=\bigcup_n \mathbf{GL}(n, A)$ (see § 2) we have thus a homomorphism $\mathbf{GL}(A)\to K^1(A)$.

Proposition 0. — (*See Whitehead* [7, Theorem 1] *and* [K, § 12]) *The map above induces an isomorphism*

$$\mathbf{GL}(A)/\mathbf{E}(A)\to K^1(A),$$

where $\mathbf{E}(A)=[\mathbf{GL}(A), \mathbf{GL}(A)]$ *and is the subgroup generated by all " elementary " matrices* (*see* § 2).

$K^1(A)$ is the *Whitehead group* of A. If α is an automorphism of a projective A-module or an element of $\mathbf{GL}(n, A)$ for some n, its class in $K^1(A)$ will be denoted by $W\alpha$, and called its " Whitehead determinant ".

Now we can state our principal results.

Theorem 1. — *Let* A *be left regular and* t *an indeterminate. Then* $K^1(A)\to K^1(A[t])$ *is an isomorphism.*

Again, by the Syzygy Theorem, we can replace t above by $t_1, \ldots, t_n$. More generally we can replace $A[t]$ by a suitable graded ring. (Theorem 1′, § 3).

If A is a division ring and $A^*=\mathbf{GL}(1, A)$ is the group of units, then we obtain (using a result of Dieudonné [3] when A is not commutative) the following weak generalization of the division algorithm (for polynomials in one variable):

Corollary. — *If* A *is a division ring, then* $\mathbf{GL}(A[t_1, \ldots, t_n])/\mathbf{E}(A[t_1, \ldots, t_n])$ *is isomorphic to the commutator quotient group of* A^*.

Using [K, § 19] we obtain also:

Corollary. — *If* A *is the ring of integers in a (finite) algebraic number field, then the commutator quotient group of* $\mathbf{SL}(A[t_1, \ldots, t_n])$ *is finite.*

Remark. — If the conjecture in § 11 of [K] could be settled affirmatively one could claim these corollaries already for $\mathbf{GL}(m, \ )$ and $\mathbf{SL}(m, \ )$, with m sufficiently large. In the last corollary one could have, in addition, that $\mathbf{GL}(m, A[t_1, \ldots, t_n])$ is a finitely generated group for $m>n+2$, a fact that would be not uninteresting already for $A=\mathbf{Z}$.

The inclusion $i: A\to A[t, t^{-1}]$ is a right inverse to the unit augmentation $f: A[t, t^{-1}]\to A$, $f(t)=1$, so the kernel of $K^1(f)$ is a direct summand of $K^1(A[t, t^{-1}])$. The next theorem, which is a kind of Künneth formula, describes this kernel.

If P is a projective A-module, then $Q=A[t, t^{-1}]\otimes_A P$ is a projective $A[t, t^{-1}]$-module, so $W(t\,.\,1_Q)\in K^1(A[t, t^{-1}])$. This defines a homomorphism

$$h: K^0(A)\to K^1(A[t, t^{-1}]),$$

and, since $f(t)=1$, $\mathrm{im}(h)\subset \ker K^1(f)$.

Theorem 2. — If A *is a left regular ring, then*

$$(h, \mathrm{K}^1(i)) \quad : \mathrm{K}^0(\mathrm{A}) \oplus \mathrm{K}^1(\mathrm{A}) \to \mathrm{K}^1(\mathrm{A}[t, t^{-1}])$$

is an isomorphism.

If B is a ring and $\mathfrak{q}$ a two-sided ideal we write $\mathrm{K}^1(\mathrm{B}, \mathfrak{q}) = \mathbf{GL}(\mathrm{B}, \mathfrak{q})/\mathbf{E}(\mathrm{B}, \mathfrak{q})$ where $\mathbf{GL}(\mathrm{B}, \mathfrak{q}) = \ker(\mathbf{GL}(\mathrm{B}) \to \mathbf{GL}(\mathrm{B}/\mathfrak{q}))$ is the " $\mathfrak{q}$-congruence subgroup ", and

$$\mathbf{E}(\mathrm{B}, \mathfrak{q}) = [\mathbf{GL}(\mathrm{B}), \mathbf{GL}(\mathrm{B}, \mathfrak{q})].$$

If the projection $f: \mathrm{B} \to \mathrm{B}/\mathfrak{q}$ has a right inverse, then the sequence

$$0 \to \mathrm{K}^1(\mathrm{B}, \mathfrak{q}) \to \mathrm{K}^1(\mathrm{B}) \to \mathrm{K}^1(\mathrm{B}/\mathfrak{q}) \to 0$$

is shown in [K, Proposition 13.2] to be exact. If we apply Theorem 2 now to $\mathrm{B} = \mathrm{A}[t, t^{-1}]$ and $\mathfrak{q} = (1-t)\mathrm{B}$, therefore, we obtain:

Corollary. — Let A *be a left regular ring,* $\mathrm{B} = \mathrm{A}[t, t^{-1}]$, *and* $\mathfrak{q} = (1-t)\mathrm{B}$. *Then, if* $\mathbf{GL}(\mathrm{B}, \mathfrak{q})$ *is the* $\mathfrak{q}$*-congruence subgroup of* $\mathbf{GL}(\mathrm{B})$, *we have*

$$\mathrm{K}^0(\mathrm{A}) \cong \mathbf{GL}(\mathrm{B}, \mathfrak{q})/[\mathbf{GL}(\mathrm{B}), \mathbf{GL}(\mathrm{B}, \mathfrak{q})].$$

This isomorphism is a little remarkable in view of the " additive " nature of the left side, and " multiplicative " character of the right.

If we apply to Theorem 2 the Syzygy Theorem, the Corollary to Grothendieck's Theorem, and induction on n, we obtain:

Corollary. — If T *is a free abelian group of rank* n *and* A[T] *the group ring over a left regular ring* A, *then*

$$\mathrm{K}^1(\mathrm{A}[\mathrm{T}]) \cong \mathrm{K}^0(\mathrm{A})^n \oplus \mathrm{K}^1(\mathrm{A}).$$

Corollary. — If T *is a free abelian group and if* A *is a left regular ring with* $\mathrm{K}^0(\mathrm{A}) \cong \mathbf{Z}$, *then* $\mathrm{K}^1(\mathrm{A}[\mathrm{T}]) \cong \mathrm{T} \oplus \mathrm{K}^1(\mathrm{A})$.

In this corollary, if $\mathrm{K}^0(\mathrm{A})$ is generated by the class $\gamma\mathrm{A}$ of A in $\mathrm{K}^0(\mathrm{A})$ — this is automatic if A is commutative — then the description above of the homomorphism h shows that the monomorphism $\mathrm{T} \to \mathrm{K}^1(\mathrm{A}[\mathrm{T}])$ in the corollary sends $t \in \mathrm{T}$ to $\mathrm{W}(t . 1_{\mathrm{A}[\mathrm{T}]})$. From the matrix point of view (see Proposition 0) it is thus induced by the inclusion $\mathrm{T} \subset \mathbf{GL}(1, \mathrm{A}[\mathrm{T}]) \subset \mathbf{GL}(\mathrm{A}[\mathrm{T}])$. This remark is pertinent in the next corollary.

An obvious direct limit argument shows that we can replace T in the last corollary by any torsion free abelian group.

The corollary applies, notably, when A is a field or $\mathrm{A} = \mathbf{Z}$. In the latter case, the result was proved, for T infinite cyclic, already in 1940 by G. Higman [4]. Just as then it yields, by virtue of J. H. C. Whitehead's theory of simple homotopy types [7], the following topological application:

Corollary. — If the simplicial map $f: \mathrm{X} \to \mathrm{Y}$ *is a homotopy equivalence of finite simplicial complexes having free abelian fundamental groups, then* f *is a simple homotopy equivalence.*

Atiyah has pointed out to us the following very agreeable interpretation of Theorem 2:

Suppose A is the coordinate ring of an affine variety X over the complex numbers $\mathbf{C}$. Then $\mathrm{A}[t, t^{-1}] = \mathrm{A} \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$ coordinatizes $\mathrm{X} \times \mathbf{C}^*$, $\mathbf{C}^* = \mathbf{C} - \{0\}$. The unit augmentation corresponds to the map $X \to X \times \mathbf{C}^*$ sending x to $(x, 1)$. Note that $\mathbf{C}^*$ is homotopic to the unit circle $\mathbf{S}^1$.

Now let X be a (suitable) topological space, and let $X \to X \times \mathbf{S}^1$ be the map described above. Then, in the setting of Atiyah-Hirzebruch [9], this induces a homomorphism $K^1(X \times \mathbf{S}^1) \to K^1(X)$ with kernel $K^2(X)$. Bott periodicity for the unitary group then says $K^2(X) \cong K^0(X)$.

Thus, Theorem 2 is a kind of algebraic analogue of unitary periodicity. We cannot, however, literally regard it as a periodicity theorem, since we lack the higher K^i to even formulate one. It seems unreasonable, moreover, that there should be any periodicity in our somewhat rarified setting.

A final remark about the layout of the paper. Theorems 1 and 2 claim something is an isomorphism. Injectivity is no problem in Theorem 1, and is achieved in Theorem 2 by constructing a left inverse for h. This map is suggested directly by a similar procedure in the Atiyah-Bott proof of the periodicity theorem [8]. To establish surjectivity we first show (§ 2), on the basis of matrix calculations, that, modulo the image, everything is congruent to an element of very explicit type. (These calculations are also needed to define the left inverse of h in Theorem 2.) Then (§ 3) we observe that if we were permitted to compute K^1 (and K^0) with all modules, and not simply with projectives, the elements of explicit type produced in § 2 could be handled. In § 4 we show that, for regular rings, one can compute K^1 (and K^0) with all modules, and this disposes of Theorem 1. The proof of Theorem 2 is executed in § 5. Finally, in § 6 we give a proof of Grothendieck's Theorem general enough for our applications.

§ 2. Criteria for Theorems 1 and 2.

$\mathbf{GL}(n, A)$ is the group of units in the algebra of $n \times n$ matrices over A. An " elementary matrix " is one fo the form $1 + ae_{ij}$, $a \in A$, $i \neq j$. Here e_{ij} is the matrix whose only non-zero coordinate is a 1 in the $(i, j)^{th}$ position. $\mathbf{E}(n, A)$ is the subgroup of $\mathbf{GL}(n, A)$ generated by the elementary matrices. We view $\mathbf{GL}(n, A) \subset \mathbf{GL}(n+1, A)$ via the identification

$$\alpha \in \mathbf{GL}(n, A) = \begin{vmatrix} \alpha & 0 \\ 0 & 1 \end{vmatrix} \in \mathbf{GL}(n+1, A).$$

This done, we set $\mathbf{GL}(A) = \bigcup_n \mathbf{GL}(n, A)$ and $\mathbf{E}(A) = \bigcup_n \mathbf{E}(n, A)$. Proposition 0 of § 1 permits us to define $K^1(A)$ provisionally by:

$$K^1(A) = \mathbf{GL}(A)/\mathbf{E}(A).$$

Let $W = W_A : \mathbf{GL}(A) \to K^1(A)$ be the canonical projection. If $f: A \to B$ is a ring homomorphism, then f induces a homomorphism, also denoted here by f, $\mathbf{GL}(A) \to \mathbf{GL}(B)$, and clearly $f(\mathbf{E}(A)) \subset \mathbf{E}(B)$. We can thus define

$$K^1(f) : K^1(A) \to K^1(B)$$

by $K^1(f)(W_A \alpha) = W_B f(\alpha)$. This makes $K^1(A)$ a covariant functor of A.

We are interested in the cases $B=A[t]$ and $B=A[t, t^{-1}]$. To avoid a repetition of the same basic argument, we will place ourselves in a somewhat general setting.

Let $A=\sum_{-\infty<n<\infty} A_n$ be a graded ring; $A_iA_j\subset A_{i+j}$. Denote by A^+ the subring $\sum_{n\geq 0} A_n$. We require certain hypotheses :

a) A^+ is generated over A_0 by A_1; equivalently, $A_iA_j=A_{i+j}$ for all $i, j\geq 0$.

b) In case $A\neq A^+$ we assume there is a unit which is homogeneous of positive degree.

Further, we suppose given a ring homomorphism $f: A\to A_0\subset A$ satisfying:

c) f is a retraction; i.e. $f(a)=a$ for all $a\in A_0$.

d) If β is a square matrix homogeneous of degree one (i.e. with all coordinates in A_1) then β and $f(\beta)$ commute.

If $A\neq A^+$ and t is a unit as in *b)*, then we can replace t by $f(t)^{-1}t$, if necessary, and assume, by *c)*, that $f(t)=1$. This done, we set T equal to the cyclic group generated by t. In case $A=A^+$ we set $T=\{1\}$. We consider $T\subset \mathbf{GL}(1, A)\subset \mathbf{GL}(A)$.

Examples. — 1) $A=A_0[t]$, or, more generally, $A=A^+$ satisfying *a)*. Then we can take f defined by $f(A_n)=0$ for all $n>0$, for example.

2) $A=A_0[t, t^{-1}]$. Define f by $f(t)=1$; this is the usual augmentation.

Proposition **(2.1)**. — *Under the hypotheses above, every element of* $\mathbf{GL}(A)$ *is congruent, modulo* $T.\mathbf{E}(A)$, *to a matrix with all coordinates in* A^+. *An element of the latter type is congruent, modulo* $\mathbf{GL}(A_0).\mathbf{E}(A^+)$, *to a matrix of the form* $1-f\beta+\beta$, *where* β *is homogeneous of degree one and (necessarily) satisfies an equation of the form* $\beta^r(1-f\beta)^s=0$, *for some* $r, s\geq 0$.

Proof. — Since $\mathbf{E}(A)$ is the commutator subgroup of $\mathbf{GL}(A)$ (Proposition 0), $T.\mathbf{E}(A)$ is normal. Hence the latter contains the subgroup generated by all conjugates of T, and this includes all diagonal matrices with elements of T on the diagonal. Evidently any matrix over A is a product of one of these and a matrix over A^+, and this is our first conclusion.

Now suppose $\alpha\in\mathbf{GL}(A)$ has coordinates in A^+, say $\alpha=\alpha_0+\alpha_1+\ldots+\alpha_d$ with α_i homogeneous of degree i. Working first modulo $\mathbf{E}(A^+)$ we shall render α an element of degree one (i.e. with $d=1$). By induction it suffices to show that if $d>1$ then we can reduce the degree of α.

Now $\alpha\in\mathbf{GL}(n, A)$ for some n, so we shall think of the α_i now as $n\times n$ matrices. Since, by *a)*, $A_d=A_{d-1}A_1$, we can write $\alpha_d=\Sigma_{j=1}^h\gamma_j a_j$ with γ_j an $n\times n$ matrix over A_{d-1}, and $a_j\in A_1$, $1\leq j\leq h$. Working now in $\mathbf{GL}((h+1)n, A)$ we transform α modulo $\mathbf{E}((h+1)n, A^+)$ as follows (where 1_n denotes the $n\times n$ identity matrix):

$$\left|\begin{array}{cccc}\alpha & 0 & \ldots & 0\\ 0 & 1_n & \ldots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1_n\end{array}\right| \to \left|\begin{array}{cccc}\alpha & \gamma_1 & \ldots & \gamma_n\\ 0 & 1_n & \ldots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1_n\end{array}\right| \to \left|\begin{array}{cccc}\alpha-\alpha_d & \gamma_1 & \ldots & \gamma_h\\ -a_1.1_n & 1_n & & 0\\ \vdots & & \ddots & \\ -a_h.1_n & 0 & & 1_n\end{array}\right|$$

The last term has degree $\leq d-1$, as desired.

It remains to handle an element $\alpha=\alpha_0+\alpha_1$. Let $\gamma=f\alpha=\alpha_0+f\alpha_1\in\mathbf{GL}(A_0)$ and set $\beta=\gamma^{-1}\alpha_1$. Then

$$\gamma(1-f\beta+\beta)=\gamma(1-\gamma^{-1}f\alpha_1+\gamma^{-1}\alpha_1)=\gamma-f\alpha_1+\alpha_1=\alpha_0+f\alpha_1-f\alpha_1+\alpha_1=\alpha.$$

Hence, modulo $\mathbf{GL}(A_0)$, α is congruent to $1-f\beta+\beta\in\mathbf{GL}(A)$.

We conclude now by showing that β satisfies the necessary equation. Since β is homogeneous of degree one it follows from *d)* that β commutes with $f\beta$ and hence also with $\delta=1-f\beta$. Write $(\delta+\beta)^{-1}=\gamma_{-m}+\ldots+\gamma_{-1}+\gamma_0+\gamma_1+\ldots+\gamma_n$, and compare degrees in the equation

$$(\delta+\beta)(\gamma_{-m}+\ldots+\gamma_n)=1.$$

In degree 0 we have $\delta\gamma_0+\beta\gamma_{-1}=1$. Multiplying this equation on the left by $\delta^r\beta^s$ we see that the desired equation (namely, $\delta^r\beta^s=0$) will follow once we know that, for each $i\geq 0$, $\beta^s\gamma_i=0$, and for each $i<0$, $\delta^r\gamma_i=0$, for suitable r and s. But these assertions follow by induction from the equations $\delta\gamma_i+\beta\gamma_{i-1}=0$ for $i\neq 0$, starting from $\beta\gamma_n=0$ and $\delta\gamma_{-m}=0$, respectively.

Consider the case $A=A^+$ and $f(A_n)=0$ for $n>0$. Then, in the proposition above $f\beta=0$, so that $\beta^s=0$ for some s. In general, a unit of the form $1+\beta$, with β nilpotent, is called *unipotent*.

Corollary (**2.2**). — *Let* $A=A_0+A_1+\ldots$ *be a graded ring generated over* A_0 *by* A_1. *Then the inclusion of* A_0 *induces a decomposition* $K^1(A)\cong K^1(A_0)\oplus H$, *and every element of* H *is represented by a unipotent* $\alpha\in\mathbf{GL}(A)$. *Hence, if* $W_A(\alpha)=0$ *for all unipotent* $\alpha\in\mathbf{GL}(A)$, *then the same is true of* A_0, *and* $K^1(A)\cong K^1(A_0)$.

Corollary (**2.3**). — *Let* $A=A_0[t]$ *be a polynomial extension, and consider the conditions*:

a) $K^1(A_0)\to K^1(A)$ *is an isomorphism.*

b) $W_{A_0}(\alpha)=0$ *for all unipotent* $\alpha\in\mathbf{GL}(A_0)$.

c) $W_A(\alpha)=0$ *for all unipotent* $\alpha\in\mathbf{GL}(A)$.

Then c) $\Leftrightarrow$ *a)* $\Rightarrow$ *b).*

Proof. — *c)* $\Rightarrow$ *a)* and *b)* is contained in Corollary 2.2.

Now assume *a)*. Let $g:A_0\to A$ be the inclusion and $f_i:A\to A_0$ the retraction defined by $f_i(t)=i$, $i=0, 1$. Then our assumption says that $K^1(g)$ is an isomorphism and that $K^1(f_0)=K^1(g)^{-1}=K^1(f_1)$. Let $\alpha=1+\beta$ be unipotent in $\mathbf{GL}(A_0)$.

Then

$$W_{A_0}(\alpha)=K^1(f_1)(W_A(1+t\beta))=K^1(f_0)(W_A(1+t\beta))=W_{A_0}(f_0(1+t\beta))=W_{A_0}(1)=0.$$

(Note that $1+t\beta$ is unipotent and hence automatically invertible.) We have thus shown *b)*. But now if α is unipotent in $\mathbf{GL}(A)$ then $f_0(\alpha)$ is unipotent in $\mathbf{GL}(A_0)$, so $W_{A_0}(f_0\alpha)=K^1(f_0)(W_A\alpha)=0$. Since $K^1(f_0)$ is an isomorphism, $W_A\alpha=0$, and this is *c)*.

Remarks. — 1) Corollary 2.2 will be the basis for our proof of the generalization, Theorem 1′, of Theorem 1, in the next section.

2) If $\mathfrak{q}$ is an ideal in A_0 and A is the associated "Rees ring", defined by $A_n = \mathfrak{q}^n$, $n \geq 0$, then the above considerations carry over without essential change for the relative groups:

$$K^1(A_0, \mathfrak{q}) \to K^1(A, \mathfrak{q}A),$$

in the sense of [K, Chap. III]. In particular, Corollary 2.3 carries over intact, and we recover Corollary 2.3 from this in the special case $\mathfrak{q} = A_0$. On the other hand we have not succeeded in generalizing Theorem 1 to the relative case.

We close this section now with the criterion to be used in the proof of Theorem 2.

Corollary **(2.4)**. — *Let* $A = A_0[T] = A_0[t, t^{-1}]$ *where* T *is an infinite cyclic group with generator t. If* $\alpha \in \mathbf{GL}(A)$ *is a polynomial in t (i.e. all coordinates lie in* $A_0[t]$*) then there is a* $\gamma \in \mathbf{GL}(A_0) . \mathbf{E}(A_0[t])$ *such that* $\alpha\gamma = 1 + (t-1)\beta$, *where* β *is a matrix over* A_0 *satisfying* $\beta^r(1-\beta)^s = 0$ *for some* $r, s \geq 0$.

§ 3. Proofs of Theorems 1 and 2.

Let $\mathscr{C}$ be a category and T an infinite cyclic group with generator t. We are interested in " $\mathscr{C}$-representations " of T. Such a representation is defined by assigning to t an automorphism α of some $M \in \text{obj}\ \mathscr{C}$. We will identify α with the representation. If $\alpha' \in \text{Aut}_{\mathscr{C}}(M')$ is another, then a morphism $\alpha \to \alpha'$ is (by definition) a $\mathscr{C}$-morphism $f : \text{dom}\ \alpha = M \to \text{dom}\ \alpha' = M'$ such that $f\alpha = \alpha' f$. We have thus defined a new category which we shall denote by $\mathscr{C}[T] = \mathscr{C}[t, t^{-1}]$. If we regard T as (the morphisms of) a category with one object, then $\mathscr{C}[T]$ can also be described as the functor category Funct(T, $\mathscr{C}$). Thus it is clear that $\mathscr{C}[T]$ is additive (resp. abelian) if $\mathscr{C}$ is. $\mathscr{C}[T]$ need not have enough projectives even if $\mathscr{C}$ does, as happens in the cases of primary interest to us when $\mathscr{C}$ is the category of finitely generated (projective) A-modules. For then $\mathscr{C}[T]$ has no non trivial projectives! However, if $\mathscr{C}$ is the category of all left A-modules, then $\mathscr{C}[T]$ " is " the category of all left A[T]-modules; hence the notation.

We shall assume $\mathscr{C}$ is given as a full sub-category of some abelian category $\mathscr{A}$; then likewise for $\mathscr{C}[T] \subset \mathscr{A}[T]$. A sequence in $\mathscr{C}[T]$ will be called " exact " if it is exact in $\mathscr{A}[T]$. Note that this is equivalent to exactness on the domains.

We will always assume that $\mathscr{C}$ contains a zero object of $\mathscr{A}$.

Definition. — Let $W = W_{\mathscr{C}} : \text{obj}\ \mathscr{C}[T] \to K^1(\mathscr{C})$ be universal for maps into an abelian group which satisfy

(A) (Additivity). If $0 \to \alpha_n \to \ldots \to \alpha_1 \to \alpha_0 \to 0$ is exact, then $\Sigma(-1)^i W\alpha_i = 0$.

(M) (Multiplicativity). If $\text{dom}\ \alpha = \text{dom}\ \beta$, then $W\alpha\beta = W\alpha + W\beta$.

This clearly defines W and K^1 up to canonical isomorphism, and their existence is clear provided the isomorphism types of obj $\mathscr{C}$ form a set, an assumption we shall always make.

Remarks. — 1) $K^0(\mathscr{C})$ is similarly defined by taking $\gamma = \gamma_{\mathscr{C}} : \text{obj}\ \mathscr{C} \to K^0(\mathscr{C})$ universal for maps into an abelian group which are required only to be additive.

2) If $\mathscr{C}$ is an abelian category, it is sufficient to require additivity for short exact sequences, the additivity on long sequences being a consequence of this. More generally,

if $\mathscr{C}$ is a full subcategory of an abelian category $\mathscr{A}$, it will suffice to require additivity on short exact sequences provided the following condition is satisfied:

If $0 \to M' \to M \to M'' \to 0$ is exact in $\mathscr{A}$ and $M, M'' \in \mathscr{C}$ then $M' \in \mathscr{C}$.

This condition permits us to break up a long exact sequence into short ones. However, without some such condition, the two definitions of additivity will not be equivalent. For example, consider the full subcategory of abelian groups with objects 0, $\mathbf{Z}_2, \mathbf{Z}_6, \mathbf{Z}_{15}, \mathbf{Z}_5$. The additivity on long exact sequences is essential in the arguments used in § 4.

3) To handle K^0 and K^1 simultaneously, as well as the relative K^1, one can proceed as follows: Let $F : \mathscr{C} \to \mathscr{C}'$ be an additive functor, and let $\mathscr{C}(F)$ be the full subcategory of $\mathscr{C}[T]$ whose objects are those α for which $F(\alpha)$ is an identity morphism. Now take $D : \text{obj}\, \mathscr{C}(F) \to K^*(\mathscr{C}, F)$ universal for axioms (A) above, and

(M'). If $\text{dom}\, \alpha = M = \text{dom}\, \beta$, then

$$D\alpha\beta + D1_M = D\alpha + D\beta.$$

The functors $M \rightsquigarrow 1_M$ and $\alpha \rightsquigarrow \text{dom}\, \alpha$ induce a decomposition $K^*(\mathscr{C}, F) = K^0(\mathscr{C}) \oplus K^1(\mathscr{C}, F)$ and D followed by the projection on $K^1(\mathscr{C}, F)$ is universal for axioms (A) and (M).

When $F = \text{Id}_{\mathscr{C}}$, $K^*(\mathscr{C}, F) = K^0(\mathscr{C})$. When F is a " constant " functor, $K^1(\mathscr{C}, F) = K^1(\mathscr{C})$. If $\mathscr{C}$ is the category of finitely generated projective left A-modules and $F = A/\mathfrak{q} \otimes_A$ for some two-sided ideal $\mathfrak{q}$, then $K^0(\mathscr{C}) = K^0(A)$ and $K^1(\mathscr{C}, F) = K^1(A, \mathfrak{q})$ in the sense of [K, Chap. III].

If one retains this point of view in Theorem 5 below, then the proof goes over intact for $K^*(\mathscr{C}) = K^0(\mathscr{C}) \oplus K^1(\mathscr{C})$ by simply changing it to accomodate axiom (M') instead of (M) at one point. For a general F the property needed is that $\mathscr{C}(F)$ be closed under " pullbacks ".

Now suppose $M \in \text{obj}\, \mathscr{C}$ and $\beta \in \text{Hom}_{\mathscr{C}}(M, M)$. We call β *$\mathscr{C}$-nilpotent* if there is a filtration $0 = M_0 \subset M_1 \subset \ldots M_n = M$ (in $\mathscr{A}$) such that M_i and M_i/M_{i-1} are in $\mathscr{C}$, and such that $\beta M_i \subset M_{i-1}$, $1 \leq i \leq n$. In this case the automorphism $\alpha = 1_M + \beta$ is called *$\mathscr{C}$-unipotent*. It is clear that the M_i are α invariant, and that α induces the identity on each M_i/M_{i-1}. By axiom (M), W annihilates identity automorphisms, so we conclude from axiom (A) and induction on n:

Lemma (**3.1**). — *If α is $\mathscr{C}$-unipotent, then* $W_{\mathscr{C}}\alpha = 0$ *in* $K^1(\mathscr{C})$.

Let A be a ring, $\mathscr{M}$ the category of finitely generated left A-modules, and $\mathscr{P}$ the category of projective modules in $\mathscr{M}$. Then Proposition 0 of the introduction is to be understood as describing an isomorphism.

$$K^1(A) = \mathbf{GL}(A)/\mathbf{E}(A) \cong K^1(\mathscr{P}).$$

Suppose M is a finitely generated A-module and $\beta \in \text{Hom}_A(M, M)$ is nilpotent, $\beta^n = 0$. Set $M_i = \text{im}\, \beta^{n-i}$, $0 \leq i \leq n$. Then M_i is also finitely generated, and we see that β is automatically $\mathscr{M}$-nilpotent. Hence $W_{\mathscr{M}}$ annihilates all unipotents, by Lemma 3.1.

In particular, the homomorphism $K^1(\mathscr{P}) \to K^1(\mathscr{M})$ annihilates elements of $K^1(\mathscr{P})$ represented by unipotents.

Now suppose $A = A_0 + A_1 + \ldots$ is a graded ring generated over A_0 by A_1. If we knew $K^1(\mathscr{P}) \to K^1(\mathscr{M})$ were a monomorphism, then the remark above together with Corollary 2.2 would imply that $K^1(A_0) \to K^1(A)$ is an isomorphism. Thus, Theorem 1′ below, which generalizes Theorem 1, is a consequence of Theorem 3 below.

Theorem 1′. — Let $A = A_0 + A_1 + \ldots$ *be a left regular graded ring generated over* A_0 *by* A_1. *Then*

$$K^1(A_0) \to K^1(A)$$

is an isomorphism.

Theorem 3. — Let A *be a left regular ring. Then*

$$K^1(\mathscr{P}) \to K^1(\mathscr{M})$$

is an isomorphism. In particular, $W_{\mathfrak{X}}(\alpha) = 0$ *for all unipotent* α.

Corollary. — If A *is a left regular ring, every unipotent matrix in* **GL**(A) *is in the commutator subgroup, the latter being generated by elementary matrices.*

Corollary. — If A *is a left regular ring and* $\mathfrak{q}$ *a nilpotent two-sided ideal, then the* $\mathfrak{q}$*-congruence subgroup* $\mathbf{GL}(A, \mathfrak{q})$ *lies in the commutator subgroup of* $\mathbf{GL}(A)$. *Hence* $K^1(A) \to K^1(A/\mathfrak{q})$ *is an isomorphism.*

The last conclusion uses the fact [K, Lemma 1.1] that $\mathbf{E}(A) \to \mathbf{E}(A/\mathfrak{q})$ is surjective. This corollary applies, notably, when A is an Artin ring of finite global dimension and $\mathfrak{q}$ is the radical. If A is commutative with a non zero nilpotent element n, then $W(1+n) \neq 0$, as may be seen by factoring the determinant homomorphism through $K^1(A)$. Theorem 3 thus provides a rather bizarre proof of the well-known fact that a commutative regular ring has zero nil radical.

§ 4. Proof of Theorem 3.

In this section, as in § 3, we will consider various full subcategories of an abelian category $\mathscr{A}$. All statements of exactness, including those used to define K^0 and K^1, are to be interpreted as holding in $\mathscr{A}$. All categories considered will be assumed to contain a zero object of $\mathscr{A}$. We remark once again that in the theorems proved here it is essential to use the definition of K^0 and K^1 involving additivity on long exact sequences as in § 3. Of course this will be irrelevant if condition *b)* of the theorems is assumed to hold for *both* subcategories.

We will deduce Theorem 3 from the following result which generalizes a theorem of Grothendieck [2, Theorem 2].

Theorem 4. — Let $\mathscr{A}$ *be an abelian category and let* $\mathscr{P} \subset \mathscr{M}$ *be full subcategories of* $\mathscr{A}$ *satisfying the following conditions:*

a) $\mathscr{P}$ *and* $\mathscr{M}$ *are closed under finite direct sums.*

b) *If* $0 \to M' \to M \to M'' \to 0$ *is exact in* $\mathscr{A}$ *and* $M, M'' \in \mathscr{M}$ *then* $M' \in \mathscr{M}$.

c) *If* M *is an object of* $\mathcal{M}$, *there is an exact sequence*

$$0 \to P_d \to \ldots \to P_0 \to M \to 0$$

with all $P_i \in \mathcal{P}$.

Then the inclusion $\mathcal{P} \subset \mathcal{M}$ *induces an isomorphism* $K^0(\mathcal{P}) \approx K^0(\mathcal{M})$.

Note that the integer d in *c)* is allowed to vary with M and is not required to be bounded.

If P is the sequence $P_d \to \ldots \to P_0$ of *c)*, the inverse isomorphism $K^0(\mathcal{M}) \to K^0(\mathcal{P})$ sends $\gamma(M)$ into $\chi(P) = \Sigma(-1)^i\gamma(P_i)$ because $\chi(P)$ clearly maps onto $\gamma(M)$ in $K^0(\mathcal{M})$. We will refer to such a sequence P as a finite $\mathcal{P}$-resolution of M.

The proof makes use of the following lemma.

Lemma 1. — Let $P \to M$ *be a finite* $\mathcal{P}$*-resolution of* $M \in \mathcal{M}$ *and let* $f : M' \to M$ *in* $\mathcal{M}$. *Then there is a finite* $\mathcal{P}$*-resolution* $P' \to M'$ *and a map* $P' \to P$ *covering* f.

Proof. — Let B be the pullback of

$$\begin{array}{ccccc} & & M' & & \\ & & \downarrow{\scriptstyle f} & & \\ P_0 & \xrightarrow{d_0} & M & \to & 0 \end{array}$$

In other words, B is the universal object for commutative diagrams

$$\begin{array}{ccccc} B & \to & M' & & \\ \downarrow & & \downarrow & & \\ P_0 & \to & M & \to & 0 \end{array}$$

Since B is the kernel of $(d_0, -f) : P_0 \oplus M' \to M$, it follows that $B \in \mathcal{M}$ because d_0, and hence $(d_0, -f)$, is an epimorphism. Also, it it easy to see that $B \to M'$ is an epimorphism. Since B has a $\mathcal{P}$-resolution by *c)*, there is an epimorphism $P'_0 \to B$ with $P'_0 \in \mathcal{P}$. Composing this with the maps $B \to M'$ and $B \to P_0$ gives us a commutative diagram

$$\begin{array}{ccccc} P'_0 & \to & M' & \to & 0 \\ \downarrow & & \downarrow & & \\ P_0 & \to & M & \to & 0 \end{array}$$

Now, if

$$\begin{array}{ccccccccccc} & & P'_{r-1} & \xrightarrow{d'_{r-1}} & \ldots & \to & P'_0 & \to & M' & \to & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ \ldots P_r & \to & P_{r-1} & \xrightarrow{d_{r-1}} & \ldots & \to & P_0 & \to & M & \to & 0 \end{array}$$

has been constructed, we continue by repeating the previous argument on

$$\begin{array}{ccccc} & & \ker d'_{r-1} & & \\ & & \downarrow & & \\ P_r & \to & \ker d_{r-1} & \to & 0 \end{array}$$

These kernels are in $\mathscr{M}$ by *b)* and induction on r. Since P is a finite resolution, we will eventually reach a point where $P_i = 0$ for $i \geq r-1$. At this point, we finish off P′ with a finite $\mathscr{P}$-resolution of $\ker d'_{r-1}$.

Proof of Theorem 4. — Let $f: C' \to C$ be a map of complexes. The mapping cone of f is the complex $C(f)$ defined by $C(f)_n = C'_{n-1} \oplus C_n$ with the differentiation given by $\begin{vmatrix} -d & 0 \\ f & d \end{vmatrix}$. There is an exact sequence $0 \to C \xrightarrow{i} C(f) \xrightarrow{j} C' \to 0$ by the injection and projection maps associated with a direct sum. Note that j is homogeneous of degree -1. In the resulting homology sequences, the connecting homomorphism is homogeneous of degree 0 and is well-known (and easily checked) to be $H(f): H(C') \to H(C)$. Therefore $H(f)$ is an isomorphism if and only if $C(f)$ has zero homology.

If C′ and C are finite complexes in $\mathscr{P}$, so is $C(f)$ and

$$(*) \qquad \chi(C(f)) = \chi(C) - \chi(C')$$

where, as above, $\chi(C) = \Sigma(-1)^i \gamma(C_i) \in K^0(\mathscr{P})$. Therefore, if $H(f)$ is an isomorphism, it follows that $\chi(C) = \chi(C')$ because $C(f)$ has 0 homology and hence is an exact sequence.

Now, suppose $P \to M$ and $P' \to M$ are two finite $\mathscr{P}$-resolutions of $M \in \mathscr{M}$. By applying Lemma 1 to the resolution $P \oplus P'$ of $M \oplus M$ and the diagonal map $M \to M \oplus M$ we obtain a finite $\mathscr{P}$-resolution $P'' \to M$ and a map $P'' \to P \oplus P'$ covering the diagonal map. Composing this with the coordinate projections yields maps $P'' \to P$, $P'' \to P'$ covering the identity map of M. Thus these maps induce isomorphisms of homology and hence $\chi(P) = \chi(P'') = \chi(P')$. This shows that the map $\varphi : \text{obj}\,\mathscr{M} \to K^0(\mathscr{P})$ by $\varphi(M) = \chi(P)$, P any finite $\mathscr{P}$-resolution of M, is well defined. We must now show that φ is additive. As we have remarked in § 3, condition *b)* shows that it will suffice to check additivity on short exact sequences.

Let $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be exact with all terms in $\mathscr{M}$. Let $P \to M$ be a finite $\mathscr{P}$-resolution of M. By Lemma 1, we can find a finite $\mathscr{P}$-resolution P′ of M′ and a map $f: P' \to P$ covering $i: M' \to M$. Let $C(f)$ be the mapping cone of f. Since P′ and P have zero homology except in dimension 0 where $H_0(P') = M'$, $H_0(P) = M$, the exact homology sequence of $0 \to P \to C(f) \to P' \to 0$ shows that $H_i(C(f)) = 0$ for $i \geq 2$, $H_1(C(f)) \approx \ker i = 0$, and $H_0(C(f)) \approx \operatorname{coker} i \approx M''$. Since $C(f)$ clearly has nothing in dimensions < 0, it follows that $C(f)$ is a finite $\mathscr{P}$-resolution of M″. The relation (*) now shows that $\varphi(M'') = \varphi(M) - \varphi(M')$.

We have now shown that φ defines a map $\varphi : K^0(\mathscr{M}) \to K^0(\mathscr{P})$. This is clearly a right inverse for the map $K^0(\mathscr{P})\to K^0(\mathscr{M})$. It is also a left inverse because if $M\in\mathscr{P}$ we can compute $\varphi(M)$ from the resolution $0\to M\to M\to 0$.

Example. — Let $\mathscr{A}$ be the category of abelian groups and $\mathscr{M}$ the full subcategory whose objects are finite direct sums of copies of $\mathbf{Z}_2$, $\mathbf{Z}_4$, and $\mathbf{Z}_8$. Let $\mathscr{P}$ be the full subcategory whose objects are finite direct sums of copies of $\mathbf{Z}_4$ and $\mathbf{Z}_8$. Then Theorem 4 applies and $K^0(\mathscr{P})\approx K^0(\mathscr{M})\approx\mathbf{Z}$ generated by $\gamma(\mathbf{Z}_2)$. However if we define K^0 in terms of short exact sequences, $K^0(\mathscr{M})$ will be unchanged while $K^0(\mathscr{P})$ will become $\mathbf{Z}\oplus\mathbf{Z}$ generated by $\gamma(\mathbf{Z}_4)$ and $\gamma(\mathbf{Z}_8)$.

Theorem 3 is an immediate corollary of the following theorem which gives an analogue of Theorem 4 for K^1. Note that condition *c)* of Theorem 4 is here replaced by a stronger condition.

Theorem 5. — Let $\mathscr{A}$ be an abelian category and let $\mathscr{P}\subset\mathscr{M}$ be full subcategories of $\mathscr{A}$ satisfying the following conditions:

a) *$\mathscr{P}$ and $\mathscr{M}$ are closed under finite direct sums.*

b) *If $0\to M'\to M\to M''\to 0$ is exact in $\mathscr{A}$ and $M, M''\in\mathscr{M}$ then $M'\in\mathscr{M}$.*

c) *If M is an object of $\mathscr{M}$, there is an epimorphism $P\to M$ with $P\in\mathscr{P}$ such that every endomorphism of M lifts to one of P.*

d) *If*

$$\dots\to P_r\xrightarrow{d_r}P_{r-1}\to\dots\to P_0\to M\to 0$$

is exact with $M\in\mathscr{M}$ and all $P_i\in\mathscr{P}$, there is an integer r such that $\ker d_r\in\mathscr{P}$.

Then the inclusion $\mathscr{P}\subset\mathscr{M}$ induces an isomorphism $K^1(\mathscr{P})\approx K^1(\mathscr{M})$.

To prove this, we must first show that Theorem 4 applies to $\mathscr{P}[T]\subset\mathscr{M}[T]$. This is done using the following lemma:

Lemma 2. — If M is an object of $\mathscr{M}$, there is an epimorphism $f: P\to M$ with $P\in\mathscr{P}$ such that every automorphism of M lifts to one of P.

Proof. — Let $g: Q\to M$ be an epimorphism as in *c)*, so that every endomorphism of M lifts. Let $P=Q\oplus Q$ and let $f: P\to M$ be the composition of $g\oplus g: P\oplus P\to M\oplus M$ with the projection $(1_M, 0): M\oplus M\to M$. If $\alpha\in\mathrm{Aut}(M)$, lift α to $(\alpha,\alpha^{-1})=\begin{vmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{vmatrix}\in\mathrm{Aut}(M\oplus M)$. By [K, Lemma 1.6], $\begin{vmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{vmatrix}$ is a product of automorphisms of the forms $\begin{vmatrix}1_M & \gamma\\ 0 & 1_M\end{vmatrix}$ and $\begin{vmatrix}1_M & 0\\ \gamma & 1_M\end{vmatrix}$ with $\gamma\in\mathrm{Hom}(M, M)$. Since γ can be lifted to $\gamma'\in\mathrm{Hom}(Q, Q)$, we can lift $\begin{vmatrix}1_M & \gamma\\ 0 & 1_M\end{vmatrix}$ to $\begin{vmatrix}1_Q & \gamma'\\ 0 & 1_Q\end{vmatrix}$ which is an automorphism of P.

Corollary. — If M is an object of $\mathscr{M}$, there is a finite $\mathscr{P}$-resolution $P\to M$ of M such that every automorphism of M lifts to one of P.

Proof. — That there is an infinite such resolution P′ follows directly from the lemma, using condition *b)* to insure that we never leave $\mathscr{M}$. If d is the differential in P′, condition *d)* shows that for some r, $\ker d_r\in\mathscr{P}$. Let $P\to M$ be

$$0\to\ker d_r\to P'_r\to\dots\to P'_0\to M\to 0$$

If α is an automorphism of M, lift it to an automorphism β' of P'. Restricting β' to P gives an automorphism β on all terms but $\ker d_r$. The 5-lemma then shows that β is also an automorphism of $\ker d_r$.

Proof of Theorem 5. — The categories $\mathscr{P}[T] \subset \mathscr{M}[T]$ clearly satisfy conditions *a)* and *b)* of Theorem 4. Let $\alpha \in \mathrm{Aut}(M)$ be an object of $\mathscr{M}[T]$. Let $P \to M$ be the finite $\mathscr{P}$-resolution given by the preceding corollary. Since we can lift α to an automorphism β of P, we see that condition *c)* of Theorem 4 is also satisfied. Therefore $K^0(\mathscr{P}[T]) \approx K^0(\mathscr{M}[T])$. By the remarks immediately after Theorem 4, the inverse φ of this isomorphism sends $W(\alpha) \to \chi(\beta)$ where β is the automorphism of P which we have just considered. If we can show that φ defines a map $K^1(\mathscr{M}) \to K^1(\mathscr{P})$ we will have the required inverse for $K^1(\mathscr{P}) \to K^1(\mathscr{M})$. To do this, it will suffice to show that the composition

$$\varphi' : K^0(\mathscr{M}[T]) \to K^0(\mathscr{P}[T]) \to K^1(\mathscr{P})$$

is multiplicative. If $\alpha, \alpha' \in \mathrm{Aut}(M)$ and $P \to M$ is as in the corollary of Lemma 2, lift α, α' to automorphisms β, β' of P. Then $\beta\beta'$ lifts $\alpha\alpha'$. If β_i denotes the restriction of β_i to P_i, we have

$$\begin{aligned}\varphi'(\alpha\alpha') = \chi(\beta\beta') &= \Sigma(-1)^i(W(\beta\beta')_i) = \Sigma(-1)^i W(\beta_i\beta_i') \\ &= \Sigma(-1)^i(W(\beta_i) + W(\beta_i')) = \varphi'(\alpha) + \varphi'(\alpha')\end{aligned}$$

Therefore φ' is multiplicative.

Corollary. — If $\mathscr{A}$ is an abelian category in which every object has finite projective dimension and if $\mathscr{P}$ is the full subcategory of projective objects, then the inclusion $\mathscr{P} \subset \mathscr{A}$ induces an isomorphism $K^1(\mathscr{P}) \approx K^1(\mathscr{A})$.

This includes Theorem 3.

The generalization to the case where $\mathscr{M} \neq \mathscr{A}$ has some useful consequences. For example, it shows that even if A is not regular, we can compute K^0 and K^1 for projective modules by using all modules of finite homological dimension. Another typical use is the following: If A is an algebra without torsion over an integral domain, the Grothendieck group of categories of A-modules can be computed using only torsion free modules. Reductions of this type are often useful because non exact functors may have exact restrictions to smaller categories.

§ 5. Proof of Theorem 2.

Let A be a ring and T an infinite cyclic group with generator t. Then $A \subset B = A[t] \subset C = A[t, t^{-1}]$, and the augmentation $f : C \to A$ by $f(t) = 1$ is a retraction.

We define $h : K^0(A) \to K^1(C)$ by $h(\gamma_A P) = W_C(t.1_Q)$, where $Q = C \otimes_A P$. Since $f(t) = 1$, $\mathrm{im}(h) \subset \ker(K^1(f))$.

By virtue of Theorem 3, Theorem 2 is a consequence of the following theorem.

Theorem 2'. — There is a homomorphism $\varphi : K^1(C) \to K^0(A)$ *such that* $\varphi . h = \mathrm{id}$. *Hence*

$$(\varphi, K^1(f)) : K^1(C) \to K^0(A) \oplus K^1(A)$$

is a split epimorphism. Moreover every element of its kernel is represented by a unipotent.

Proof. — We shall start with some general remarks preparatory to the construction of φ. If δ is an endomorphism of B^n, let $M(\delta)=\operatorname{coker}(\delta)=B^n/\delta B^n$, viewed as an A-module. If δ' is an endomorphism of $B^{n'}$, then clearly

(1) $$M(\delta\oplus\delta')\cong M(\delta)\oplus M(\delta').$$

If $n'=n$, then $\delta'\delta$ is defined and there is an exact sequence

$$0\to\delta' B^n/\delta'\delta B^n\to B^n/\delta'\delta B^n\to B^n/\delta' B^n\to 0.$$

If δ' is a monomorphism, then $\delta' B^n/\delta'\delta B^n\cong B^n/\delta B^n$, so we have an exact sequence

(2) $$0\to M(\delta)\to M(\delta'\delta)\to M(\delta')\to 0.$$

Finally we note that:

(3) $$M(t.1_{B^n})\cong A^n \quad\text{and}\quad M(t^N.1_B)\cong A^N.$$

Let $\alpha\in\mathbf{GL}(n, C)$. Then $t^N\alpha$ is a polynomial in t for large N, so it defines an endomorphism δ of B^n. In order not to interrupt our discussion we postpone the proof of :

(*) $M(\delta)$ is a finitely generated projective A-module.

Admitting this, $\gamma M(\delta)=\gamma_A M(\delta)\in K^0(A)$ is defined, so we can write

$$\psi_n(\alpha)=\gamma M(\delta)-nN\gamma A.$$

To show that ψ_n is independent of N, we note that the endomorphism of B^n defined by $t^{N+1}\alpha$ can be written $\delta'\delta$, where $\delta'=t.1_{B^n}$. By (2) and (3), then, we have $\gamma M(\delta'\delta)-n(N+1)\gamma A=\gamma M(\delta)-nN\gamma A+\gamma M(t.1_{B^n})-n\gamma A=\gamma M(\delta)-nN\gamma A.$

If $t^N\alpha$ and $t^{N'}\alpha'$ are polynomials, inducing δ and δ' on B^n, then $t^{N+N'}\alpha'\alpha$ induces $\delta'\delta$ on B^n, so we have, using (2),

$$\begin{aligned}\psi_n(\alpha'\alpha)&=\gamma M(\delta'\delta)-n(N'+N)\gamma A\\&=\gamma M(\delta')-nN'\gamma A+\gamma M(\delta)-nN\gamma A\\&=\psi_n(\alpha')+\psi_n(\alpha).\end{aligned}$$

Hence ψ_n is a homomorphism. ($\mathbf{GL}(n, C)\to\operatorname{Aut}(C^n)$ is an a antihomomorphism, but $K^0(A)$ is abelian.)

Consider now $\alpha\oplus 1_C\in\mathbf{GL}(n+1, C)$; $t^N(\alpha\oplus 1_C)$ is a polynomial inducing $\delta\oplus t^N.1_B$ on B^{n+1}. It follows from (1) and (3) that

$$\begin{aligned}\psi_{n+1}(\alpha\oplus 1_C)&=\gamma M(\delta\oplus t^N.1_B)-(n+1)N\gamma A\\&=\gamma M(\delta)-nN\gamma A+\gamma M(t^N.1_B)-N\gamma A\\&=\gamma M(\delta)-nN\gamma A=\psi_n(\alpha).\end{aligned}$$

Hence ψ_n and ψ_{n+1} are compatible with the inclusions $\mathbf{GL}(n, C)\subset\mathbf{GL}(n+1, C)$, so they define a homomorphism $\psi:\mathbf{GL}(C)\to K^0(A)$, and this induces

$$\varphi: K^1(C)=\mathbf{GL}(C)/\mathbf{E}(C)\to K^0(A),$$

since $K^0(A)$ is abelian and $\mathbf{E}(C)=[\mathbf{GL}(C),\mathbf{GL}(C)]$ (Proposition 0).

Now suppose P is a finitely generated projective A-module. We must show that

$\varphi(h(\gamma_A P)) = \gamma_A P$. We can write $P \oplus Q \cong A^n$ for some n. This induces $CP \oplus CQ \cong C^n$, where CM denotes $C \otimes_A M$, and permits us to represent $(t . 1_{CP}) \oplus (1_{CQ})$ by a matrix α in $\mathbf{GL}(n, C)$. Our choice of basis guarantees that α is already a polynomial. Now

$$\begin{aligned} h(\gamma_A P) &= W_C(t . 1_{CP}) \\ &= W_C((t . 1_{CP}) \oplus (1_{CQ})) = W_C(\alpha). \end{aligned}$$

Hence $\varphi(h(\gamma_A P)) = \varphi(W_C(\alpha)) = \psi_n(\alpha)$. Let δ denote the restriction of α to $B^n = BP \oplus BQ$. Then $\delta = t . 1_{BP} \oplus 1_{BQ}$, so $M(\delta) = BP/t.BP \cong (B/t.B) \otimes_A P \cong A \otimes_A P \cong P$. Thus

$$\psi_n(\alpha) = \gamma_A P - n \circ \gamma_A A = \gamma_A P,$$

as desired.

It remains to prove (*). With $\alpha \in \mathbf{GL}(n, C)$, suppose $t^N \alpha$ is a polynomial inducing, say, δ on B^n. By Corollary 2.4 there is a $\gamma \in \mathbf{GL}(n, A) . \mathbf{E}(n, B)$ such that $t^N \alpha \gamma = 1 + (t-1)\beta$, with β a matrix over A satisfying $\beta^r (1-\beta)^s = 0$ for some $r, s \geq 0$. Now $t^N \alpha \gamma$ induces $\delta\gamma$ on B^n, where γ, belonging to $\mathbf{GL}(n, B)$, defines an automorphism of B^n. In particular $M(\delta) \cong M(\delta\gamma) = M(1 + (t-1)\beta)$.

Lemma. — *If* β *is an endomorphism of a module* M, *and* $\beta^r (1-\beta)^s = 0$, *then* $M = M_0 \oplus M_1$, *where* $M_i = \bigcup_n \ker(i - \beta)^n$, $i = 0, 1$.

Proof. — If $x \in M_0 \cap M_1$, then $\beta^n x = 0 = (1-\beta)^m x$ for some n, m. Since 1 is a linear combination of β^n and $(1-\beta)^m$ we conclude that $x = 0$.

Similarly $1 = \beta^r f + (1-\beta)^s g$, where f and g are integral polynomials in β, so we have, for $x \in M$, $x = \beta^r f x + (1-\beta)^s g x \in M_0 \oplus M_1$.

We apply this lemma to the endomorphism, also denoted β, of A^n defined by the matrix β above. Then $A^n = P_0 \oplus P_1$ and $\beta = \beta_0 \oplus \beta_1$, $\beta_i = \beta | P_i$, with β_0 and $1_{P_1} - \beta_1$ nilpotent. Abbreviating $B \otimes_A M$ by BM we have $B^n = BA^n \cong BP_0 \oplus BP_1$ and $1_{B^n} + (t-1)\beta \cong (1_0 + (t-1)\beta_0) \oplus (t\beta_1 + (1_1 - \beta_1))$, where $1_i = 1_{BP_i}$ and β and β_i are identified with their extensions to BP and BP_i, respectively. The first term is unipotent, so, in particular, an automorphism of BP_0. In the second term β_1 is a unipotent automorphism, so we can write $t\beta_1 + (1_1 - \beta_1) = \beta_1(t . 1_1 + (\beta_1^{-1} - 1_1)) = \beta_1(t . 1_1 + \nu)$ with $\nu = (\beta_1^{-1} - 1_1)$ nilpotent.

We can now compute $M(\delta)$. As noted above

$$M(\delta) \cong M(1 + (t-1)\beta) = \operatorname{coker}(1_0 + (t-1)\beta_0) \oplus \operatorname{coker}(\beta_1(t . 1_1 + \nu)) \cong \operatorname{coker}(t . 1_1 + \nu).$$

Let $\pi = t . 1_1 + \nu$; we will show $\operatorname{coker}(\pi) \cong P_1$ (as A-modules), and this will establish (*). $P_1 (= 1 \otimes P_1)$ is an A-submodule of BP_1, so it suffices to show that $BP_1 = P_1 \oplus \operatorname{im}(\pi)$. If $0 \neq x \in BP_1 =$ "$P_1[t]$", say $x = x_0 + x_1 t + \ldots + x_n t^n$, with $x_i \in P_1$, $x_n \neq 0$, then $\pi x = tx + \nu x$ has "degree" $n + 1 > 0$, since ν is of degree zero. Hence $P_1 \cap \operatorname{im}(\pi) = 0$. Next we note that $BP_1 = P_1 + t.BP_1 = P_1 + (\pi - \nu)BP_1 \subset P_1 + \operatorname{im}(\pi) + \operatorname{im}(\nu) = Q + \operatorname{im}(\nu)$ with

$$Q = P_1 + \operatorname{im}(\pi).$$

Hence $BP_1 = Q + \nu(Q + \nu BP_1) = Q + \nu^2 BP_1 = \ldots = Q + \nu^n BP_1 = Q$, since $\nu^n = 0$ for some n.

To complete the proof of Theorem 2′ we must show that $K^1(C)$ is generated by the image $K^1(A)$, the images of elements of the form $t.1_Q$, and the classes of unipotent automorphisms. Any element of $K^1(C)$ is $W_C(\alpha)$ for some α in some $\mathbf{GL}(n, C)$. Now $\alpha = (t^{-N}.1).(t^N\alpha)$, where $t^N\alpha$ is a polynomial, as above, so it suffices to catch $t^N\alpha$. But, in the notation above,

$$\begin{aligned} t^N\alpha &= (t^N\alpha\gamma)\gamma^{-1} = (1+(t-1)\beta)\gamma^{-1} \\ &= [(1_0+(t-1)\beta_0)\oplus(t\beta_1+(1_1-\beta_1))]\gamma^{-1} \\ &= [u\oplus(t.1_1)\beta_1(1_1+t^{-1}\nu)]\gamma^{-1}. \end{aligned}$$

Here γ^{-1} and β_1 come from A, and u, β_1, and $1_1+t^{-1}\nu$ are unipotent, so the proof is complete.

§ 6. Proof of Grothendieck's Theorem.

We begin with some general remarks on projective modules over graded rings. Let $A=\Sigma_0^\infty A_i$ be a graded ring. If $\mathcal{G}_A$ and $\mathcal{M}_A$ are the categories of graded and ordinary A-modules, there is an exact functor $\mathcal{G}_A \to \mathcal{M}_A$ obtained by forgetting the grading. We will denote this functor by $M \rightsquigarrow \overline{M}$. If M is a graded A-module, let $M(n)$ be the graded A-module defined by $M(n)_i = M_{n+i}$. A free graded A-module is by definition a direct sum of modules of the form $A(n)$. Since $\overline{A(n)} = A$, the forgetting functor preserves free modules and hence preserves projective modules. Conversely:

Lemma 1. — $M \in \mathcal{G}_A$ *is projective if and only if* $\overline{M} \in \mathcal{M}_A$ *is projective.*

Proof. — Let $f: F \to M$ be an epimorphism with F free. If $\overline{M}$ is projective, there is a map $g: \overline{M} \to \overline{F}$ splitting the epimorphism $\overline{F} \to \overline{M}$. Let $g_0: M \to F$ be the component of g of degree 0. Then clearly $fg_0 = 1_M$ so M is projective. The converse is trivial.

Corollary. — *If* $M \in \mathcal{G}_A$ *has projective dimension n in* $\mathcal{G}_A$ *then* $\overline{M}$ *has projective dimension n in* $\mathcal{M}_A$

Proof. — If

$$0 \to N \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

in $\mathcal{G}_A$ with all P_i projective, then

$$0 \to \overline{N} \to \overline{P}_{n-1} \to \ldots \to \overline{P}_0 \to \overline{M} \to 0$$

in $\mathcal{M}_A$ and $\overline{N}$ is projective if and only if N is.

Let us call a graded module M bounded below if there is an integer N such that $M_n = 0$ for $n < -N$. These form an abelian subcategory $\mathcal{G}_A^b$ of $\mathcal{G}_A$ and the preceding arguments clearly apply to $\mathcal{G}_A^b$ also since any $M \in \mathcal{G}_A^b$ is a quotient of a free module $F \in \mathcal{G}_A^b$.

For any graded module M, define $D_i(M) \subset M_i$ by $D_i(M) = \Sigma_j A_j M_{i-j}$ over all $j<i$, and define $Q_i(M) = M_i/D_i(M)$. Then Q_i is additive and right exact. We may regard $\Sigma_i Q_i(M)$ as a graded module over A_0. If $M \in \mathcal{G}_A^b$ and $Q(M)=0$ then $M=0$ since if n is least such that $M_n \neq 0$ then $D_n(M)=0$ and $M_n = Q_n(M)$. If $M \approx A(n)$

is free then $Q_i(M) = 0$ for $i \neq -n$, $Q_{-n}(M) = A_0$. Thus Q sends free modules into free modules and hence sends projective modules into projective modules. Note that a graded A_0-module Q is projective if and only if each Q_n is projective as an ordinary A_0-module.

If Q is a graded A_0-module, then $A \otimes_{A_0} Q$ is a graded A-module where $(A \otimes_{A_0} Q)_n = \Sigma_i A_i \otimes_{A_0} Q_{n-i}$ over all i. Clearly $A \otimes_{A_0} Q$ is isomorphic to $\Sigma_n (A \otimes_{A_0} Q_n)(-n)$ where, in this sum, Q_n is regarded as an ordinary A_0-module. We now determine all projectives of $\mathscr{G}^b_A$.

Lemma 2. — The functor $Q \rightsquigarrow A \otimes_{A_0} Q$ *establishes a one to one correspondence between isomorphism classes of graded projective* A_0*-modules which are bounded below and isomorphism classes of graded projective* A*-modules which are bounded below. Its inverse is given by the functor* Q.

It should be emphasized that those functors do not give an isomorphism of categories since there may be many maps of $A \otimes_{A_0} Q$ which do not come from maps of Q.

Proof. — It is clear that the functors preserve the boundedness condition and that $Q(A \otimes_{A_0} Q) = Q$. All that remains is to show that if P is a projective A-module which is bounded below and $Q = Q(P)$ then there is a (non-natural) isomorphism $P \approx A \otimes_{A_0} Q$. If we regard P and $Q = Q(P)$ as graded A_0-modules, the map $f : P \rightarrow Q$ given by the definition of Q is clearly an epimorphism of graded A_0-modules. Since Q is A_0-projective, there is an A_0-map $g : Q \rightarrow P$ such that $fg = 1_Q$. Now, g defines an A-map $h : A \otimes_{A_0} Q \rightarrow P$ and obviously $Q(h) : Q(A \otimes_{A_0} Q) \approx Q(P)$. Since Q is right exact, this shows that $Q(\operatorname{coker} h) = 0$ and therefore $\operatorname{coker} h = 0$ since all modules involved are bounded below. This shows that h is an epimorphism. Since P is projective, h splits and the additivity of Q shows that $Q(\ker h) \approx \ker[Q(A \otimes_{A_0} Q) \rightarrow Q(P)] = 0$. Therefore $\ker h = 0$.

We now turn to the actual proof of Grothendieck's theorem. From now on we shall assume all graded modules to be zero in all negative dimensions. We shall also assume that A is noetherian and that all modules are finitely generated. Note that a graded module M is finitely generated if and only if $\overline{M}$ is because if $m_1, \ldots, m_n$ generate $\overline{M}$, the homogeneous components of the m_i generate M. In particular, the finitely generated graded A-modules form an abelian category if A is noetherian.

Let the polynomial ring $A[t]$ be graded by $A[t]_n = \sum_{i+j=n} A_i t^j$. We shall, moreover, identify A with $A[t]/(1-t)A[t]$.

Lemma 3. — Every A*-module* M *is isomorphic to* $A \otimes_{A[t]} N$ *for some graded* $A[t]$*-module* N.

Proof. — Write $M \approx A^n/R$, and say R is generated by $\alpha_i = (a(i,1), \ldots, a(i,n))$, $1 \leq i \leq m$. Let d be a bound for the degrees of the $a(i,j)$. If $a = a_0 + a_1 + \ldots + a_d$ is an element of degree $\leq d$ in A write $a' = a_0 t^d + a_1 t^{d-1} + \ldots + a_d \in A[t]_d$. Then let R' be the submodule of $A[t]^n$ generated by the $\alpha'_i = (a(i,1)', \ldots, a(i,n)')$, $1 \leq i \leq m$, and set $N = A[t]^n/R'$. Clearly N is a graded $A[t]$-module. The exact sequence $R' \rightarrow A[t]^n \rightarrow N \rightarrow 0$ yields $A \otimes_{A[t]} R' \rightarrow A^n \rightarrow A \otimes_{A[t]} N \rightarrow 0$, and it is clear that the image in A^n of $A \otimes_{A[t]} R'$ is just R. This proves the lemma.

Lemma 4. — $A \otimes_{A[t]}$ *is exact on the category of graded* $A[t]$*-modules.*

Proof. — The lemma amounts to the assertion that if M is a graded A[t]-module and M′ a graded submodule, then $(1-t)M \cap M' = (1-t)M'$. Let $m = m_0 + m_1 + \ldots \in M$ and suppose $(1-t)m \in M'$; i.e. $m_0 + (m_1 - tm_0) + \ldots + (m_n - tm_{n-1}) + \ldots \in M'$. Then it follows by an obvious induction that each $m_i \in M'$.

Since the functor taking P into $A \otimes_{A_0} P$ is exact on the category of projective A_0-modules P, it defines a map $K^0(A_0) \to K^0(A)$ where $K^0(A)$ refers to the category of ordinary projective A-modules. We also have a ring homomorphism $A \to A_0$ which is the identity on A_0 and sends $A_i \to 0$ for $i \neq 0$. The functor sending P into $P \otimes_A A_0$ gives a map $K^0(A) \to K^0(A_0)$ which is clearly a left inverse for $K^0(A_0) \to K^0(A)$.

Theorem 6. — *Let* $A = A_0 + A_1 + \ldots$ *be a graded left regular ring. Then* $K^0(A_0) \to K^0(A)$ *is an isomorphism,* $K^0(A)$ *refering to the category of ordinary* A*-modules.*

Proof. — Since the homomorphism in question has a left inverse, we need only show that it is surjective. Let P be a projective A-module. Choose a graded A[t]-module N, such that $P = A \otimes_{A[t]} N$ (Lemma 2). Choose a finite resolution

$$0 \to P_d \to \ldots \to P_0 \to N \to 0$$

by graded projective A[t]-modules. This is possible by our hypothesis on A, the Syzygy Theorem, and Lemma 1, Corollary.

By Lemma 4,

$$(*) \qquad 0 \to A \otimes_{A[t]} P_d \to \ldots \to A \otimes_{A[t]} P_0 \to A \otimes_{A[t]} N \approx P \to 0$$

is exact. Now by Lemma 2 (applied to A[t]) each P_i is (disregarding the grading) a direct sum of modules of the form $A[t] \otimes_{A_0} Q$ with Q A_0-projective. Since $A \otimes_{A[t]} (A[t] \otimes_{A_0} Q) \approx A \otimes_{A_0} Q$, we conclude that each $A \otimes_{A[t]} P_i$ is a direct sum of modules of the latter type. This combined with the exact sequence (*) clearly proves the theorem.

Corollary. — *If* A *is a left regular ring, then the map* $K^0(A) \to K^0(A[t])$ *sending* P *into* $A[t] \otimes_A P$ *is an isomorphism.*

Corollary. — *If* A *is a left regular ring, then the map* $K^0(A) \to K^0(A[t, t^{-1}])$ *sending* P *into* $A[t, t^{-1}] \otimes_A P$ *is an isomorphism.*

Proof. — The map has a left inverse induced by the ring homomorphism $A[t, t^{-1}] \to A$ sending t into 1. The map also factors through $K^0(A[t])$ by means of the ring homomorphisms

$$A \to A[t] \to A[t, t^{-1}].$$

By the theorem, it will suffice to show that $K^0(A[t]) \to K^0(A[t, t^{-1}])$ is surjective. Let P be a projective $A[t, t^{-1}]$-module. We can define P by a system of generators and relations. By multiplying each relation by a suitable power of t we can insure that no negative powers occur. Therefore these relations considered over A[t] define a module M such that $P \approx A[t, t^{-1}] \otimes_{A[t]} M$. Now M has a finite projective resolution over A[t]. Tensoring this resolution with $A[t, t^{-1}]$ resolves P by projective modules coming from A[t]. In this last step we need the fact that $A[t, t^{-1}]$ is A[t]-flat, which follows from the usual results about localization. Alternatively we can observe that $A[t, t^{-1}]$ is the direct limit of the free modules $A[t]t^n$ as $n \to -\infty$.

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