ESAIM: Modélisation mathématique et analyse numérique

Ziwen Jiang

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ESAIM: Modélisation mathématique et analyse numérique, tome 33, nº 3 (1999), p. 531-546

<http://www.numdam.org/item?id=M2AN_1999__33_3_531_0>

Mathematical Modelling and Numerical Analysis M2AN, Vol 33, N° 3, 1999, p 531–546
Modelisation Mathematique et Analyse Numerique

$L^\infty(L^2)$ AND $L^\infty(L^\infty)$ ERROR ESTIMATES FOR MIXED METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

ZIWEN JIANG[1]

Abstract. Error estimates in $L^\infty(0,T,L^2(\Omega))$, $L^\infty(0,T,L^2(\Omega)^2)$, $L^\infty(0,T,L^\infty(\Omega))$, $L^\infty(0,T,L^\infty(\Omega)^2)$, Ω in $\mathbb{R}^2$, are derived for a mixed finite element method for the initial-boundary value problem for integro-differential equation

$$u_t = \operatorname{div}\left\{a \nabla u + \int_0^t b_1 \nabla u \mathrm{d}\tau + \int_0^t \mathbf{c} u \mathrm{d}\tau\right\} + f$$

based on the Raviart-Thomas space $\mathbf{V}_h \times W_h \subset \mathbf{H}(\operatorname{div},\Omega) \times L^2(\Omega)$ Optimal order estimates are obtained for the approximation of u, u_t in $L^\infty(0,T,L^2(\Omega))$ and the associated velocity $\mathbf{p}$ in $L^\infty(0,T,L^2(\Omega)^2)$, $\operatorname{div}\mathbf{p}$ in $L^\infty(0,T,L^2(\Omega))$ Quasi optimal order estimates are obtained for the approximation of u in $L^\infty(0,T,L^\infty(\Omega))$ and $\mathbf{p}$ in $L^\infty(0,T,L^\infty(\Omega)^2)$

Résumé. Les estimations d'erreur dans $L^\infty(0,T,L^2(\Omega))$, $L^\infty(0,T,L^2(\Omega)^2)$, $L^\infty(0,T,L^\infty(\Omega))$ et $L^\infty(0,T,L^\infty(\Omega)^2)$ avec Ω sous espace de $\mathbb{R}^2$, sont obtenues par une méthode mixte d'éléments finis à partir de la valeur initiale du probleme sur la frontière de l'équation intégro-differentielle

$$u_t = \operatorname{div}\left\{a \nabla u + \int_0^t b_1 \nabla u \mathrm{d}\tau + \int_0^t \mathbf{c} u \mathrm{d}\tau\right\} + f$$

basée sur l'espace de Raviart-Thomas $\mathbf{V}_n \times W_h \subset \mathbf{H}(\operatorname{div},\Omega) \times L^2(\Omega)$ Les estimations d'ordre optimal sont obtenues à partir de l'approximation de u, u_t dans $L^\infty(0,T,L^2(\Omega))$, la vitesse associée $\mathbf{p}$ dans $L^\infty(0,T,L^2(\Omega)^2)$, $\operatorname{div}\mathbf{p}$ dans $L^\infty(0,T,L^2(\Omega))$ Les estimations d'ordre quasi-optimal sont obtenues à partir de l'approximation de u dans $L^\infty(0,T,L^2(\Omega))$ et de $\mathbf{p}$ dans $L^\infty(0,T,L^\infty(\Omega)^2)$

AMS Subject Classification. 35k15, 35k20, 45k05

Received May 14, 1997 Revised August 5, 1998

Keywords and phrases Error estimates, mixed finite element, integro-differential equations, parabolic type

[1] Department of Mathematics, Shandong Normal University, Jinan, Shandong 250014, People's Republic of China

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. For fixed $0 < T < \infty$, we shall consider the following integro-differential equation of parabolic type:

$$\begin{cases} u_t(\mathbf{x},t) = \operatorname{div}\{a(\mathbf{x},t) \nabla u(\mathbf{x},t) + \int_0^t b_1(\mathbf{x},t,\tau) \nabla u(\mathbf{x},\tau)d\tau & (\mathbf{x},t) \in \Omega \times (0,T], \\ \qquad + \int_0^t \mathbf{c}(\mathbf{x},t,\tau)u(\mathbf{x},\tau)d\tau\} + f(\mathbf{x},t), & \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times [0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where u_t denotes the time derivative of the function u, ∇u denotes the gradient of the function u, and $\operatorname{div}\mathbf{v}$ denotes the divergence of the vector function $\mathbf{v}$; $a, b_1, \mathbf{c}, f$, and u_0 are known functions.

Integro-differential equations can arise from many physical processes in which it is deficiency (the local characteristic) of the usual diffusion equations (see, for example [1]). For linear or nonlinear integro-differential equations, the standard finite element methods have received considerable attention and are well studied (see, for example [1,2,6–8,15]). It is known that the mixed finite element method computes both the scalar (pressure) and vector (flux) functions simultaneously with comparable accuracy, be it directly or through postprocessing. Recently, mixed methods have been formulated by several authors and are well studied for second-order elliptic problems, parabolic problems, and second-order hyperbolic problems (see, for example [3–5, 9–14, 16]). In this paper, we shall study mixed methods to approximate the solution of the problems (1.1). In order to achieve this aim, we assume that the problem (1.1) is uniquely solvable for any $f \in C^1(0,T;L^2(\Omega))$ and $u_0 \in H^s(\Omega)$. For the function a, we assume that there exist constants $c_0, c_1 > 0$ such that $0 < c_0 \le a \le c_1$. We also assume that the functions a, b_1, and $\mathbf{c}$ are smooth and bounded together with their derivatives. Here and in what follows, we will not write the independent variables $\mathbf{x}, t, \tau$ for any function unless it is necessary. Vectors will be expressed in boldface.

For $1 \le s \le \infty$ and k any nonnegative integer, let

$$W^{k,s}(\Omega) = \{f \in L^s(\Omega) \quad | \quad D^q f \in L^s(\Omega) \quad |q| \le k\}$$

denote the Sobolev spaces endowed with the norm

$$\|f\|_{k,s;\Omega} = \left(\sum_{|q|\le k} \|D^q f\|^s_{L^s(\Omega)}\right)^{1/s}$$

(the subscript Ω will always be omitted unless necessary to avoid ambiguity). Let $H^k(\Omega) = W^{k,2}(\Omega)$ with the norm $\|\cdot\|_k = \|\cdot\|_{k,2}$ (the notation $\|\cdot\|$ will mean $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$). We shall denote by $(\cdot,\cdot)$ the inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$, that is $(\eta,\theta) = \int_\Omega \eta\theta d\mathbf{x}$ or $(\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{v}\cdot\mathbf{w}d\mathbf{x}$.

Let $\mathbf{V} = \mathbf{H}(\operatorname{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 \quad | \quad \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$ normed by $\|\mathbf{v}\|_{\mathbf{V}} = \|\mathbf{v}\| + \|\operatorname{div}\mathbf{v}\|$, and let $W = L^2(\Omega)$.

Let χ be a normed space with the norm $\|\cdot\|_\chi$. $L^q(0,T;\chi)$ denote the space of the maps of $[0,T]$ into χ and define the following norms for $1 \le q < \infty$ and suitable functions $v : [0,T] \to \chi$:

$$\|v\|_{L^q(0,T;\chi)} = \left(\int_0^T \|v(t)\|^q_\chi dt\right)^{1/q}.$$

Take $C^k(0,T;\chi)$ to be the space of k-times continuously differentiable maps of $[0,T]$ into χ, and endowed with the norms $\|\cdot\|_{L^q(0,T;\chi)}$ for $1 \le q < \infty$.

For $q = \infty$, the usual modification is made.

To formulate the weak form of (1.1) appropriate for the mixed method, let

$$\mathbf{p} = -a \bigtriangledown u - \int_0^t b_1 \bigtriangledown u \mathrm{d}\tau - \int_0^t \mathbf{c} u \mathrm{d}\tau,$$

and set

$$\alpha(\mathbf{x},t) = a^{-1}(\mathbf{x},t), \quad b(\mathbf{x},t,\tau) = \alpha(\mathbf{x},t) b_1(\mathbf{x},t,\tau),$$
$$\beta(\mathbf{x},t,\tau) = -\bigtriangledown b(\mathbf{x},t,\tau) + \alpha(\mathbf{x},t)\mathbf{c}(\mathbf{x},t,\tau).$$

Then, the problem (1.1) can be written in the mixed form of the first order system:

$$\begin{cases} u_t + \mathrm{div}\mathbf{p} = f, & \text{in} \quad \Omega \times (0,T], \\ \alpha \mathbf{p} + \nabla u + \int_0^t \nabla(bu)\mathrm{d}\tau + \int_0^t \beta u \mathrm{d}\tau = \mathbf{0}, & \text{in} \quad \Omega \times (0,T], \\ u = 0, & \text{on} \quad \partial\Omega \times [0,T], \\ u(0) = u_0, & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

The weak form of (1.2) [or (1.1)] is obtained by seeking a solution $\{\mathbf{p},u\} : [0,T] \to \mathbf{V} \times W$ such that

$$\begin{cases} (u_t, w) + (\mathrm{div}\mathbf{p}, w) = (f,w), & \forall w \in W, \quad 0 < t \le T, \\ (\alpha\mathbf{p}, \mathbf{v}) - (u + \int_0^t bu\mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta u \mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}, \quad 0 < t \le T, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

In order to define an adequate finite element approximation procedure for $\{\mathbf{p},u\}$, we consider the finite-dimensional subspace $\mathbf{V}_h \times W_h$ of $\mathbf{V} \times W$ associated with a quasiuniform partition $\mathcal{T}_h$ of Ω into triangles of diameter not greater than h $(0 < h < 1)$. The boundary elements of $\mathcal{T}_h$ are allowed to have one curved edge. We choose $\mathbf{V}_h \times W_h$ as the Raviart-Thomas space [5, 11, 12] of index $k \ge 0$ and introduce the L^2-projection $p_h : W \to W_h$, and the Raviart-Thomas projection $\pi_h : H^1(\Omega)^2 \to \mathbf{V}_h$, which have the following useful commuting property:

$$\mathrm{div} \circ \pi_h = p_h \circ \mathrm{div} : H^1(\Omega)^2 \to W_h. \tag{1.4}$$

These projections have the following approximation properties [5, 11, 12]:

$$\|w - p_h w\|_{-s} \le \mathrm{ch}^{l+s} \|w\|_l, \qquad 0 \le l, \qquad s \le k+1, \tag{1.5}$$

$$\|w - p_h w\|_{0,q} \le \mathrm{ch}^{l} \|w\|_{l,q}, \qquad 0 \le l \le k+1, \qquad 1 \le q \le +\infty, \tag{1.6}$$

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_{0,q} \le \mathrm{ch}^{l} \|\mathbf{v}\|_{l,q}, \qquad \frac{1}{q} < l \le k+1, \qquad 1 \le q \le +\infty, \tag{1.7}$$

$$\|\mathrm{div}(\mathbf{v} - \pi_h \mathbf{v})\| \le \mathrm{ch}^{l} \|\mathrm{div}\mathbf{v}\|_l, \qquad 0 \le l \le k+1. \tag{1.8}$$

Our *continuous-in-time mixed finite element approximation* to (1.3) is defined by the determining of a pair $\{\mathbf{p}_h, u_h\} : [0,T] \to \mathbf{V}_h \times W_h$ such that

$$\begin{cases} (u_{h,t}, w) + (\mathrm{div}\mathbf{p}_h, w) = (f,w), & \forall w \in W_h, \quad 0 < t \le T, \\ (\alpha\mathbf{p}_h, \mathbf{v}) - (u_h + \int_0^t bu_h\mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta u_h \mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}_h, \quad 0 < t \le T, \\ \mathbf{p}_h(0) = \tilde{\mathbf{p}}_h(0), \qquad u_h(0) = \tilde{u}_h(0), \end{cases} \tag{1.9}$$

where $\tilde{\mathbf{p}}_h(0)$ and $\tilde{u}_h(0)$ are defined in (3.1) for $t = 0$.

We shall establish the existence and uniqueness of the solution of (1.9) in Section 2; moreover, we shall show that the differences $u - u_h$ and $\mathbf{p} - \mathbf{p}_h$ are of optimal order in $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega)^2)$, and are of quasi-optimal order in $L^\infty(0,T;L^\infty(\Omega))$ and $L^\infty(0,T;L^\infty(\Omega)^2)$ in Section 4. In order to obtain the above error estimates, we shall define a generalized mixed elliptic projection associated with our equations and study the error estimates of the generalized mixed elliptic projection in Section 3. In Section 4, we shall also show the optimal order estimates in $L^\infty(0,T;L^2(\Omega))$ of $\mathrm{div}(\mathbf{p}-\mathbf{p}_h)$ and $(u-u_h)_t$.

In this paper, c will be used to denote various positive constants and combinations of positive constants that are independent of h and t unless otherwise stated. In the remaining sections, r will be a fixed integer.

2. Existence and uniqueness

In this section, we shall demonstrate the existence and uniqueness of the solution of (1.9).

In fact, if $\mathbf{V}_h = \mathrm{span}\{\Psi_1, \Psi_2, \cdots, \Psi_m\}$ and $W_h = \mathrm{span}\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$, let

$$\mathbf{p}_h(\mathbf{x},t) = \sum_{j=1}^{m} \alpha_j(t)\Psi_j(\mathbf{x}), \quad u_h(\mathbf{x},t) = \sum_{i=1}^{n} \beta_i(t)\varphi_i(\mathbf{x}),$$

where $\alpha_j(0), j = 1,2,\cdots,m$, and $\beta_i(0), i = 1,2,\cdots,n$, are known by the solvability and uniqueness of (3.1) which has been demonstrated in Section 3, and let

$$\begin{gathered} a_{ki} = (\varphi_k, \varphi_i), \quad b_{kj} = (\varphi_k, \mathrm{div}\Psi_l), \quad c_{lj}(t) = (\Psi_l, \alpha\Psi_j), \quad d_{li}(t,\tau) = (b\varphi_i, \mathrm{div}\Psi_l) - (\beta\varphi_i, \Psi_l), \\ \mathbf{f}(t) = ((f(t),\varphi_1), \cdots, (f(t),\varphi_n)), \quad \alpha(t) = (\alpha_1(t), \cdots, \alpha_m(t)), \quad \overline{\beta}(t) = (\beta_1(t), \cdots, \beta_n(t)), \\ A = (a_{ki})_{n\times n}, \quad B = (b_{kj})_{n\times m}, \quad C(t) = (c_{lj}(t))_{m\times m}, \quad D(t,\tau) = (d_{li}(t,\tau))_{m\times n}. \end{gathered}$$

Then for $0 < t \le T$, (1.9) can be written as

$$\begin{cases} A\overline{\beta}'(t) + B\alpha(t) = \mathbf{f}(t), \\ C(t)\alpha(t) - B^T\overline{\beta}(t) - \int_0^t D(t,\tau)\overline{\beta}(\tau)\mathrm{d}\tau = \mathbf{0}, \end{cases} \tag{2.1}$$

where A and $C(t), 0 < t \le T$ are positive definite matrices. It is easy to see that (2.1) is actually a system of integro-differential equation of Volterra type. From the positive definiteness of A and $C(t). 0 < t \le T$, we see that the system (2.1) with the initial value $\alpha(0)$ and $\overline{\beta}(0)$ possesses a unique solution $\{\alpha(t), \overline{\beta}(t)\}$. Consequently, the existence and uniqueness of the solution of (1.9) has been demonstrated.

3. A generalized mixed elliptic projection

In the study of mixed methods for parabolic problems, we usually introduce a mixed elliptic projection associated with our equations. Modifying this idea according to our integro-differential equations, we define a map $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\} : [0,T] \to \mathbf{V}_h \times W_h$ such that

$$\begin{cases} (\mathrm{div}(\mathbf{p} - \tilde{\mathbf{p}}_h), w) = 0, & \forall w \in W_h, \quad 0 \le t \le T, \\ (\alpha(\mathbf{p} - \tilde{\mathbf{p}}_h), \mathbf{v}) - (u - \tilde{u}_h + \int_0^t b(u - \tilde{u}_h)\mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta(u - \tilde{u}_h)\mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}_h, \quad 0 \le t \le T. \end{cases} \tag{3.1}$$

Before collecting the results of the error analysis of $\tilde{u}_h$ and $\tilde{\mathbf{p}}_h$, let us demonstrate the existence and uniqueness of the solution of (3.1). Since (3.1) is linear, it suffices to show that the associated homogeneous system has only the trivial solution, which is

$$\begin{cases} (\mathrm{div}\tilde{\mathbf{p}}_h, w) = 0, & \forall w \in W_h, \quad 0 \le t \le T, \\ (\alpha\tilde{\mathbf{p}}_h, \mathbf{v}) - (\tilde{u}_h + \int_0^t b\tilde{u}_h\mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta\tilde{u}_h\mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}_h, \quad 0 \le t \le T. \end{cases} \tag{3.2}$$

Choosing $w = \mathrm{div}\tilde{\mathbf{p}}_h$ and $\mathbf{v} = \tilde{\mathbf{p}}_h$ in (3.2), then we have $\mathrm{div}\tilde{\mathbf{p}}_h = 0$ and

$$c_1^{-1}\|\tilde{\mathbf{p}}_h\|^2 \le (\alpha\tilde{\mathbf{p}}_h, \tilde{\mathbf{p}}_h) = -\left(\int_0^t \beta\tilde{u}_h\mathrm{d}\tau, \tilde{\mathbf{p}}_h\right) \le c\int_0^t \|\tilde{u}_h\|\mathrm{d}\tau\|\tilde{\mathbf{p}}_h\|,$$

which implies that

$$\|\tilde{\mathbf{p}}_h\| \le c \int_0^t \|\tilde{u}_h\| \mathrm{d}\tau. \tag{3.3}$$

On the other hand, from [12] we know for $\mathbf{V}_h \times W_h$ that there exists a positive constant c independent h such that

$$\|w_h\| \le c \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(w_h, \mathrm{div}\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}}, \quad \forall w_h \in W_h. \tag{3.4}$$

Hence from (3.4) and (3.2) we have

$$\|\tilde{u}_h\| \le c \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\tilde{u}_h, \mathrm{div}\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \le c(\|\tilde{\mathbf{p}}_h\| + \int_0^t \|\tilde{u}_h\| \mathrm{d}\tau). \tag{3.5}$$

Combining (3.3) with (3.5),

$$\|\tilde{u}_h\| \le c \int_0^t \|\tilde{u}_h\| \mathrm{d}\tau. \tag{3.6}$$

Using Gronwall's lemma, we have from (3.6) and (3.3) that $\|\tilde{u}_h\| = 0$, and $\|\tilde{\mathbf{p}}_h\| = 0$. Hence $\tilde{u}_h \equiv 0$ and $\tilde{\mathbf{p}}_h \equiv \mathbf{0}$. So, the existence and uniqueness of the solution of (3.1) has been demonstrated and $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\}$ in (3.1) is well defined.

Now, let us study some properties of $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\}$. Let $\mathbf{p}_1 = \mathbf{p} - \tilde{\mathbf{p}}_h, u_1 = p_h u - \tilde{u}_h$, and $u_2 = u - p_h u$.

First, we consider the properties of $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\}$ when $b = b(t,\tau)$ in (3.1). Note that $b = b(t,\tau)$ and $p_h(bu) = bp_h u$, then we rewrite (3.1) as

$$\begin{cases} (\mathrm{div}\mathbf{p}_1, w) = 0, & \forall w \in W_h, \quad 0 \le t \le T, \\ (\alpha \mathbf{p}_1, \mathbf{v}) - (u_1 + \int_0^t b u_1 \mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta(u_1 + u_2)\mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}_h, \quad 0 \le t \le T. \end{cases} \tag{3.7}$$

Lemma 3.1. *If $\mathbf{p}_1, u_1$, and u_2 satisfy the relation (3 7), and assume that Ω is 2-regular (see the definition of 2-regularity in* [5], *p. 42), then for all $0 \le t \le T$, there exists a constant $c > 0$, independent of h and t, such that*

$$\|u_1\| \le c \left\{ h\|\mathbf{p}_1\| + h^{2-\delta_{k0}} \|\mathrm{div}\mathbf{p}_1\| + \int_0^t (h\|u_2\| + \|u_2\|_{-1}) \mathrm{d}\tau \right\},$$

where for $k = 0$, $\delta_{k0} = 1$; $k \ge 1$, $\delta_{k0} = 0$.

Proof. For $\psi \in L^2(\Omega)$, let $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\mathrm{div}(a \bigtriangledown \phi) = \psi \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial\Omega. \tag{3.8}$$

Then we have $\|\phi\|_2 \le c\|\psi\|$. For $0 \le t \le T$, using (1.4) and (3.7),

$$\begin{aligned} (u_1, \psi) &= (u_1, \mathrm{div}(a \bigtriangledown \phi)) = (u_1, \mathrm{div}(\pi_h(a \bigtriangledown \phi))) \\ &= (\alpha \mathbf{p}_1 + \textstyle\int_0^t \beta(u_1 + u_2)\mathrm{d}\tau, \pi_h(a \bigtriangledown \phi)) - (\textstyle\int_0^t b u_1 \mathrm{d}\tau, \mathrm{div}(\pi_h(a \bigtriangledown \phi))). \end{aligned} \tag{3.9}$$

Noting that $\alpha = a^{-1}$ and (1.6–1.7) together with (3.7)

$$\begin{aligned} (\alpha \mathbf{p}_1, \pi_h(a \bigtriangledown \phi)) &= (\alpha \mathbf{p}_1, \pi_h(a \bigtriangledown \phi) - a \bigtriangledown \phi) + (\mathbf{p}_1, \bigtriangledown \phi) \\ &= (\alpha \mathbf{p}_1, \pi_h(a \bigtriangledown \phi) - a \bigtriangledown \phi) + (\mathrm{div}\mathbf{p}_1, p_h\phi - \phi) \\ &\le c\|\mathbf{p}_1\| \|\pi_h(a \bigtriangledown \phi) - a \bigtriangledown \phi\| + \|\mathrm{div}\mathbf{p}_1\| \|p_h\phi - \phi\| \\ &\le c(h\|\mathbf{p}_1\| + h^{2-\delta_{k0}} \|\mathrm{div}\mathbf{p}_1\|) \|\phi\|_2 \\ &\le c(h\|\mathbf{p}_1\| + h^{2-\delta_{k0}} \|\mathrm{div}\mathbf{p}_1\|) \|\psi\|. \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
(\int_0^t \beta u_1 \mathrm{d}\tau, \pi_h(a \bigtriangledown \phi)) &\le \| \int_0^t \beta u_1 \mathrm{d}\tau \| \|\pi_h(a \bigtriangledown \phi)\| \\
&\le c \int_0^t \|u_1\| \mathrm{d}\tau (\|\pi_h(a \bigtriangledown \phi) - a \bigtriangledown \phi\| + \|a \bigtriangledown \phi\|) \\
&\le c \int_0^t \|u_1\| \mathrm{d}\tau (h \| \bigtriangledown \phi\|_1 + \| \bigtriangledown \phi\|) \\
&\le c \int_0^t \|u_1\| \mathrm{d}\tau \|\psi\|. \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
(\int_0^t \beta u_2 \mathrm{d}\tau, \pi_h(a \bigtriangledown \phi)) &= (\int_0^t \beta u_2 \mathrm{d}\tau, \pi_h(a \bigtriangledown \phi) - a \bigtriangledown \phi) + (\int_0^t \beta u_2 \mathrm{d}\tau, a \bigtriangledown \phi) \\
&\le c \int_0^t (h\|u_2\| + \|u_2\|_{-1}) \mathrm{d}\tau \| \bigtriangledown \phi\|_1 \\
&\le c \int_0^t (h\|u_2\| + \|u_2\|_{-1}) \mathrm{d}\tau \|\psi\|. \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
-(\int_0^t b u_1 \mathrm{d}\tau, \mathrm{div}(\pi_h(a \bigtriangledown \phi))) &= -(\int_0^t b u_1 \mathrm{d}\tau, p_h(\mathrm{div}(a \bigtriangledown \phi))) \\
&\le c \int_0^t \|u_1\| \mathrm{d}\tau \| \bigtriangledown \phi\|_1 \\
&\le c \int_0^t \|u_1\| \mathrm{d}\tau \|\psi\|. \qquad (3.13)
\end{aligned}
$$

Combining (3.9–3.13), we have for $0 \le t \le T$ that

$$
\begin{aligned}
\|u_1\| &= \sup_{\psi \in L^2(\Omega), \psi \ne 0} \frac{(u_1, \psi)}{\|\psi\|} \\
&\le c \left\{ h\|\mathbf{p}_1\| + h^{2-\delta_{k0}} \|\mathrm{div}\mathbf{p}_1\| + \int_0^t (h\|u_2\| + \|u_2\|_{-1}) \mathrm{d}\tau \right\} + c \int_0^t \|u_1\| \mathrm{d}\tau.
\end{aligned}
$$

Using Gronwall's lemma, we have

$$
\|u_1\| \le c \left\{ h\|\mathbf{p}_1\| + h^{2-\delta_{k0}} \|\mathrm{div}\mathbf{p}_1\| + \int_0^t (h\|u_2\| + \|u_2\|_{-1}) \mathrm{d}\tau \right\}, 0 \le t \le T.
$$

The proof is complete. □

Using Lemma 3.1, we have the following theorem:

Theorem 3.2. *Let $\{\mathbf{p}, u\}$ be the solution of (1.3) and $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\}$ be that of (3.1). Assume that $\{\mathbf{p}, u\}$ is sufficiently smooth and that Ω is 2-regular. Then for $0 \le t \le T$, there exist constants $c > 0$, independent of h*

and t, such that

$$\|p_h u - \tilde{u}_h\| \le \mathrm{ch}^{r+1-\delta_{k0}} \left\{ \|\mathbf{p}\|_r + \int_0^t \|u\|_r \mathrm{d}\tau \right\}, \qquad 1 \le r \le k+1,$$

$$\|\mathrm{div}(\mathbf{p} - \tilde{\mathbf{p}}_h)\| \le \mathrm{ch}^r \|\mathbf{p}\|_{r+1}, \qquad 0 \le r \le k+1,$$

$$\|\mathbf{p} - \tilde{\mathbf{p}}_h\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1)\mathrm{d}\tau\}, & k=0, \\ \mathrm{ch}^r\{\|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r)\mathrm{d}\tau\}, & k \ge 1, \end{cases} \qquad 2 \le r \le k+1,$$

$$\|u - \tilde{u}_h\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}\|_1 + \|u\|_1 + \int_0^t \|u\|_1 \mathrm{d}\tau\}, & k=0, \\ \mathrm{ch}^r\{\|\mathbf{p}\|_{r-1} + \|u\|_r + \int_0^t \|u\|_{r-1}\mathrm{d}\tau\}, & k \ge 1, \end{cases} \qquad 2 \le r \le k+1,$$

$$\|u - \tilde{u}_h\|_{0,\infty} \le \mathrm{ch}^r \left\{ \|\mathbf{p}\|_r + \|u\|_{r,\infty} + \int_0^t \|u\|_r \mathrm{d}\tau \right\}, \qquad k \ge 1, \quad 2 \le r \le k+1.$$

Proof. For $0 \le t \le T$, the proof proceeds in three steps:

(i) We consider the estimate $\|\mathrm{div}(\mathbf{p} - \tilde{\mathbf{p}}_h)\|$. Noting that (3.7),

$$\begin{aligned} (\mathrm{div}\mathbf{p}_1, \mathrm{div}\mathbf{p}_1) &= (\mathrm{div}\mathbf{p}_1, \mathrm{div}(\mathbf{p} - \pi_h \mathbf{p})) \\ &\le \|\mathrm{div}\mathbf{p}_1\| \|\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p})\|, \end{aligned}$$

then we have from (1.8) that

$$\|\mathrm{div}(\mathbf{p} - \tilde{\mathbf{p}}_h)\| = \|\mathrm{div}\mathbf{p}_1\| \le \mathrm{ch}^r \|\mathbf{p}\|_{r+1}, \quad 0 \le r \le k+1. \tag{3.14}$$

(ii) We consider the estimates $\|p_h u - \tilde{u}_h\|$ and $\|\mathbf{p} - \tilde{\mathbf{p}}_h\|$. Choosing $\mathbf{v} = \pi_h \mathbf{p} - \tilde{\mathbf{p}}_h$ and $w = \mathrm{div}\mathbf{v}$ in (3.7), and noting that (1.4), we have $\mathrm{div}(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h) \equiv 0$ at first. Hence

$$\begin{aligned} c_1^{-1} \|\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h\|^2 &\le (\alpha(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h), \pi_h \mathbf{p} - \tilde{\mathbf{p}}_h) \\ &= -\left(\int_0^t \beta(u_1 + u_2)\mathrm{d}\tau + \alpha(\mathbf{p} - \pi_h \mathbf{p}), \pi_h \mathbf{p} - \tilde{\mathbf{p}}_h \right) \\ &\le c \left(\int_0^t (\|u_1\| + \|u_2\|)\mathrm{d}\tau + \|\mathbf{p} - \pi_h \mathbf{p}\| \right) \|\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h\|. \end{aligned}$$

Noting that (1.6–1.7), we have

$$\|\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h\| \le c \left\{ h^r (\|\mathbf{p}\|_r + \int_0^t \|u\|_r \mathrm{d}\tau) + \int_0^t \|u_1\| \mathrm{d}\tau \right\}, \quad 1 \le r \le k+1,$$

and so

$$\|\mathbf{p} - \tilde{\mathbf{p}}_h\| \le c \left\{ h^r \left(\|\mathbf{p}\|_r + \int_0^t \|u\|_r \mathrm{d}\tau \right) + \int_0^t \|u_1\| \mathrm{d}\tau \right\}, \quad 1 \le r \le k+1. \tag{3.15}$$

Combining Lemma 3.1 and (1.5) together with (3.14–3.15), and using Gronwall's lemma, we have

$$\|p_h u - \tilde{u}_h\| = \|u_1\| \le \mathrm{ch}^{r+1-\delta_{k0}} \left(\|\mathbf{p}\|_r + \int_0^t \|u\|_r \mathrm{d}\tau \right), \quad 1 \le r \le k+1, \tag{3.16}$$

$$\|\mathbf{p} - \tilde{\mathbf{p}}_h\| = \|\mathbf{p}_1\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1)\mathrm{d}\tau\}, & k=0, \\ \mathrm{ch}^r\{\|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r)\mathrm{d}\tau\}, & k \ge 1, \ 2 \le r \le k+1. \end{cases}$$

(iii) Now we consider the estimates $\|u-\tilde{u}_h\|$ and $\|u-\tilde{u}_h\|_{0,\infty}$.
When $k=0$, we have from (1.6) and (3.16) that

$$\|u-\tilde{u}_h\| \le \|u-p_h u\| + \|u_1\| \le \mathrm{ch}\left(\|\mathbf{p}\|_1 + \|u\|_1 + \int_0^t \|u\|_1 \mathrm{d}\tau\right).$$

When $k \ge 1$, we have from (1.6) and (3.16) that,

$$\|u-\tilde{u}_h\| \le \mathrm{ch}^r\left(\|u\|_r + \|\mathbf{p}\|_{r-1} + \int_0^t \|u\|_{r-1}\mathrm{d}\tau\right), \quad 2 \le r \le k+1.$$

We now use the following "inverse estimates": For $2 \le \nu \le \theta$,

$$\|p_h u - \tilde{u}_h\|_{0,\theta} \le \mathrm{ch}^{\frac{2}{\theta}-\frac{2}{\nu}} \|p_h u - \tilde{u}_h\|_{0,\nu}. \tag{3.17}$$

Then we have from (1.6) and (3.16–3.17) that,

$$\|u-\tilde{u}_h\|_{0,\infty} \le \mathrm{ch}^r\left(\|u\|_{r,\infty} + \|\mathbf{p}\|_r + \int_0^t \|u\|_r \mathrm{d}\tau\right), \quad 2 \le r \le k+1, \quad k \ge 1.$$

The proof is complete. □

Theorem 3.3. *Under the conditions of Theorem 3.2. Then for $0 \le t \le T$, there exist constants $c > 0$, independent of h and t, such that*

$$\|\mathrm{div}(\mathbf{p}-\tilde{\mathbf{p}}_h)_t\| \le \mathrm{ch}^r \|\mathbf{p}_t\|_{r+1}, \quad 0 \le r \le k+1,$$

$$\|(\mathbf{p}-\tilde{\mathbf{p}}_h)_t\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_t\|_1 + \|u\|_1 + \|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1)\mathrm{d}\tau\}, & k=0, \\ \mathrm{ch}^r\{\|\mathbf{p}_t\|_r + \|u\|_r + \|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r)\mathrm{d}\tau\}, & k \ge 1, \quad 2 \le r \le k+1, \end{cases}$$

$$\|(u-\tilde{u}_h)_t\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_t\|_1 + \|u_t\|_1 + \|u\|_1 + \|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1)\mathrm{d}\tau\}, & k=0, \\ \mathrm{ch}^r\{\|\mathbf{p}_t\|_r + \|u_t\|_r + \|u\|_r + \|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r)\mathrm{d}\tau\}, & k \ge 1, \quad 2 \le r \le k+1. \end{cases}$$

Proof. We differentiate (3.1) to obtain

$$\begin{cases} (\mathrm{div}(\mathbf{p}-\tilde{\mathbf{p}}_h)_t, w) = 0, & \forall w \in W_h, \quad 0 \le t \le T, \\ (\alpha(\mathbf{p}-\tilde{\mathbf{p}}_h)_t, \mathbf{v}) - ((u-\tilde{u}_h)_t, \mathrm{div}\mathbf{v}) = (b(u-\tilde{u}_h) + \int_0^t b_t(u-\tilde{u}_h)\mathrm{d}\tau, \mathrm{div}\mathbf{v}) & \\ \qquad -(\beta(u-\tilde{u}_h) + \int_0^t \beta_t(u-\tilde{u}_h)\mathrm{d}\tau + \alpha_t(\mathbf{p}-\tilde{\mathbf{p}}_h), \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}_h, \quad 0 \le t \le T. \end{cases} \tag{3.18}$$

Choosing $w = \mathrm{div}(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h)_t$ in (3.18) and using (1.4), we have

$$\|\mathrm{div}(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h)_t\|^2 = (\mathrm{div}(\mathbf{p}-\tilde{\mathbf{p}}_h)_t,\ \mathrm{div}(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h)_t) = 0,$$

hence

$$\mathrm{div}(\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h)_t \equiv 0 \tag{3.19}$$

and

$$\|\mathrm{div}(\mathbf{p}-\tilde{\mathbf{p}}_h)_t\| = \|\mathrm{div}(\mathbf{p}-\pi_h \mathbf{p})_t\| \le \mathrm{ch}^r \|\mathbf{p}_t\|_{r+1}, \quad 0 \le r \le k+1. \tag{3.20}$$

Noting that (3.18–3.19) and using the ε inequality, similar to the estimate of $\|\pi_h \mathbf{p} - \tilde{\mathbf{p}}_h\|$ in the proof of Theorem 3.2, we have

$$\|(\mathbf{p}-\tilde{\mathbf{p}}_h)_t\| \le c\left(\|(\mathbf{p}-\pi_h \mathbf{p})_t\| + \|u-\tilde{u}_h\| + \int_0^t \|u-\tilde{u}_h\|\mathrm{d}\tau + \|\mathbf{p}-\tilde{\mathbf{p}}_h\|\right). \tag{3.21}$$

Let $\psi \in L^2(\Omega)$ and $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy the relation (3.8), then $\|\phi\|_2 \le c\|\psi\|$ and

$$((u - \tilde{u}_h)_t, \psi) = ((p_h u - \tilde{u}_h)_t, \operatorname{div}(a \bigtriangledown \phi)) + ((u - p_h u)_t, \psi). \tag{3.22}$$

Noting that (1.4), (3.18), and (3.21–3.22), similar to the proof of Lemma 3.1, we can also have

$$\begin{aligned} \|(u - \tilde{u}_h)_t\| \quad &\le \quad c\{\|(u - p_h u)_t\| + \|(\mathbf{p} - \pi_h \tilde{\mathbf{p}})_t\| + \|\mathbf{p} - \tilde{\mathbf{p}}_h\| \\ &\quad + \|u - \tilde{u}_h\| + \textstyle\int_0^t \|u - \tilde{u}_h\| \mathrm{d}\tau\}. \end{aligned} \tag{3.23}$$

Combining (1.6–1.7) with Theorem 3.2, we have from (3.21) and (3.23) that

$$\|(\mathbf{p} - \tilde{\mathbf{p}}_h)_t\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_t\|_1 + \|u\|_1 + \|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1) \mathrm{d}\tau\}, & k = 0, \\ \mathrm{ch}^r\{\|\mathbf{p}_t\|_r + \|u\|_r + \|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau\}, & k \ge 1, \quad 2 \le r \le k+1, \end{cases}$$

$$\|(u - \tilde{u}_h)_t\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_t\|_1 + \|u_t\|_1 + \|u\|_1 + \|\mathbf{p}\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1) \mathrm{d}\tau\}, & k = 0, \\ \mathrm{ch}^r\{\|\mathbf{p}_t\|_r + \|u_t\|_r + \|u\|_r + \|\mathbf{p}\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau\}, & k \ge 1, \quad 2 \le r \le k+1. \end{cases}$$

The proof is complete. □

Differentiating (3.18), similar to the proof of Theorem 3.3, we can also show by the results of Theorem (3.2–3.3) that the following theorem holds and we omit its proof.

Theorem 3.4. *Under the conditions of Theorem 3.2. Then for $0 \le t \le T$, there exist constants $c > 0$, independent of h and t, such that*

$$\|\operatorname{div}(\mathbf{p} - \tilde{\mathbf{p}}_h)_{tt}\| \le \mathrm{ch}^r \|\mathbf{p}_{tt}\|_{r+1}, \quad 0 \le r \le k+1,$$

$$\|(\mathbf{p} - \tilde{\mathbf{p}}_h)_{tt}\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_{tt}\|_1 + \|\mathbf{p}_t\|_1 + \|u_t\|_1 + \|\mathbf{p}\|_1 + \|u\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1) \mathrm{d}\tau\}, & k = 0, \\ \mathrm{ch}^r\{\|\mathbf{p}_{tt}\|_r + \|\mathbf{p}_t\|_r + \|u_t\|_r + \|\mathbf{p}\|_r + \|u\|_r \\ \qquad + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau\}, & k \ge 1,\ 2 \le r \le k+1, \end{cases}$$

$$\|(u - \tilde{u}_h)_{tt}\| \le \begin{cases} \mathrm{ch}\{\|\mathbf{p}_{tt}\|_1 + \|u_{tt}\|_1 + \|\mathbf{p}_t\|_1 + \|u_t\|_1 + \|\mathbf{p}\|_1 + \|u\|_1 + \int_0^t (\|\mathbf{p}\|_1 + \|u\|_1) \mathrm{d}\tau\}, & k = 0, \\ \mathrm{ch}^r\{\|\mathbf{p}_{tt}\|_r + \|u_{tt}\|_r + \|\mathbf{p}_t\|_r + \|u_t\|_r + \|\mathbf{p}\|_r + \|u\|_r \\ \qquad + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau\}, & k \ge 1, \quad 2 \le r \le k+1. \end{cases}$$

Secondly, we consider the properties of $\{\tilde{\mathbf{p}}_h, \tilde{u}_h\}$ when $b = b(\mathbf{x}, t, \tau)$ in (3.1) and have the following Theorem.

Theorem 3.5. *Assume that $b = b(\mathbf{x}, t, \tau)$ and that the conditions of Theorem 3.2 hold. Then for $0 \le t \le T$, there exist constants $c > 0$, independent of h and t, such that*

$$\|\operatorname{div} D_t^j (\mathbf{p} - \tilde{\mathbf{p}}_h)\| \le \mathrm{ch}^r \|D_t^j \mathbf{p}\|_{r+1}, \quad j = 0, 1, 2,$$

where $0 \le r \le k+1$, and for $k \ge 1$, $2 \le r \le k+1$,

$$\begin{aligned} \|u - \tilde{u}_h\| \quad &\le \quad \mathrm{ch}^r \left\{ \|\mathbf{p}\|_{r-1} + \|u\|_r + \int_0^t \|u\|_{r-1} \mathrm{d}\tau \right\}, \\ \|D_t^j (u - \tilde{u}_h)\| \quad &\le \quad \mathrm{ch}^r \left\{ \sum_{i=0}^{j} (\|D_t^i \mathbf{p}\|_r + \|D_t^i u\|_r) + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau \right\}, \ j = 1, 2, \\ \|D_t^j (\mathbf{p} - \tilde{\mathbf{p}}_h)\| \quad &\le \quad \mathrm{ch}^r \left\{ \sum_{i=0}^{j} \|D_t^i \mathbf{p}\|_r + \sum_{i=0}^{j-1} \|D_t^i u\|_r + \int_0^t (\|\mathbf{p}\|_{r-1} + \|u\|_r) \mathrm{d}\tau \right\}, \ j = 0, 1, 2, \end{aligned}$$

for $k=0$,

$$\begin{aligned}
\|u-\tilde{u}_h\| &\le \mathrm{ch}\left\{\|\mathbf{p}\|_1+\|u\|_1+\int_0^t\|u\|_1\mathrm{d}\tau\right\},\\
\|D_t^j(u-\tilde{u}_h)\| &\le \mathrm{ch}\left\{\sum_{i=0}^{j}(\|D_t^i\mathbf{p}\|_1+\|D_t^iu\|_1)+\int_0^t(\|\mathbf{p}\|_1+\|u\|_1)\mathrm{d}\tau\right\},\ j=1,2,\\
\|D_t^j(\mathbf{p}-\tilde{\mathbf{p}}_h)\| &\le \mathrm{ch}\left\{\sum_{i=0}^{j}\|D_t^i\mathbf{p}\|_1+\sum_{i=0}^{j-1}\|D_t^iu\|_1+\int_0^t(\|\mathbf{p}\|_1+\|u\|_1)\mathrm{d}\tau\right\},\ j=0,1,2.
\end{aligned}$$

Proof. We know the results of Theorem 3.3–3.4 hold in the case of $b=b(t,\tau)$ by the results of Theorem 3.2 and (3.1), then for Theorem 3.5 it suffices to prove the results Theorem 3.2 hold in the case $b=b(\mathbf{x},t,\tau)$. In fact, (3.1) can be rewritten as

$$\begin{cases} (\mathrm{div}\mathbf{p}_1,w)=0, & \forall w\in W_h,\quad 0\le t\le T,\\ (\alpha\mathbf{p}_1,\mathbf{v})-(u_1+\int_0^t bu_1\mathrm{d}\tau+\int_0^t bu_2\mathrm{d}\tau,\mathrm{div}\mathbf{v})+(\int_0^t\beta(u_1+u_2)\mathrm{d}\tau,\mathbf{v})=0, & \forall \mathbf{v}\in\mathbf{V}_h,\quad 0\le t\le T.\end{cases} \tag{3.24}$$

If we take $\mathbf{v}=\pi_h\mathbf{p}-\tilde{\mathbf{p}}_h$ and $w=\mathrm{div}\mathbf{v}$ in (3.24) and (3.7), noting that $\mathrm{div}(\pi_h\mathbf{p}-\tilde{\mathbf{p}}_h)=0$, then (3.24) is the same with (3.7). From the proof of Theorem 3.2 we need only prove the result of Lemma 3.1 holds in the case $b=b(\mathbf{x},t,\tau)$.

Comparing (3.24) with (3.7) we can find the difference of (3.24) and (3.7) only in the term $(\int_0^t bu_2\mathrm{d}\tau,\mathrm{div}\mathbf{v})$, so the right hand side of (3.9) has another term $(\int_0^t bu_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi)))$. Since

$$\begin{aligned}
\int_0^t bu_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi)) &= \int_0^t(b-p_h^0b)u_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi))+\sum_{e\in\mathcal{T}_h}\left(\int_0^t(p_h^0b)u_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi))\right)_e\\
&= \int_0^t(b-p_h^0b)u_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi)),
\end{aligned}$$

where p_h^0b is the interpolation of b and a piecewise constant function on the elements of $\mathcal{T}_h$. Then we have by the properties of the operators p_h^0, p_h and π_h that

$$\int_0^t bu_2\mathrm{d}\tau,\mathrm{div}(\pi_h(a\bigtriangledown\phi))\le \mathrm{ch}\int_0^t\|u_2\|\mathrm{d}\tau\|\phi\|_2. \tag{3.25}$$

(3.25) together with the proof of Lemma 3.1 imply the result of Lemma 3.1 holds in the case $b=b(\mathbf{x},t,\tau)$. The proof is complete. □

4. Continuous-in-time mixed finite element approximation

In this section, we will first use the generalized mixed elliptic projection $\{\tilde{\mathbf{p}}_h,\tilde{u}_h\}$ to derive the optimal $L^\infty(L^2)$ error estimates and then use the projection $\{\tilde{\mathbf{p}}_h,\tilde{u}_h\}$ together with the regularized Green's functions defined by Wang in [16] to derive the quasi-optimal $L^\infty(L^\infty)$ error estimates for the continuous-in-time mixed finite element approximation.

Theorem 4.1. *Let* $\{\mathbf{p},u\}$ *be the solution of (1.3) and* $\{\mathbf{p}_h,u_h\}$ *be that of (1.9). Assume that for* $1\le r\le k+1,\mathbf{p}\in L^\infty(0,T;H^r(\Omega)^2),\mathbf{p}_t\in L^2(0,T;H^r(\Omega)^2)$, $u,u_t\in L^2(0,T;H^r(\Omega))$ *and that* Ω *is 2-regular. Then for*

$0 \le t \le T$, there exists a constant $c > 0$ independent of h and t such that

$$\|\mathbf{p} - \mathbf{p}_h\| \le \mathrm{c}h^r \left\{ \|\mathbf{p}\|_r + \left(\int_0^t (\|u_t\|_r^2 + \|\mathbf{p}_t\|_r^2 + \|u\|_r^2 + \|\mathbf{p}\|_r^2)\mathrm{d}s \right)^{1/2} \right\}.$$

Moreover, if in addition $u \in L^\infty(0,T; H^r(\Omega))$. Then

$$\|u - u_h\| \le \mathrm{c}h^r \left\{ \|u\|_r + \|\mathbf{p}\|_r + \left(\int_0^t (\|u_t\|_r^2 + \|\mathbf{p}_t\|_r^2 + \|u\|_r^2 + \|\mathbf{p}\|_r^2)\mathrm{d}s \right)^{1/2} \right\}.$$

Proof. Let $u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h) = u_3 + u_4$ and $\mathbf{p} - \mathbf{p}_h = (\mathbf{p} - \tilde{\mathbf{p}}_h) + (\tilde{\mathbf{p}}_h - \mathbf{p}_h) = \mathbf{p}_3 + \mathbf{p}_4$. Then for $0 < t \le T$, form (1.3) and (1.9) together with (3.1), we see that

$$\begin{cases} (u_{4,t}, w) + (\mathrm{div}\mathbf{p}_4, w) = -(u_{3,t}, w), & \forall w \in W_h, \\ (\alpha \mathbf{p}_4, \mathbf{v}) - (u_4 + \int_0^t b u_4 \mathrm{d}\tau, \mathrm{div}\mathbf{v}) + (\int_0^t \beta u_4 \mathrm{d}\tau, \mathbf{v}) = 0, & \forall \mathbf{v} \in \mathbf{V}_h. \end{cases} \tag{4.1}$$

With the choices $\mathbf{v} = \mathbf{p}_4$ and $w = u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau$ in (4.1), we see that

$$\left(u_{4,t}, u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau \right) + \left(\alpha \mathbf{p}_4 + \int_0^t \beta u_4 \mathrm{d}\tau, \mathbf{p}_4 \right) = -\left(u_{3,t}, u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau \right). \tag{4.2}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u_4, \int_0^t p_h(bu_4)\mathrm{d}\tau \right) = \left(u_{4,t}, \int_0^t p_h(bu_4)\mathrm{d}\tau \right) + \left(u_4, p_h(bu_4) + \int_0^t p_h(b_t u_4)\mathrm{d}\tau \right), \tag{4.3}$$

we have from (4.2–4.3) that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_4\|^2 + c_1^{-1}\|\mathbf{p}_4\|^2 &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_4\|^2 + (\alpha\mathbf{p}_4, \mathbf{p}_4) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(u_4, \int_0^t p_h(bu_4)\mathrm{d}\tau \right) + \left(u_4, p_h(bu_4) + \int_0^t p_h(b_t u_4)\mathrm{d}\tau \right) \\ &\quad - \left(\int_0^t \beta u_4 \mathrm{d}\tau, \mathbf{p}_4 \right) - \left(u_{3,t}, u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau \right) \\ &\le -\frac{\mathrm{d}}{\mathrm{d}t}\left(u_4, \int_0^t p_h(bu_4)\mathrm{d}\tau \right) + c\left(\|u_4\|^2 + \int_0^t \|u_4\|^2\mathrm{d}\tau + \|u_{3,t}\|^2 \right) + c_1^{-1}\|\mathbf{p}_4\|^2. \end{aligned}$$

Integrating with respect to t from 0 to t and noting that $u_4(0) = 0$, then we have

$$\begin{aligned} \frac{1}{2}\|u_4\|^2 &\le -\left(u_4, \int_0^t p_h(bu_4)\mathrm{d}\tau \right) + c\int_0^t \left(\|u_4\| + \int_0^s \|u_4\|\mathrm{d}\tau + \|u_{3,t}\| \right)^2 \mathrm{d}s \\ &\le \frac{1}{4}\|u_4\|^2 + c\int_0^t (\|u_4\|^2 + \|u_{3,t}\|^2)\mathrm{d}s. \end{aligned}$$

Using Gronwall's lemma, we have from Theorem 3.5 that

$$\|u_4\| \le c\left(\int_0^t \|u_{3,t}\|^2 \mathrm{d}s \right)^{1/2} \le \mathrm{c}h^r \left(\int_0^t \sum_{i=0}^{1} (\|D_t^i u\|_r^2 + \|D_t^i \mathbf{p}\|_r^2)\mathrm{d}s \right)^{1/2}. \tag{4.4}$$

In order to estimate $\|\mathbf{p}_4\|$, we differentiate the second equation of (4.1) to obtain

$$(\alpha\mathbf{p}_{4,t}+\alpha_t\mathbf{p}_4,\mathbf{v})-\left(u_{4,t}+bu_4+\int_0^t b_t u_4 \mathrm{d}\tau,\mathrm{div}\mathbf{v}\right)+\left(\beta u_4+\int_0^t \beta_t u_4\mathrm{d}\tau,\mathbf{v}\right)=0,\quad \forall \mathbf{v}\in\mathbf{V}_h. \tag{4.5}$$

Taking $\mathbf{v}=\mathbf{p}_4$ in (4.5) and $w=u_{4,t}+p_h(bu_4)+\int_0^t p_h(b_t u_4)\mathrm{d}\tau$ in the first equation of (4.1), then similar to the estimate of $\|u_4\|$, we have

$$\|\mathbf{p}_4\|\le c\left(\int_0^t \|u_{3,t}\|^2\mathrm{d}s\right)^{1/2}\le \mathrm{c}h^r\left(\int_0^t\sum_{i=0}^1(\|D_t^i u\|_r^2+\|D_t^i\mathbf{p}\|_r^2)\mathrm{d}s\right)^{1/2}. \tag{4.6}$$

Theorem 3.5, (4.4) and (4.6) complete the proof of Theorem 4.1. □

Theorem 4.2. *Let $\{\mathbf{p},u\}$ be the solution of (1.3) and $\{\mathbf{p}_h,u_h\}$ be that of (1.9). Assume that for $1\le r\le k+1$, $\{\mathbf{p},u\},\{\mathbf{p}_t,u_t\}\in L^\infty(0,T;H^r(\Omega)^2)\times L^\infty(0,T;H^r(\Omega))$, $\{\mathbf{p}_{tt},u_{tt}\}\in L^2(0,T;H^r(\Omega)^2)\times L^2(0,T;H^r(\Omega))$ and that Ω is 2-regular. Then for $0\le t\le T$, there exists a constant $c>0$ independent of h and t such that*

$$\begin{aligned}\|(u-u_h)_t\| &\le \mathrm{c}h^r\left\{\sum_{i=0}^1(\|D_t^i u\|_r+\|D_t^i\mathbf{p}\|_r+\|D_t^i u(0)\|_r+\|D_t^i\mathbf{p}(0)\|_r)\right.\\ &\quad\left.+\left(\int_0^t\sum_{i=0}^2(\|D_t^i u\|_r^2+\|D_t^i\mathbf{P}\|_r^2)\mathrm{d}s\right)^{1/2}\right\}.\end{aligned}$$

Moreover, if in addition $\mathrm{div}\mathbf{p}\in L^\infty(0,T;H^r(\Omega)),\mathbf{p}\in L^\infty(0,T;H^r(\Omega)^2)$*. Then*

$$\begin{aligned}\|\mathrm{div}(\mathbf{p}-\mathbf{p}_h)\| &\le \mathrm{c}h^r\left\{\sum_{i=0}^1(\|D_t^i u\|_r+\|D_t^i\mathbf{p}\|_{r+1-i}+\|D_t^i u(0)\|_r+\|D_t^i\mathbf{p}(0)\|_r)\right.\\ &\quad\left.+\left(\int_0^t\sum_{i=0}^2(\|D_t^i u\|_r^2+\|D_t^i\mathbf{P}\|_r^2)\mathrm{d}s\right)^{1/2}\right\}.\end{aligned}$$

Proof. We differentiate (4.1) to obtain

$$\left\{\begin{array}{ll}(u_{4,tt},w)+(\mathrm{div}\mathbf{p}_{4,t},w)=-(u_{3,tt},w), & \forall w\in W_h,\\ (\alpha\mathbf{p}_{4,t}+\alpha_t\mathbf{p}_4,\mathbf{v})-\left(u_{4,t}+bu_4+\int_0^t b_t u_4\mathrm{d}\tau,\mathrm{div}\mathbf{v}\right)\left(\beta u_4+\int_0^t\beta_t u_4\mathrm{d}\tau,\mathbf{v}\right)=0, & \forall\mathbf{v}\in\mathbf{V}_h.\end{array}\right. \tag{4.7}$$

where $\mathbf{p}_4(0)=\mathbf{0}$ and $u_{4,t}(0)$ satisfies the following relations

$$(u_{4,t}(0),w)=-(u_{3,t}(0),w),\quad \forall w\in W_h. \tag{4.8}$$

Taking $w=u_{4,t}(0)$ in (4.8), then we have

$$\|u_{4,t}(0)\|\le\|u_{3,t}(0)\|. \tag{4.9}$$

Now, let us choosing $w=u_{4,t}+p_h(bu_4)+\int_0^t p_h(b_t u_4)\mathrm{d}\tau$ and $\mathbf{v}=\mathrm{div}\mathbf{p}_{4,t}$ in (4.7). Noting that $u_4(0)=0$ and (4.9), similar to the estimate of $\|u_4\|$, we have from Theorem 3.5, (4.4), and (4.6) that

$$\|u_{4,t}\|\le \mathrm{c}h^r\left\{\sum_{i=0}^1(\|D_t^i u(0)\|_r+\|D_t^i\mathbf{p}(0)\|_r)+\left(\int_0^t\sum_{i=0}^2(\|D_t^i u\|_r^2+\|D_t^i\mathbf{p}\|_r^2)\mathrm{d}s\right)^{1/2}\right\}. \tag{4.10}$$

From (4.1), (4.10), and Theorem 3.5, we have

$$\begin{aligned} \|\mathrm{div}\mathbf{p}_4\| &\leq \|u_{4,t}\| + \|u_{3,t}\| \\ &\leq ch^r \left\{ \sum_{i=0}^{1} (\|D_t^i u\|_r + \|D_t^i \mathbf{p}\|_r) + \sum_{i=0}^{1} (\|D_t^i u(0)\|_r + \|D_t^i \mathbf{p}(0)\|_r) \right. \\ &\quad \left. + \left(\int_0^t \sum_{i=0}^{2} (\|D_t^i u\|_r^2 + \|D_t^i \mathbf{p}\|_r^2) \mathrm{d}s \right)^{1/2} \right\}. \end{aligned} \tag{4.11}$$

Theorem 3.5, (4.11) and (4.10) complete the proof of Theorem 4.2. □

We now consider the $L^\infty(L^\infty)$ error estimates. For this purpose we first introduce two pairs of regularized Green's functions at $\mathbf{z} \in \Omega$ by (see [16])

$$\begin{cases} \mathbf{G}_1 + \nabla \lambda_1 = \mathbf{0} & \text{in } \Omega, \\ \mathrm{div}\mathbf{G}_1 = \delta_1^h & \text{in } \Omega, \\ \lambda_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

and

$$\begin{cases} \alpha\mathbf{G}_2 + \nabla \lambda_2 = \delta_2^h & \text{in } \Omega, \\ \mathrm{div}\mathbf{G}_2 = 0 & \text{in } \Omega, \\ \lambda_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

where δ_1^h and δ_2^h are the regularized Dirac functions at $\mathbf{z} \in \Omega$ satisfying

$$(w, \delta_1^h) = w(\mathbf{z}), \quad \forall w \in W_h, \qquad (\mathbf{v}, \delta_2^h) = \mathbf{v}(\mathbf{z}), \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

Choosing right point $\mathbf{z}$ respectively can yield (see Eq. (3.4) in [16])

$$\|w\|_{0,\infty} \leq 2|(w, \delta_1^h)|, \tag{4.14}$$

$$\|\mathbf{v}\|_{0,\infty} \leq 2|(\mathbf{v}, \delta_2^h)|. \tag{4.15}$$

Let $\{\mathbf{G}_1^h, \lambda_1^h\} \in \mathbf{V}_h \times W_h$ be the mixed finite element approximation of $\{\mathbf{G}_1, \lambda_1\}$ and $\{\mathbf{G}_2^h, \lambda_2^h\} \in \mathbf{V}_h \times W_h$ be that of $\{\mathbf{G}_2, \lambda_2\}$. Wang proved that (see Eqs. (3.14, 3.20, 3.12b, 5.44) in [16])

$$|\mathbf{G}_1^h| \leq c|\ln h|^{\frac{1}{2}}, \tag{4.16}$$

$$|\mathbf{G}_2^h|_{0,1} \leq c|\ln h|, \tag{4.17}$$

$$|\lambda_2 - \lambda_2^h| \leq c, \tag{4.18}$$

$$|\lambda_2| \leq c(1 + |\ln h|^{\frac{1}{2}}). \tag{4.19}$$

From (4.18–4.19), we have

$$\|\lambda_2^h\| \leq c(1 + |\ln h|^{\frac{1}{2}}). \tag{4.20}$$

Theorem 4.3. *Let $\{\mathbf{p}, u\}$ be the solution of (1.3) and $\{\mathbf{p}_h, u_h\}$ be that of (1.9). Assume that for $1 \leq r \leq k+1$, $\{\mathbf{p}, u\} \in L^\infty(0,T; H^r(\Omega)^2) \times L^\infty(0,T; W^{r,\infty}(\Omega))$, $\{\mathbf{p}_t, u_t\}, \{\mathbf{p}_{tt}, u_{tt}\} \in L^2(0,T; H^r(\Omega)^2) \times L^2(0,T; H^r(\Omega))$ and that Ω is 2-regular. Then for $0 \leq t \leq T$ and $0 < h \leq 1/3$, there exists a constant $c > 0$ independent of h*

and t such that

$$\|u-u_h\|_{0,\infty} \le \mathrm{c}h^r|\ln h|^{\frac{1}{2}} \left\{ \|u\|_{r,\infty} + \|\mathbf{p}\|_r + \left(\int_0^t (\|u_t\|_r^2 + \|\mathbf{p}_t\|_r^2 + \|u\|_{r,\infty}^2 + \|\mathbf{p}\|_r^2)\mathrm{d}s\right)^{1/2} \right\}.$$

Moreover, if in addition $\{\mathbf{p}_t, u_t\} \in L^\infty(0,T;H^r(\Omega)^2) \times L^\infty(0,T;H^r(\Omega))$, $\mathbf{p} \in L^\infty(0,T;W^{r,\infty}(\Omega)^2)$. *Then*

$$\begin{aligned} \|\mathbf{p}-\mathbf{p}_h\|_{0,\infty} \le\ & \mathrm{c}h^r|\ln h|^{\frac{3}{2}} \Big\{ \|u_t\|_r + \|\mathbf{p}_t\|_r + \|u\|_{r,\infty} + \|\mathbf{p}\|_{r,\infty} + \|u_t(0)\|_r + \|\mathbf{p}_t(0)\|_r + \|u(0)\|_r + \|\mathbf{p}(0)\|_r \\ & + \left(\int_0^t (\|u_{tt}\|_r^2 + \|\mathbf{p}_{tt}\|_r^2 + \|u_t\|_r^2 + \|\mathbf{p}_t\|_r^2 + \|u\|_{r,\infty}^2 + \|\mathbf{p}\|_{r,\infty}^2)\mathrm{d}s\right)^{1/2} \Big\}. \end{aligned}$$

Proof. For $0 \le t \le T$, we have from (4.12) and (4.1) that

$$\begin{aligned} \left(u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau, \delta_1^h\right) &= \left(u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau, \mathrm{div}\mathbf{G}_1\right) \\ &= \left(u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau, \mathrm{div}\mathbf{G}_1^h\right) \\ &= \left(u_4 + \int_0^t bu_4\mathrm{d}\tau, \mathrm{div}\mathbf{G}_1^h\right) \\ &= (\alpha\mathbf{p}_4, \mathbf{G}_1^h) + \left(\int_0^t \beta u_4\mathrm{d}\tau, \mathbf{G}_1^h\right) \\ &\le c\left(\|\mathbf{p}_4\| + \int_0^t \|u_4\|\mathrm{d}\tau\right)\|\mathbf{G}_1^h\|, \end{aligned}$$

which, combining with (4.14), yields

$$\left\|u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau\right\|_{0,\infty} \le c\left(\|\mathbf{p}_4\| + \int_0^t \|u_4\|\mathrm{d}\tau\right)\|\mathbf{G}_1^h\|,$$

hence

$$\begin{aligned} \|u_4\|_{0,\infty} &\le \left\|u_4 + \int_0^t p_h(bu_4)\mathrm{d}\tau\right\|_{0,\infty} + \left\|\int_0^t p_h(bu_4)\mathrm{d}\tau\right\|_{0,\infty} \\ &\le c\left(\|\mathbf{p}_4\| + \int_0^t \|u_4\|\mathrm{d}\tau\right)\|\mathbf{G}_1^h\| + c\int_0^t \|u_4\|_{0,\infty}. \end{aligned}$$

Using Gronwall's lemma, we have

$$\|u_4\|_{0,\infty} \le c\left(\|\mathbf{p}_4\| + \int_0^t \|u_4\|\mathrm{d}\tau\right)\|\mathbf{G}_1^h\|. \tag{4.21}$$

Noting that $\mathrm{div}\mathbf{G}_2^h = 0$, we have from (4.13) and (4.1) that

$$\begin{aligned}(\mathbf{p}_4, \delta_2^h) &= (\mathbf{p}_4, \alpha\mathbf{G}_2 + \nabla\lambda_2) \\ &= (\mathbf{p}_4, \alpha(\mathbf{G}_2 - \mathbf{G}_2^h)) + (\alpha\mathbf{p}_4, \mathbf{G}_2^h) + (\mathbf{p}_4, \nabla\lambda_2) \\ &= -(\mathrm{div}\mathbf{p}_4, \lambda_2^h) - \left(\int_0^t \beta u_4 \mathrm{d}\tau, \mathbf{G}_2^h\right) \\ &= ((u-u_h)_t, \lambda_2^h) - \left(\int_0^t \beta u_4 \mathrm{d}\tau, \mathbf{G}_2^h\right) \\ &\leq c\|(u-u_h)_t\|\|\lambda_2^h\| + c\int_0^t \|u_4\|_{0,\infty}\mathrm{d}\tau\|\mathbf{G}_2^h\|_{0,1},\end{aligned}$$

which, combining with (4.15) and (4.21), yields

$$\|\mathbf{p}_4\|_{0,\infty} \leq c\left\{\|(u-u_h)_t\|\|\lambda_2^h\| + \int_0^t (\|\mathbf{p}_4\| + \|u_4\|)\mathrm{d}\tau\|\mathbf{G}_1^h\|\|\mathbf{G}_2^h\|_{0,1}\right\}. \tag{4.22}$$

On the other hand, (3.1) can be rewritten as

$$\begin{cases} (\mathrm{div}(\pi_h\mathbf{p} - \tilde{\mathbf{p}}_h), w) = 0, & \forall w \in W_h, \\ (\alpha(\mathbf{p} - \tilde{\mathbf{p}}_h), \mathbf{v}) - \left(p_h u - \tilde{u}_h + \int_0^t bu_3\mathrm{d}\tau, \mathrm{div}\mathbf{v}\right) + \left(\int_0^t \beta u_3 \mathrm{d}\tau, \mathbf{v}\right) = 0, & \forall \mathbf{v} \in \mathbf{V}_h. \end{cases} \tag{4.23}$$

Similar to the proof of (4.21) and (4.22), we can also have from (4.23) that

$$\|p_h u - \tilde{u}_h\|_{0,\infty} \leq c\left\{\left(\|\mathbf{p} - \tilde{\mathbf{p}}_h\| + \int_0^t \|u_3\|\mathrm{d}\tau\right)\|\mathbf{G}_1^h\| + \int_0^t \|u - p_h u\|_{0,\infty}\mathrm{d}\tau\right\}, \tag{4.24}$$

$$\|\pi_h\mathbf{p} - \tilde{\mathbf{p}}_h\|_{0,\infty} \leq c\left(\|\mathbf{p} - \pi_h\mathbf{p}\|_{0,\infty} + \int_0^t \|u_3\|_{0,\infty}\mathrm{d}\tau\right)\|\mathbf{G}_2^h\|_{0,1}. \tag{4.25}$$

Noting that (4.21, 4.4, 4.6, 4.16), we have

$$\|u_4\|_{0,\infty} \leq \mathrm{c}h^r|\ln h|^{\frac{1}{2}}\left(\int_0^t (\|u_t\|_r^2 + \|u\|_r^2 + \|\mathbf{p}_t\|_r^2 + \|\mathbf{p}\|_r^2)\mathrm{d}\tau\right)^{1/2}. \tag{4.26}$$

Combining (4.22) with (4.4, 4.6, 4.16–4.17, 4.20), and Theorem 4.2, we have

$$\begin{aligned}\|\mathbf{p}_4\|_{0,\infty} \leq\ & \mathrm{c}h^r(1 + |\ln h|^{\frac{1}{2}} + |\ln h|^{\frac{3}{2}})\left\{\sum_{i=0}^{1}(\|D_t^i u\|_r + \|D_t^i\mathbf{p}\|_r\right. \\ & \left.+\|D_t^i u(0)\|_r + \|D_t^i\mathbf{p}(0)\|_r) + \left(\int_0^t \sum_{i=0}^{2}(\|D_t^i u\|_r^2 + \|D_t^i\mathbf{p}\|_r^2\right)\mathrm{d}\tau)^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.27}$$

Combining (4.24) with Theorem 3.5, (4.16), and (1.6), we have

$$\|u_3\|_{0,\infty} \leq \mathrm{c}h^r\left(\|u\|_{r,\infty} + \int_0^t \|u\|_{r,\infty}\mathrm{d}\tau\right) + \mathrm{c}h^r|\ln h|^{\frac{1}{2}}\left(\|\mathbf{p}\|_r + \int_0^t(\|u\|_r + \|\mathbf{p}\|_r)\mathrm{d}\tau\right). \tag{4.28}$$

Combining (4.25) with (4.28), (4.17), and (1.7), we have

$$\|\mathbf{p}_3\|_{0,\infty} \leq \mathrm{c}h^r(\|\mathbf{p}\|_{r,\infty} + |\ln h|\left(\|\mathbf{p}\|_{r,\infty} + \int_0^t \|u\|_{r,\infty}\mathrm{d}\tau\right) + |\ln h|^{\frac{3}{2}}\int_0^t(\|u\|_r + \|\mathbf{p}\|_r)\mathrm{d}\tau). \tag{4.29}$$

(4.26–4.29) complete the proof of Theorem 4.3. □

The author thanks the referees for some useful and helpful suggestions.

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