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# PROPAGATION OF ELASTIC SURFACE WAVES ALONG A CYLINDRICAL CAVITY OF ARBITRARY CROSS SECTION (*)

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Abstract. — *The propagation of elastic surface waves, guided by the free surface of an infinitely long cylinder, of arbitrary cross section, is formulated as an eigenvalue problem, for an unbounded self adjoint operator.*

*We prove the existence of a hierarchy of guided modes. Two of them propagate for any value of the wave number, whereas all of the others only exist beyond a cut-off wave number. For any fixed value of the wave number, only a finite number of modes propagate.*

Résumé. — *Nous formulons la propagation des ondes élastiques, guidées par la surface libre d'une cavité cylindrique de section arbitraire, sous la forme d'un problème de valeurs propres pour un opérateur auto-adjoint, non-borné.*

*Nous prouvons l'existence d'une famille dénombrable de modes guidés. Les deux premiers se propagent pour toute valeur du nombre d'onde, alors que les suivants n'existent qu'au-delà d'un nombre d'onde de coupure. Pour chaque valeur fixée du nombre d'onde, seul un nombre fini de modes se propagent.*

## 1. INTRODUCTION

Acoustic surface waves have been defined by Lagasse *et al.* [15] as « solutions of the wave equation which lead to a concentration of energy near the free surface of a half-space ». For elastic waves, it has been known since Rayleigh [21], one century ago, that a wave may be guided by the surface of a half plane, if this surface is stress free. The case of a half-plane is well known : Schulenberger [26] studied the unperturbed case, whereas Dermenjian and Guillot [11] have studied the scattering by a bounded heterogeneity. These results have important applications to seismology, as the Rayleigh wave is primarily responsible for earthquakes.

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Another field of application, that of transmission devices, deals mainly with topographic waveguides The half-plane is locally deformed, and the deformation should trap the energy of the waves, thus acting as a transducer They are discussed in Lagasse *et al* [15]

In this work, we study another kind of guides the waves will be guided by the (stress-free) surface of an infinitely long hollow cylinder, in an isotropic homogeneous half-space The reference medium is now the whole space, which makes this a model problem for more difficult situations

The case of a circular cylinder has been treated by Bostrom *et al* [8], see also Biot [7] and Mindlin [18], but the general case has received relatively little attention Burden [9] recently proposed a numerical method, and Wilson *et al* [29] studied the high frequency limit As far as we know, no attention has been devoted to a theoretical characterization of the surface waves, and of their properties

In this work we shall give such a characterization Our main result is the existence of a hierarchy of guided waves, each propagating only beyond a threshold wave number Two of them even propagate without cut-off We prove that the thresholds increase to infinity, so that for any fixed value of the wave number, there exists only a finite number of modes We also study the high frequency propagation Wilson *et al* [29] had conjectured that the speed of every mode approaches the speed of the Rayleigh wave in the high frequency limit

We prove a first result in that direction, but think it is true only in the case of a smooth boundary Indeed, Lagasse [14] proved numerically the existence of a wave guided by an infinitely long wedge, whose speed is less than the Rayleigh speed

Before proceeding to more specific matters, we want to stress two points

— Firstly, the importance of the free surface condition For the same problem, with homogeneous Dirichlet boundary conditions, *no guided modes exist*

— Secondly, the fact that these waves are specific to elastodynamics Indeed, the corresponding problem, in a homogeneous medium, for Laplace's or Maxwell's equations, has no solution For instance, in optical waveguides, the guide is heterogeneous, and a guided wave only occurs if the index of the heart is greater than that of the cladding (see Marcuse [16])

For our analysis we follow a rather classical approach, used for example by Bamberger-Bonnet [3] for optical fibers we look for solutions of the linear elastodynamics equations, propagating along the $x_3$ direction, with wave number $\beta$, harmonic in time with frequency $\omega$, whose energy is finite in each section of the guide and satisfying the free surface condition along the boundary of the cylinder Such solutions only exist if $\beta$ and $\omega$ satisfy a dispersion relation Our point of view is to take $\beta$ as a parameter, making $\omega^2$ appear as an eigenvalue of an unbounded self-adjoint operator, with non compact resolvent. This enables us to use powerful results from spectral theory to describe the spectrum of this operator. With the help of perturbation techniques we determine its essential spectrum, and show it contains no eigenvalues. We are then able to characterize the eigenvalues via the Min-Max principle. The main point of our analysis is the construction of suitable test functions. In our context, it is natural that these are given by a field related to the Rayleigh wave, taking into account the curvature of the surface.

This article is organized as follows : in the next section, we state in mathematical terms the problem we wish to address. We also recall, and extend, some results from the circular case. Section 3 will be concerned with the determination of the essential spectrum of our operator, and of a priori bounds for the eigenvalues. Our main results are contained in section 4, where we prove the existence of the eigenvalues. We also study the cut-off wave numbers, and the high frequency behavior of the modes.

These results were announced in Bamberger *et al.* [6]. See also [5] for details on the proofs we do not give in this paper.

## 2. STATEMENT OF THE PROBLEM

### 2.1. The Physical Problem

We consider a bounded open set $\mathcal{O} \subset R^2$, and its complement, denoted by $\Omega$, which we suppose connected (but notice it need not be connected). It all what follows, we make the following hypothesis :

(HS) $\Omega$ is a lipschitz domain, locally on one side of its boundary. $\Gamma = \partial\Omega$.

This means (*cf.* Adams [2]) that there exists a covering of $\Omega$ by $(N+1)$ open sets $(\Omega_0, \Omega_1, \dots, \Omega_{N+1})$, with $\Omega_j$ bounded for $j \geq 1$, such that :

- $\Omega_0 \cap \Gamma = \varnothing$, $\Gamma_j \subset \bigcup_{j \geq 1} \Omega_j$,
- there exists functions $f_j \in W^{1,\infty}(R)$, $1 < j \leq N+1$, with $f_j(0) = 0$ such that :

$$\Omega_j \cap \Omega = \{(x_1, x_2)/x_2 > f_j(x_1), |x_1| < a_j\}$$

$$\Gamma_j = \Gamma \cap \Omega_j = \{(x_1, x_2)/x_2 = f_j(x_1), |x_1| < a_j\} .$$

This hypothesis will not be sufficient for some of our results, and we indicate in each case, the precise smoothness that is required.

The problem we wish to address is that of the propagation of elastic waves in the cylindrical exterior region $\Omega \times R$. Our hypothesis are those of linear elastodynamics, in a homogeneous isotropic medium (*cf.* Miklowitz [17]).

This medium is described by its Lame constants $\lambda$ and $\mu$ and its density $\rho$, that are all positive constants. We also define the velocities of the $P$ and $S$ waves :

$$V_P = \left( \frac{\lambda + 2\mu}{\rho} \right)^{1/2}, \quad V_S = \left( \frac{\mu}{\rho} \right)^{1/2}. \tag{2.1}$$

We denote the position by $x = (x_1, x_2) \in \Omega$, $\bar{x} = (x, x_3) \in \Omega \times R$, and the displacement field by :

$$U_j(x_1, x_2, x_3, t) = (U_j(\bar{x}, t))_{j=1,2,3}.$$

We are looking for displacement fields of the form :

$$U_j(\bar{x}, t) = \tilde{u}_j(x)\, e^{i(\omega t - \beta x_3)} \quad j = 1, 2, 3$$

that are harmonic in time, with frequency $\omega$, and that propagate along the direction $x_3$, with a wave number $\beta$. Such a wave has a phase velocity given by :

$$V = \frac{\omega}{\beta}.$$

To work with real coefficients and real valued functions and is useful to introduce the new unknown : $u_1 = \tilde{u}_1$, $u_2 = \tilde{u}_2$, $u_3 = i\tilde{u}_3$ and to define the « strain tensor » associated to $u = (u_1, u_2, u_3)$ by :

$$\begin{cases} \varepsilon_{ij}^{\beta}(u) = \dfrac{1}{2}\left( \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right) & i, j = 1, 2 \\ \varepsilon_{i3}^{\beta}(u) = \dfrac{1}{2}\left( \dfrac{\partial u_3}{\partial x_i} + \beta u_i \right) = \varepsilon_{3i}^{\beta}(u) & i = 1, 2 \\ \varepsilon_{33}^{\beta}(u) = -\beta u_3. \end{cases} \tag{2.2}$$

The corresponding stress tensor is given by Hooke's law :

$$\sigma_{ij}^{\beta}(u) = \lambda \operatorname{tr} \varepsilon^{\beta}(u)\, \delta_{ij} + 2\mu \varepsilon_{ij}^{\beta}(u) \quad i, j = 1, 2, 3. \tag{2.3}$$

From elastodynamics equations [17], it is easy to see that the equations of motion become :

$$\begin{cases} \dfrac{1}{\rho}\left( -\displaystyle\sum_{j=1}^{2} \dfrac{\partial \sigma_{ij}^{\beta}(u)}{\partial x_j} + \beta \sigma_{i3}^{\beta}(u) \right) = \omega^2 u_i & i = 1, 2 \\ \dfrac{1}{\rho}\left( -\displaystyle\sum_{j=1}^{2} \dfrac{\partial \sigma_{3j}^{\beta}(u)}{\partial x_j} - \beta \sigma_{33}^{\beta}(u) \right) = \omega^2 u_3. \end{cases} \tag{2.4}$$

Recall that we are looking for fields satisfying the free surface condition on $\Gamma$ :

$$\sigma^{\beta}(u) \, . \, n = 0 \, . \tag{2.5}$$

A solution of (2.4)-(2.5) is called a guided mode if it satisfies :

$$\int_{\Omega} |u|^2 \, dx < \infty \, .$$

This condition means that the energy of the wave is finite in each cross section of the guide, and this in turn implies that the displacements are confined to the vicinity of $\Gamma$.

## 2.2. The Circular Case

Before studying the general case, we wish to recall the results obtained by Boström and Burden [8] in the case where $\Gamma$ is a circle. In that case, the dispersion relation has an analytical formula (even though it still has to be solved numerically).

For this computation, it is natural to use polar coordinates. Following Miklowitz [17], we use scalar potentials $\tilde{\phi}$, $\tilde{\psi}$ and $\tilde{\eta}$ such that :

$$u = \operatorname{grad} \tilde{\phi} + \operatorname{curl} \, (\tilde{\psi} e_3 + \operatorname{curl} \, (\tilde{\eta} e_3)) \, . \tag{2.7}$$

They satisfy scalar wave equations at a frequency $\omega$ :

$$- V_i^2 \, \Delta \tilde{\chi} + \beta^2 \, V_i^2 \, \tilde{\chi} - \omega^2 \, \tilde{\chi} = 0$$

with $V_i = V_P$ for $\tilde{\chi} = \tilde{\phi}$, and $V_i = V_S$ for $\tilde{\chi} = \tilde{\phi}$ or $\tilde{\chi} = \tilde{\eta}$.

Then, we look for potentials of the form :

$$\left\{ \begin{aligned} \tilde{\phi}(r, \theta) &= \phi(r) \cos \, (n\theta) \\ \tilde{\psi}(r, \theta) &= \psi(r) \sin \, (n\theta) \\ \tilde{\eta}(r, \theta) &= \eta(r) \cos \, (n\theta) \, . \end{aligned} \right. \tag{2.8}$$

Introducing the adimensional quantities :

$$\alpha_P^2 = 1 - V^2/V_P^2, \quad \alpha_S^2 = 1 - V^2/V_S^2 ; \; V = \omega/\beta \, .$$

We see that these functions satisfy modified Bessel's equations :

$$(2.9)\qquad \begin{cases} -\dfrac{d^2\phi}{dr^2}-\dfrac{1}{r}\dfrac{d\phi}{dr}+\dfrac{n^2}{r^2}\phi+\alpha_P^2\,\beta^2\,\phi=0 \\ -\dfrac{d^2\psi}{dr^2}-\dfrac{1}{r}\dfrac{d\psi}{dr}+\dfrac{n^2}{r^2}\psi+\alpha_S^2\,\beta^2\,\psi=0 \\ -\dfrac{d^2\eta}{dr^2}-\dfrac{1}{r}\dfrac{d\eta}{dr}+\dfrac{n^2}{r^2}\eta+\alpha_S^2\,\beta^2\,\eta=0\,. \end{cases}$$

These 3 functions are coupled through the boundary conditions.

The only square integrable solutions of (2.9), for $r > a$, are the modified Bessel function $K_n$, which implies that $(\alpha_S, \alpha_P)$ are real and thus that $0 < \omega < \beta V_S$. So we have :

$$\phi(r) = A_n\, K_n(\alpha_P\,\beta r)\,,\quad \psi(r) = B_n\, K_n(\alpha_S\,\beta r)\,,\quad \eta(r) = C_n\, K_n(\alpha_S\,\beta r)\,.$$

Then imposing the boundary condition (2.5), we obtain a homogeneous linear system, in $(A_n, B_n, C_n)$, with a matrix $M_n(\beta, \omega)$. The dispersion relation is given by :

$$(2.10)\qquad \det M_n(\beta, \omega) = 0 \qquad 0 < \omega < \beta V_S\,.$$

This equation has been found by Mindlin [18], and also by Borström-Burden [8], who were the first to solve it numerically. The coefficients of $M_n(\beta, \omega)$ can be expressed in terms of the function

$$S_n(z) = \frac{K_n(z)}{zK_{n+1}(z)}$$

which is regular at $z = 0$, and bounded.

The numerical study in [8] gives the following results, summarized on figure 1 :

For each $n$, there exists a wave number $\beta_n^*$, called the $n$-th threshold such that :

— for $\beta < \beta_n^*$, equation (2.10) has no solution ;

— for $\beta \geq \beta_n^*$, equation (2.10) has a unique solution $\omega_n(\beta)$, with the following properties :

- $\omega_n(\beta_n^*) = V_S\,\beta_n^*$
- the function $\beta \to \omega_n(\beta)/\beta$ is decreasing
- $\lim_{\beta \to \infty} \omega_n(\beta)/\beta = V_R$, where $V_R$ is the velocity of the Rayleigh wave in the half space (see section 4.1).

Moreover, the value $n = 1$ is exceptional in that $\beta_1^* = 0$, i.e. the corresponding wave propagates for all values of the wave-number.

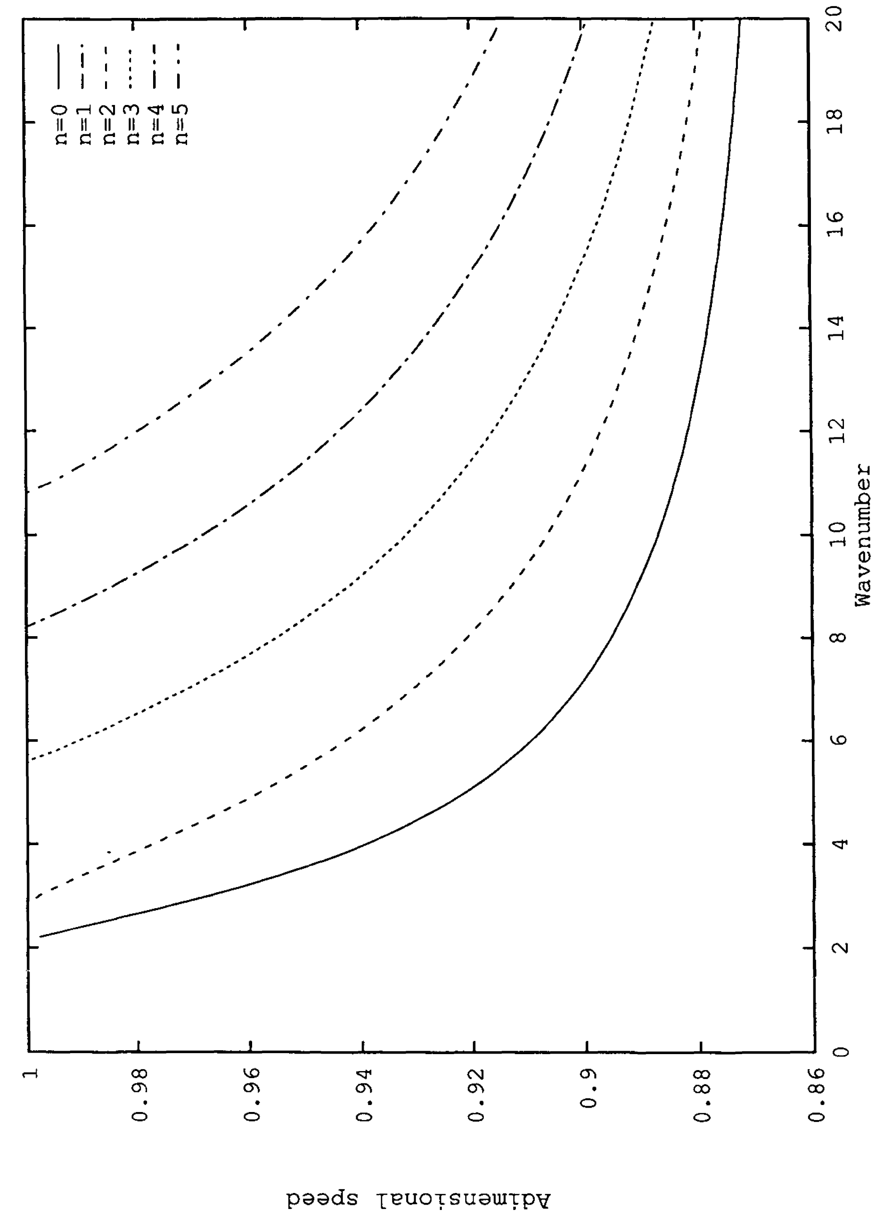
n=0
n=1
n=2
n=3
n=4
n=5
0
2
4
6
8
10
12
14
16
18
20
Wavenumber
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
Adimensional speed

**Figure 1.**

The high frequency behavior had been first remarked by Mindlin [18]. It can be proved using the implicit function theorem, see Bamberger *et al.* [5].

We have no proof of the other results we mentioned, but note that they can be deduced from the general analysis to follow. However, there is a result concerning the thresholds we shall need in part 5, to prove the general case :

PROPOSITION 2.1 : *There exist $n_0 \in \mathbb{N}^*$ and $C > 0$, such that, for $n \geq n_0$, and $\beta < Cn$,* (2.10) *has no solution. Thus, $\beta_n^* \geq Cn$. Therefore, for each fixed $\beta$, equation* (2.10) *has a finite number of solutions.*

*Proof* (Sketch) : Make the change of variables defined by $\beta = nx$, and introduce the function :

$$F\left(\frac{1}{n}, x, V\right) = \det M_n(\beta, \omega) \quad \text{with} \quad V = \frac{\omega}{\beta}.$$

The asymptotic behavior of $K_n(nz)$, for large $n$, (Abramowitz and Stegun [1]) implies that : $F(0, 0, V) = F_0 < 0$, $F_0$ being independant on $V$. Now, the uniform continuity of $F$ ($K_\nu$ is analytic in $\nu$) gives the existence of $\alpha > 0$ and $C > 0$ such that :

$$\varepsilon < \alpha , \quad x < C \Rightarrow \forall V \quad [0, V_S] , \quad F(\varepsilon, x, V) < 0 .$$

Coming back to the original variables $\beta$ and $n$ gives the result. □

### 2.3. Mathematical Formulation

We may now give the precise setting for the study of problem (2.2)-(2.5). Our main tool will be the spectral theory of self-adjoint operators in Hilbert space, thus we find it convenient to work in the usual Sobolev spaces framework. We refer the reader to Adams [2] for the precise definitions.

We first define the two spaces $H = L^2(\Omega)^3$ and $V = H^1(\Omega)^3$, $H$ being endowed with the (weighted) inner product :

$$(u, v) = \int_\Omega \rho u \,.\, v \, dx \qquad (u, v) \in H \times H . \tag{2.11}$$

We then define on operator $A(\beta)$ in $H$ by the formulas :

$$\left\{ \begin{aligned} (A(\beta)\, u)_i &= \frac{1}{\rho}\left(-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}^\beta(u)}{\partial x_j} + \beta \sigma_{i3}^\beta(u)\right) \quad i = 1, 2 \\ (A(\beta)\, u)_3 &= \frac{1}{\rho}\left(-\sum_{j=1}^{3} \frac{\partial \sigma_{3j}^\beta(u)}{\partial x_j} - \beta \sigma_{33}^\beta(u)\right) \end{aligned} \right. \tag{2.12}$$

with domain :

$$D(A(\beta)) = \{u \in H, A(\beta) u \in H, \sigma^{\beta}(u) . n = 0\} .$$

Note that the free surface condition (2.5) is part of the definition of the domain of $A(\beta)$.

Now, the abstract formulation of (2.2)-(2.5) is :

$$A(\beta) u = \omega^2 u \qquad u \in D(A(\beta)) . \tag{2.13}$$

To give the variational formulation of (2.13), we apply Green's formula, and define a bilinear form on $V \times V$ by :

$$a(\beta ; u, v) = \int_{\Omega} \lambda \operatorname{tr} \varepsilon^{\beta}(u) . \operatorname{tr} \varepsilon^{\beta}(v) \, dx + 2 \sum_{i,j} \int_{\Omega} \mu \varepsilon^{\beta}_{ij}(u) \, \varepsilon^{\beta}_{ij}(v) . \tag{2.14}$$

Of course, $a(\beta)$ is the bilinear form associated with $A(\beta)$, in the sense of Reed-Simon [22], since

$$\forall (u, v) \in D(A(\beta)) \times V \qquad (A(\beta) u, v) = a(\beta ; u, v)$$

and the variational formulation of (2.13) is :

$$a(\beta ; u, v) = \omega^2 (u, v) \quad u \in V , \quad \forall v \in V . \tag{2.15}$$

Since $a(\beta)$ is obviously symmetric and positive on $V$, we have :

PROPOSITION 2.2 : *$A(\beta)$ is a positive, and self-adjoint operator in $H$, with dense domain. $V$ is the form-domain of $A(\beta)$.*

Equation (2.13) appears as an eigenvalue problem for the unbounded self adjoint operator $A(\beta)$. Note that, as $\Omega$ is not bounded, $A(\beta)$ will not have compact resolvent, and the existence of eigenvalues is not guaranteed a priori. Also note an important feature of our approach : $\beta$ is a parameter, and we look for functions $\omega(\beta)$. This is quite natural from a mathematical point of view but is contrary to the mechanical approach, where the frequency is the primary datum, and one seeks values of the wave number giving rise to a guided wave. One important point, which we do not deal with in this work, is the invertibility of the curves $\beta \to \omega(\beta)$. For indications on this point, in the context of optical fibers, see Bamberger-Bonnet [3].

## 3. SPECTRAL PROPERTIES OF $A(\beta)$

In this section, we give the structure of the spectrum of $A(\beta)$, but we delay the proof of existence of the eigenvalues to the next section. All these results are based on suitable decompositions of the bilinear form $a(\beta)$.

The main results of this section are theorem 3.1, which determines the essential spectrum of $A(\beta)$, and theorem 3.2, giving lower and upper bounds on the eigenvalues.

### 3.1. Decomposition of the bilinear Form

To motivate what follows, we begin by recalling that, in the whole space, $A(\beta)$ would be an operator with purely continuous spectrum $[\beta^2 V_S^2, +\infty[$ (this can be shown simply by Fourier transform). Since the boundary introduces a compact perturbation (in a sense to be made precise later), we will show that the essential spectrum of $A(\beta)$ is still $[\beta^2 V_S^2, +\infty[$. This is why we try to obtain an expression for $a(\beta, v, v)$ that lets the term $\beta^2 V_S^2 |v|^2$ appear. This is contained in :

PROPOSITION 3.1 : i) $a(\beta)$ *admits the following decomposition :*

$$\forall v \in V, \quad a(\beta\,; v, v) = \beta^2 V_s^2 |v|^2 + p(\beta\,; v, v) + b(\beta\,; v, v)$$

*with* :

$$p(\beta\,; v, v) = 2\,\mu\beta \int_\Gamma (v_1 n_1 + v_2 n_2)\, v_3\, d\sigma$$

$$b(\beta\,; v, v) = \lambda \int_\Omega \left| \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right|^2 dx + 2\,\mu \int_\Omega \left( \left| \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right|^2 \right) dx$$

$$+ \mu \int_\Omega \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 dx - 2(\lambda + \mu)\,\beta \int_\Omega v_3 \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) dx$$

$$+ (\lambda + \mu)\,\beta^2 \int_\Omega |v_3|^2\, dx + \mu \int_\Omega |\nabla v_3|^2\, dx\,.$$

ii) *The form* $b(\beta)$ *is positive.*

*Proof:* We start from (2.14), and integrate by parts the terms $\varepsilon_{i3}^\beta(v)$, $i = 1, 2$. For instance :

$$\int_\Omega |\varepsilon_{13}^\beta(v)|^2\, dx = \int_\Omega \left( \left| \frac{\partial v_3}{\partial x_1} \right|^2 + \beta^2 |v_1|^2 - 2\,\beta v_3 \frac{\partial v_1}{\partial x_1} \right) dx + \tag{3.1}$$

$$+ 2\,\beta \int_\Gamma v_1 n_1 v_3\, d\gamma\,.$$

We also expand $|\mathrm{tr}\, \varepsilon^\beta(v)|$ and add everything up. This gives i).

To prove that $b$ is positive, we keep (3.1), but do not expand $|\text{tr } \varepsilon^{\beta}(v)|$. We have :

$$b(\beta ; v, v) = \lambda \int_{\Omega} |\text{tr } \varepsilon^{\beta}(v)|^2 \, dx + \mu \int_{\Omega} \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 dx + \mu \int_{\Omega} |\nabla v_3|^2 \, dx$$
$$+ \mu \sum_{r=1}^{2} \int_{\Omega} \left( 2 \left| \frac{\partial v_i}{\partial x_i} \right|^2 + \frac{\beta^2}{2} |v_3|^2 + 2 \beta v_3 \frac{\partial v_3}{\partial x_i} \right) dx .$$

The conclusion follows from the fact that each term in the sum is :

$$2 \mu \int_{\Omega} \left| \frac{\partial v_i}{\partial x_i} - \frac{\beta}{2} v_3 \right|^2 dx . \qquad \square$$

*Remark : p* involves only boundary values of $v$. If our problem were posed in the whole space, or if we had imposed Dirichlet conditions instead of (2.5), this term would disappear. The min-max principle (see section 4) implies that in these two cases, there are no eigenvalues. Thus, we see the important role of the free surface condition (2.5) for the existence of guided waves. □

In the next result, we deal with traces of functions in $V$. We denote the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$ by $\gamma$. (See Adams [2].)

COROLLARY 3.1 : *The bilinear from* $p(\beta)$ *is compact in* $V$ *: from any sequence* $(v_n)_{n \in N}$, $v_n \to v$ *weakly in* $V$ *we may extract a subsequence (still denoted by* $v_n$*), s.t. :* $p(\beta, v_n, v_n) \to p(\beta, v, v)$.

*Proof :* By continuity of the trace, $\gamma_0 v_n \to \gamma_0 v$ in $H^{1/2}(\Gamma)^3$. Since the boundary $\Gamma$ is bounded, $H^{1/2}(\Gamma)^3$ is compactly embedded in $L^2(\Gamma)^3$. Thus we extract a subsequence such that $\gamma_0 v_n \to \gamma_0 v$ strongly in $L^2(\Gamma)$ so that $p(\beta, v_n, v_n)$ converges to $p(\beta, v, v)$. □

The next result gives a coerciveness inequality, which we will use in various parts of the paper.

PROPOSITION 3.2 : i) *There exists a strictly positive function* $\gamma_0(\beta)$, *depending only on* $\Omega$, *such that :*

$$\forall v \in V , \quad a(\beta ; v, v) \geq \gamma_0(\beta) \, V_S^2 |v|^2 .$$

ii) $\gamma_0$ *satisfies the inequality :*

$$\gamma_0(\beta) \geq \text{Max } (0, C_0 \beta^2 - C_1)$$

*for some positive constants* $C_0$, $C_1$.

*Proof* The result is a generalized Korn's inequality, for an unbounded domain The proof is rather technical, so that we do not give the details, refering the reader to Bamberger *et al* [5]

i) is proved by contradiction, and the use of corollary 3 1

For ii), we modify Nitsche's proof [20] of the classical Korn's inequality It depends on the construction of suitable extension operators The same method has been applied by Santosa to prove existence for elastodynamics problems in a cylindrical domain [24]

Note that the proof remains valid for Lipschitz boundaries [20] □

### 3.2. The Essential Spectrum

We recall a few definitions from spectral theory Following Reed-Simon [22], a number $\Lambda \in R_+$ belongs to the spectrum of $A(\beta)$, $\sigma(A(\beta))$, iff $A(\beta) - \Lambda I$ has no bounded inverse An eigenvalue is any number $\Lambda$ such that $A(\beta) - \Lambda I$ is not one-to-one The discrete spectrum is the set of eigenvalues of finite multiplicity, that are isolated points of the spectrum Finally, the essential spectrum $\sigma_{\mathrm{ess}}(A(\beta))$ is the complementary set of the discrete spectrum (in the spectrum) We shall make use of the following characterizations (*cf* Weidmann [27]) of respectively the spectrum and the essential spectrum of $A(\beta)$

(3 3) $\Lambda \in \sigma(A(\beta))$ iff there exists a sequence $(v_n)_{n \in N} \in D(A(\beta))$, such that $|v_n| = 1$, $\lim_{n \to \infty} |A(\beta) v_n - \Lambda v_n| = 0$

(3 4) $\Lambda \in \sigma_{\mathrm{ess}}(A(\beta))$ iff $\Lambda \in \sigma(A(\beta))$ and the sequence $v_n$ from (3 3) has no strongly convergent subsequence in $H$

A well known theorem of Weyl (Reed-Simon [22], Schechter [25]) asserts that $\sigma_{\mathrm{ess}}(A(\beta))$ is invariant through compact perturbations In order to use corollary 3 1, we would like to consider $A(\beta)$ as a perturbation of the « whole space » case, where it is well known that $A(\beta)$ has purely continuous spectrum Unfortunately, this is not possible directly, as the two operators are not defined on the same domain This is why we give a direct proof

THEOREM 3 1

$$\sigma_{\mathrm{ess}}(A(\beta)) = [\beta^2 V_s^2, \ +\infty[$$

*Proof* i) Let $\Lambda \in \sigma(A(\beta))$, $\Lambda < \beta^2 V_S^2$, and let $(v_n)_{n \in N}$ be a sequence as in (3 3) We will show that $(v_n)$ has a subsequence converging strongly in $H$ We first extract a weakly convergent subsequence, and then use the compactness result (corollary 3 1) to conclude

After multiplication by $v_n$, we deduce from (3.3) that :

$$a(\beta ; v_n, v_n) = (w_n, v_n) + \Lambda |v_n|^2$$

where $w_n = A(\beta) v_n - \Lambda v_n$ convergences strongly to 0 in $H$.

Thus $a(\beta ; v_n, v_n)$ is bounded, since $|v_n| = 1$, which implies thanks to corollary 3.2 that $\|v_n\|_V$ is bounded. We may extract a subsequence (still denoted by $v_n$), such that $v_n \to v$ in $V$ weakly.

So, for any $w$ in $V$, we have :

$$\lim_{n \to \infty} a(\beta ; v_n, w) - \Lambda(v_n, w) = a(\beta ; v, w) - \Lambda(v, w) .$$

But, as $a(\beta ; v_n, w) - \Lambda(v_n, w) = (w_n, w)$, by strong convergence of $w_n$ in $H$, we get

$$a(\beta ; v, w) - \Lambda(v, w) = 0 \quad \forall w \in V$$

which proves that $v \in D(A(\beta))$ and that $A(\beta) v = \Lambda v$.

Taking $w = v$ leads in particular to :

$$a(\beta ; v, v) = \Lambda |v|^2 . \tag{3.5}$$

To prove strong convergence, we use proposition 3.1 to establish that :

$$b(\beta ; v^n, v^n) + p(\beta ; v^n, v^n) + (\beta^2 V_s^2 - \Lambda)|v^n|^2 \to 0 \tag{3.6}$$

and we also have (3.5), that we can write :

$$b(\beta ; v, v) + p(\beta ; v, v) + (\beta^2 V_s^2 - \Lambda)|v|^2 = 0 . \tag{3.7}$$

We now use corollary 3.1 to extract another subsequence such that $p(\beta ; v^n, v^n) \to p(\beta, v, v)$. Defining

$$b_\Lambda(\beta, v, v) = b(\beta ; v, v) + (\beta^2 V_s^2 - \Lambda)|v|^2$$

and substracting (3.6) from (3.7), we see that $b_\Lambda(\beta ; v^n, v^n) \to b_\Lambda(\beta ; v, v)$. Then thanks to the weak convergence of $v^n$ to $v$ in the space $V$ where $b_\Lambda$ is continuous we get :

$$b_\Lambda(\beta ; v - v^n, v - v^n) \to 0$$

and since $\beta^2 V_s^2 - \Lambda > 0$, this implies strong convergence.

ii) We now show that, given $\Lambda \geqslant \beta^2 V_s^2$, we may exhibit a sequence $v_n$ satisfying (3.3), with no convergent subsequence. The principle is simply to take the $S$ wave, which works in the whole space, and to use suitable cut-off functions. Once again, we omit most of the technical details, refering the reader to Bamberger *et al.* [5].

For $\Lambda > \beta^2 V_S^2$, we write $\Lambda = V_S^2(\beta^2 + |k|^2)$, $k = (k_1, k_2)$, and we define the *SH* wave [17] :

$$v_S(k) = \frac{1}{(|k|^2 + \beta^2)^{1/2}} \begin{pmatrix} \beta \\ 0 \\ -k_1 \end{pmatrix} e^{ik.x} .$$

In the distributional sense, we have

$$A(\beta)\, v_S(k) = \Lambda v_S(k) .$$

We only need to truncate $v_S(k)$ to bring it in $D(A(\beta))$ : we define a cut-off sequence $\psi_n \in C^\infty(R_+)$ such that :

$$\begin{cases} \psi_n(r) = 0 & \text{for} \quad r \geqslant n + 1 ; \\ \psi_n(r) = 1 & r \leqslant n \\ 0 \leqslant \psi_n(r) \leqslant 1 & \forall r \in R_+ \\ \|\psi_n\|_{2,\infty} \leqslant C & \forall n \end{cases}$$

and we set :

$$v_n = C_n\, \psi_n([(x_1 - 2n)^2 + x_2^2]^{1/2})\, v_S(k) ,$$

$C_n$ being chosen such that $|v_n| = 1$.

One easily checks that $\operatorname{supp} v_n \subset \{(x_1 - 2n)^2 + x_2^2 \leqslant (n+1)^2\}$ and $\operatorname{supp} \nabla v_n \subset \{n^2 \leqslant (x_1 - 2n)^2 + x_2^2 \leqslant (n+1)^2\}$ so that, for $n$ large enough, $v_n$ belongs to $D(A(\beta))$. This, with the property of $\psi_n$ implies that $|A(\beta)\, v_n - \Lambda v_n| \to 0$. Moreover, since supp $v_n$ goes to infinity, $v_n \rightharpoonup 0$ and it has no strongly convergence subsequence. □

### 3.3. Bounds on the Eigenvalues

THEOREM 3.2 : *Any eigenvalue* $\omega^2$ *satisfies* $\gamma_0(\beta)\, V_S^2 \leqslant \omega^2 \leqslant \beta^2 V_S^2$.

*Proof :* The first inequality follows trivially from Proposition 3.2. To prove the second one, we will use Rellich's theorem. Define, for $u$ an eigenfunction :

$$\operatorname{curl}^\beta u = \begin{cases} \dfrac{\partial u_3}{\partial x_2} - \beta u_2 \\ \dfrac{\partial u_3}{\partial x_1} - \beta u_1 \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{cases}$$

which satisfies :

$$\Delta(\operatorname{curl}^{\beta} u) = \left(\beta^2 - \frac{\omega^2}{V_S^2}\right) \operatorname{curl}^{\beta} u . \tag{3.7}$$

Rellich's theorem (*cf.* Rellich [23], Wilcox [28]) implies that for $R$ such that $\mathcal{O} \subset B(0, R)$ we have $\operatorname{curl}^{\beta} u = 0$ if $|x| > R$.

From (3.7), we deduce that $\operatorname{curl}^{\beta} u$ is analytic in $\Omega$. Since we suppose $\Omega$ connected, we have $\operatorname{curl}^{\beta} u = 0$ in $\Omega$. Then we may write that $(\operatorname{curl}^{\beta} u)_1 n_2 - (\operatorname{curl}^{\beta} u)_2 n_1 = 0$ on $\Gamma$. Remarking that when $u$ belongs to $D(A(\beta))$, $\Delta u_3$ belongs to $L^2(\Omega)$ gives a sense to $\left.\frac{\partial u_3}{\partial n}\right|_{\Gamma}$ as an element of $H^{-1/2}(\Gamma)$. Then this last equality can be rewritten :

$$\frac{\partial u_3}{\partial n} - \beta(u_1 n_1 + u_2 n_2) = 0 .$$

But the third boundary condition (2.5) is

$$\frac{\partial u_3}{\partial n} + \beta(u_1 n_1 + u_2 n_2) = 0$$

and this implies :

$$\begin{aligned} \frac{\partial u_3}{\partial n} &= 0 \quad \text{on } \Gamma \\ u_1 n_1 + u_2 n_2 &= 0 \quad \text{on } \Gamma . \end{aligned} \tag{3.8}$$

To conclude, we return to the third equation (2.4), which reduces to :

$$\begin{cases} - V_P^2 \Delta u_3 + (\beta^2 V_P^2 - \omega^2) u_3 = 0 & \text{in } \Omega \\ \dfrac{\partial u_3}{\partial n} = 0 & \text{on } \Gamma . \end{cases}$$

If $\beta^2 V_S^2 < \omega^2 < \beta^2 V_P^2$, we conclude using Lax-Milgram's lemma, whereas if $\omega^2 > \beta^2 V_P^2$, we invoke Rellich's theorem again. Finally, if $\omega^2 = \beta^2 V_P^2$, we cannot have a harmonic function in $L^2(\Omega)$. In all cases, we conclude that $u_3 = 0$, and then $u_1 = u_2 = 0$. □

We have now described the structure of $\sigma(A(\beta))$ :

- $]-\infty, \gamma_0(\beta) V_S^2[$ is contained in the resolvent set of $A(\beta)$.
- Any $\omega^2 \in \sigma(A(\beta)) \cap [\gamma_0(\beta) V_S^2, \beta^2 V_S^2[$ is an eigenvalue of $A(\beta)$, with finite multiplicity.

- $[\beta^2 V_S^2, +\infty[$ is the essential spectrum of $A(\beta)$.
- There are no eigenvalues embedded in the essential spectrum.

## 4. THE DISPERSION RELATION

We are now able to prove the main results of this paper, namely the existence of a countable set of curves $\omega_m(\beta)$ that are eigenvalues of $A(\beta)$, for large enough $\beta$. We actually prove more : there exists numbers $\beta_m^*$ (the thresholds), such that, for $\beta \geqslant \beta_m^*$, $A(\beta)$ has at least $m$ eigenvalues, whereas for $\beta < \beta_m^*$, $A(\beta)$ has at most $(m-1)$ eigenvalues. These numbers go to infinity with $m$, so that for $\beta$ fixed, $A(\beta)$ has only a finite number of eigenvalues. Lastly, $\beta_1^* = \beta_2^* = 0$, which means that $A(\beta)$ always has at least 2 eigenvalues. The proof of these results, with several corollaries, occupy the next two sections.

The main tool for the existence part is the celebrated Min-Max principle (see Reed-Simon [22], Dunford-Schwartz [12]), which characterizes the eigenvalues below the bottom of the essential spectrum of a self-adjoint operator. It is standard for compact operators, but the extension to operators with continuous spectrum is much less known. It appears in Dunford Schwartz ([12], p. 1543), and its precise statement is in Reed-Simon [22]. We find it useful to give this statement again.

We begin by defining the Rayleigh quotient of $A(\beta)$ :

$$\forall v \in D(A(\beta)), \quad v \neq 0 \qquad R(\beta, V) = \frac{(A(\beta)\, v, v)}{|v|^2}$$

which in fact extends to $V$ by :

$$\forall v \in V, \quad v \neq 0 \qquad R(\beta, v) = \frac{a(\beta\,; v, v)}{|v|^2}\,.$$

Then, for $m = 1, 2, \ldots$, we set :

$$s_m(\beta) = \sup_{\substack{(V_{m-1} \subset V) \\ \dim V_{m-1} = m-1}} \ \inf_{\substack{v \in V_{m-1}^{\perp} \\ v \neq 0}} R(\beta, v) \tag{4.1}$$

where $V_{m-1}^{\perp} = \{v \in V / (v, w) = 0, \forall w \in V_{m-1}\}$. It is known that $s_m(\beta)$ is an increasing sequence, converging to the lower bound of $\sigma_{\text{ess}}(A(\beta))$ :

$$\lim_{m \to \infty} s_m(\beta) = \inf \sigma_{\text{ess}}(A(\beta)) = \beta^2 V_s^2\,.$$

The Min-Max principle characterizes those eigenvalues of $A(\beta)$ below $\beta^2 V_s^2$ :

PROPOSITION 4.1 : *For each m, the following alternative holds :*

1) $s_m(\beta) < \beta^2 V_S^2$. *In that case* $A(\beta)$ *has m eigenvalues (including multiplicities) below* $\beta^2 V_S^2$, *and* $s_m(\beta)$ *is the m-th eigenvalue.*

2) $s_m(\beta) = \beta^2 V_S^2$. *In that case, for* $n \geq m$, $s_n(\beta) = \beta^2 V_S^2$ *and* $A(\beta)$ *has at most* $m-1$ *eigenvalues below* $\beta^2 V_S^2$.

*Proof :* Reed-Simon ([22], vol. IV, p. 76).

To prove the existence of eigenvalues, it is convenient to use an alternative expression of $s_m(\beta)$ (Dunford-Schwartz [12]) :

$$s_m(\beta) = \inf_{\substack{V_m \subset V \\ \dim V_m = m}} \sup_{\substack{v \in V_m \\ v \neq 0}} R(\beta, v) . \tag{4.2}$$

This result reduces the proof of existence of eigenvalues to the construction of suitable test functions satisfying (4.2).

We will use two different kinds of such functions. In both cases, we will be guided by what is known from the physics of the problem, for instance in the circular case :

— At high frequency (for large $\omega$, or $\beta$), we know the waves behave like Rayleigh waves. Thus it is natural to try to use a (truncated) Rayleigh wave on the surface. This will enable us to prove the existence of $m$ eigenvalues, for each $m$, if $\beta$ is large enough.

— On the other hand, when $\beta \to 0$, the waves are less and less guided, behaving like a constant. Taking a (truncated) constant test function will give us the existence of two eigenvalues for any $\beta$.

### 4.1. Existence of the Eigenvalues

The proof depends on properties of the $2D$ Rayleigh wave, which we recall briefly (see Joly [13], Schulenberger [26] for details).

We consider the propagation of elastic waves in the half-plane $R_+^2 = \{(x, z), z > 0\}$, and we seek fields of the form

$$\tilde{v}(x, z, t) = v(z)\, e^{i(\beta x - \omega t)} \quad v = (u, w) .$$

We are still looking for guided waves, i.e., we require $v \in H^1(R_+)^2$. The bilinear form associated to this problem is :

$$\tilde{a}(\beta, v, v) = \lambda \int_0^\infty \left| \frac{dw}{dz} + \beta u \right|^2 dz + 2\mu \int_0^\infty \left( \left| \frac{dw}{dz} \right|^2 + \beta^2 |u|^2 \right) dz + \mu \int_0^\infty \left| \frac{du}{dz} + \beta w \right|^2 dz . \tag{4.3}$$

The Rayleigh wave is the (unique) solution of the variational eigenvalue problem :

$$\tilde{a}(\beta\,;v^R, v) = \omega^2 \int_0^{+\infty} \rho v^R\, v\, dz \quad \forall v \in H^1(R_+)\,.$$

Its frequency satisfies $\omega_R(\beta) = \beta V_R$, $V_R$ being the solution in $[0, V_S]$ of Rayleigh's equation :

$$\left(2 - \frac{V^2}{V_S^2}\right)^2 - 4\left(1 - \frac{V^2}{V_S^2}\right)^{1/2}\left(1 - \frac{V^2}{V_P^2}\right)^{1/2} = 0\,.$$

The expression of the displacement field itself is not important for our purposes. It suffices to know that it can be normalized in such a way that :

$$\text{(4.4)} \qquad \int_0^{\infty} \rho\,|v^R|^2\, dz = 1 \quad \text{and} \quad \int_0^{\infty} \left|\frac{dv^R}{dz}\right|^2 dz \leqslant C\beta^2$$

for some constant $C$.

We now suppose that $\Omega$ contains a domain

$$\Omega_0 = \{(x_1, x_2),\ |x_1| < a,\ x_2 > f(x_1)\}$$

where $f \in C^1(-a, a)$, $f(0) = f'(0) = 0$. We set :

$$K(a) = \sup_{|x_1| \leqslant a} |f'(x_1)|$$

and note that since $f'(0) = 0$, $K(a) \to 0$ as $a \to 0$. Thus by making a small enough, we may bound $K(a)$.

The requirement that $x_2$ has to take all values $> f(x_1)$ might be overcome by a truncation process.

We now restrict our attention to test functions in (4.2), of the form :

$$\text{(4.4)} \qquad v(x_1, x_2) = \phi(x_1)(0, w^R(x_2 - f(x_1)), u^R(x_2 - f(x_1)))$$

with $\phi \in H_0^1(-a, a)$ to be fixed later.

We will show that, first taking a small enough, then letting $\beta$ be large enough, we can make Rayleigh's quotient for such $v$'s be strictly less than $\beta^2 V_S^2$.

THEOREM 4.1 : *Assume that $\Gamma$ is locally $C^1$. For each $m \in N^*$, there exists $\beta_m^* \geqslant 0$, such that for $\beta \geqslant \beta_m^*$, $s_m(\beta) < \beta^2 V_S^2$. Thus, for $\beta \geqslant \beta_m^*$, $s_m(\beta)$ is the $m$-th eigenvalue of $A(\beta)$.*

*Proof :* By the Min Max principle (Proposition 4.1), we just have to find an $m$-dimensional subspace $V_m \subset V$, such that :

$$\forall v \in V_m, \quad a(\beta ; v, v) - \beta^2 V_s^2 |v|^2 < 0 .$$
$$v \neq 0$$

For this, we fixe any functions $(\phi_1, ..., \Phi_m) \in H_0^1(-a, a)$, and define $\Phi_m = \text{span}\ (\phi_1, ..., \Phi_m)$. Then, we let $V_m = \{v \text{ defined in (4.4)}, \phi \in \Phi_m\}$. We also define, on $H_0^1(-a, a)$ the bilinear form associated with the Laplacian $-\dfrac{d^2}{dx_1^2}$:

$$q(a, \phi, \phi) = \int_{-a}^{a} |\phi'|^2 \, dx_1 .$$

The first step is now to bound $a(\beta)$ on $V_m$:

LEMMA 4.1 : *There exists $C > 0$ such that for $v$ given by* (4.4), $\phi \in H_0^1(-a, a)$ :

$$a(\beta, v, v) \leq \beta^2 (V_R^2 + CK(a)^2) |\phi|^2 + 2 V_S^2 q(a, \phi, \phi) ,$$

*where* $|\phi|^2 = \displaystyle\int_{-a}^{a} |\phi(x_1)|^2 \, dx_1$.

*Proof :* It is a simple computation. Insert (4.4) in (2.14). There comes :

$$\begin{aligned} a(\beta, v, v) = \lambda & \int_{\Omega_0} |\phi(x_1)|^2 \left| \frac{dw^R}{dz} + \beta u^R \right|^2 (x_2 - f(x_1)) \, dx_1 \, dx_2 \\ & + 2 \mu \int_{\Omega_0} |\phi(x_1)|^2 \left[ \left| \frac{dw^R}{dz} \right|^2 + \beta^2 |u^R|^2 \right] (x_2 - f(x_1)) \, dx_1 \, dx_2 \\ & + \mu \int_{\Omega_0} |\phi(x_1)|^2 \left| \frac{du^R}{dz} + \beta w^R \right|^2 (x_2 - f(x_1)) \, dx_1 \, dx_2 \\ & + \mu \int_{\Omega_0} \left| \frac{\partial}{\partial x_1} (\phi(x_1) \, u^R(x_2 - f(x_1))) \right|^2 dx_1 \, dx_2 \\ & + \mu \int_{\Omega_0} \left| \frac{\partial}{\partial x_1} (\phi(x_1) \, w^R(x_2 - f(x_1))) \right|^2 dx_1 \, dx_2 . \end{aligned}$$

We first transform the first 3 integrals by the change of variables $(x_1, x_2) \to (x_1, x_2 - f(x_1))$, and Fubini's theorem. The first three terms are nothing but :

$$\left( \int_{-a}^{a} |\phi(x_1)|^2 \, dx_1 \right) \tilde{a}(\beta\, ; v^R, v^R) = \beta^2 V_R^2 |\phi|^2$$

because of (4.4).

We now show how to bound the 4-th integral :

$$I = \int_{\Omega_0} \left| \frac{\partial}{\partial x_1} (\phi(x_1)\, u^R(x_2 - f(x_1))) \right|^2 dx_1\, dx_2$$
$$= \int_{\Omega_0} \left| \phi'(x_1)\, u^R(x_2 - f(x_1)) - \phi(x_1)\, f'(x_1) \frac{du^R}{dz} (x_2 - f(x_1)) \right|^2 dx_1\, dx_2 .$$

We also transform this by setting $z = x_2 - f(x_1)$, so that :

$$I = \iint_{(-a,a) \times R_+} \left| \phi'(x_1)\, u^R(z) - \phi(x_1)\, f'(x_1) \frac{du^R}{dz} (z) \right|^2 dx_1\, dz$$
$$\leqslant 2 \iint_{(-a,a) \times R^+} \left( |\phi'|^2\, |u^R|^2 + |\phi|\, |f'|^2 \left| \frac{du^R}{dz} \right|^2 \right) dx_1\, dz$$
$$\leqslant \frac{2}{\rho} \left( \int_0^{+\infty} \rho\, |u^R|^2 \, dz \right) q(a, \phi, \phi) + 2\, K(a)^2 \left( \int_0^{+\infty} \left| \frac{du^R}{dz} \right|^2 dz \right) |\phi|^2$$

we do the same for the last integral, and add up the results, which proves the lemma, thanks to 4.4. □

Proceeding to prove the theorem, we have (since $|v|^2 = |\phi|^2$ because of (4.4))

$$a(\beta, v, v) - \beta^2 V_S^2 |v|^2 \leqslant \beta^2 (V_R^2 + CK(a)^2 - V_S^2) |\phi|^2 + V_S^2\, q(a, \phi, \phi) .$$

Denoting by $\lambda_1(a), \ldots, \lambda_m(a), \ldots$ the eigenvalues of $-\dfrac{d^2}{dx_1^2}$ on $H_0^1(-a, a)$, we have :

$q(a, \phi, \phi) \leqslant \lambda_m(a)$ on the $m$ dimensional subspace $\Phi_m$. Thus :

$$\text{(4.5)} \quad a(\beta, v, v) - \beta^2 V_S^2 |v|^2 \leqslant [\beta^2 (V_R^2 + CK(a)^2 - V_S^2) + \lambda_m(a)\, V_S^2]\, |\phi|^2$$

It is now easy to make the right hand side negative : first choose $a$ small enough such that :

$$K(a)^2 < \frac{V_S^2 - V_R^2}{C} .$$

This is possible, since $V_R < V_S$. $a$ is now kept fixed. Then, if $\beta$ is large enough (viz. $\beta^2 > \dfrac{\lambda_m(a)\, V_S^2}{V_S^2 - V_R^2 - CK(a)^2}$), the right hand side of (4.5) is $< 0$, proving the theorem. □

*Remark :* Another direct proof of the proceeding result is given in [5]. This proof doesn't not use the Rayleigh wave. □

The dispersion relation characterizing the eigenvalues splits into a countable family of relations :

$$(4.6) \qquad \begin{aligned} \omega_m(\beta)^2 &= s_m(\beta) \\ s_m(\beta) &< \beta^2\, V_S^2 . \end{aligned}$$

Theorem 4.1 allows us to define the thresholds :

$$(4.7) \qquad \beta_m^* = \inf \left\{\beta_m,\ \forall \beta > \beta_m,\ s_m(\beta) < \beta^2\, V_S^2\right\} .$$

The sequence $\beta_m^*$ is non decreasing, and we show in the next section that it actually goes to infinity. Our conjecture is that the thresholds $\beta_m^*$ are equivalent to $m$. This is based partly on numerical evidence obtained in the circular case (*cf.* Bamberger *et al.,* [5] Boström-Burden [8]), and partly on the estimates we have obtained in the preceeding proof. If they are sharp, they imply $(\beta_m^*) \sim \mathrm{Cte} \times m$. We have not been able to prove this. However as $\lambda_m(a) = \dfrac{m^2\, \pi^2}{4\, a^2}$, the proof given above shown that $\beta_m^* \leqslant C(a)\, m$ with $C(a) = \dfrac{\pi V_S}{2\, a(V_S^2 - V_R^2 - CK(a)^2)^{1/2}}$.

### 4.2. Properties of the Thresholds

This section brings together several results about the thresholds $\beta_m^*$. We begin by proving that they are « cut-off wavenumbers » : for $\beta < \beta_m^*$, $s_m(\beta)$ is not an eigenvalue of $A(\beta)$ and for $\beta > \beta_m^*$, $s_m(\beta)$ is an eigenvalue. (We do not know wether $s_m(\beta_m^*)$ is or not an eigenvalue of $A(\beta_m^*)$. The answer is not given by the min-max principle). Then we prove a comparison result, which enables us to show that the thresholds go to infinity with $m$, or equivalently that, for fixed $\beta$, $A(\beta)$ has a finite number of eigenvalues. At last, we show that the first two thresholds are 0, i.e. $s_1(\beta)$ and $s_2(\beta)$ are always eigenvalues of $A(\beta)$.

PROPOSITION 4.2 : *The function* $\beta \to s_m(\beta) - \beta^2\, V_S^2$ *is non-increasing.*

*Proof :* We adapt an idea from Reed-Simon [22], also used in a similar context by Bamberger *et al.* [3], [4]. We start from Proposition 3.1, and introduce the operator $I_\beta$ in $V$ defined by :

$$I_\beta(v_1, v_2, v_3) = (v_1, v_2, \beta v_3) .$$

Then, i) of Proposition 3.1 may be written as :

$$(4.8) \quad a(\beta, v, v) - \beta^2 V_S^2 |v|^2 = \tilde{b}(I_\beta v, I_\beta v) + \frac{1}{\beta^2} \int_\Omega |\nabla(I_\beta v)_3|^2 \, dx$$

where $\tilde{b}(w, w)$ does not depend on $\beta$. As we have :

$$|v|^2 = |(I_\beta v)_1|^2 + |(I_\beta v)_2|^2 + \frac{1}{\beta^2} |(I_\beta v)_3|^2$$

we can write :

$$(4.9) \qquad R(\beta, v) - \beta^2 V_S^2 = \frac{A(I_\beta v) + \beta^2 B(I_\beta v)}{C(I_\beta v) + \beta^2 D(I_\beta v)} \underset{\text{def}}{=} F(\beta, I_\beta v)$$

$A$, $B$, $C$, $D$ being quadratic forms on $V$, with $A$, $C$ and $D$ positive. Next we observe that, if $V_m$ is an arbitrary subspace of $V$ of dimension $m$, the same is true of $I_\beta V_m$, so that :

$$s_m(\beta) - \beta^2 V_S^2 = \inf_{\substack{V_m \subset V \\ \dim V_m = m}} \sup_{w \in V_m} F(\beta, w) .$$

The main point of this transformation is that $F(\beta, w)$ is a very simple function of $\beta$. In the two cases refered to above, it was either a linear or quadratic function. In our case it is a homographic function, and we know that $F(0, w) = \dfrac{A(w)}{C(w)} \geq 0 \ \forall w \in V$. If $C(w)$ and $D(w)$ are strictly positive, we have one of the cases illustrated in Figure 2. These figures do not include the cases $C(w) = 0$ or $D(w) = 0$, but in any case the function $\tilde{F}(\beta, w) = \text{Min}\,(0, F(\beta, w))$ is non-increasing. On the other hand, since $s_m(\beta) \leq \beta^2 V_S^2$, there must exist, for each $\beta$, some $w \in V$, such that $\tilde{F}(\beta, w) = F(\beta, w)$, so that :

$$s_m(\beta) - \beta^2 V_S^2 = \inf_{\substack{V_m \subset V \\ \dim V_m = m}} \sup_{w \in V_m} \tilde{F}(\beta, w)$$

and $s_m(\beta) - \beta^2 V_S^2$, being an inf-sup of decreasing functions must also be non-increasing. □

COROLLARY 4.1 : *The thresholds $\beta_m^*$ have the alternative characterization :*

$$\beta_m^* = \sup \left\{\beta_m, \forall \beta \leq \beta_m, s_m(\beta) = \beta^2 V_S^2\right\} .$$

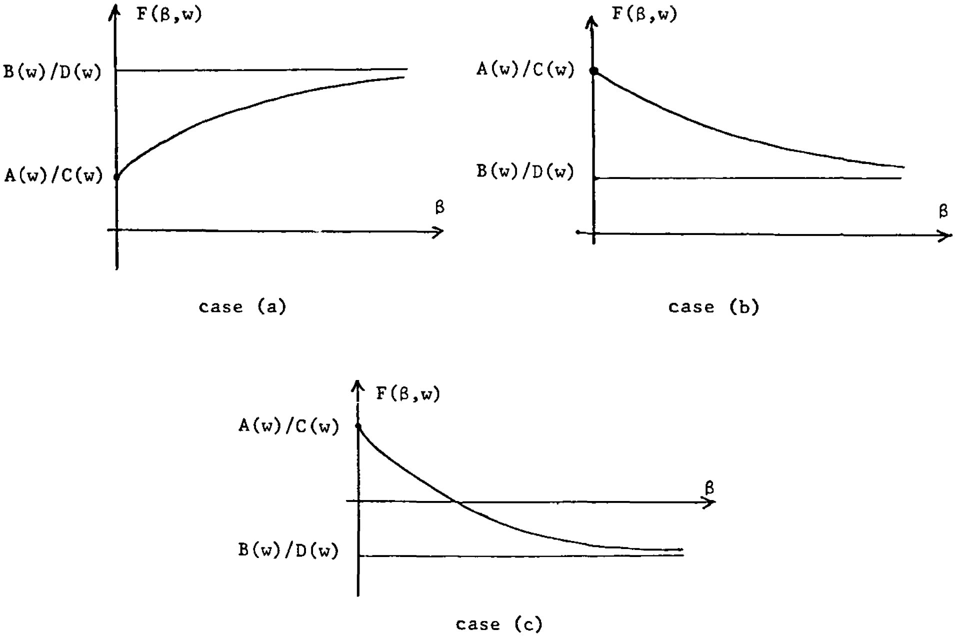


**Figure 2.**

*Proof :* Let $\beta_m^0$ be the rhs of the quantity in the corollary. It is clear that, by definition, $\beta_m^0 \leq \beta_m^*$. Let us now prove that $\beta_m^0 = \beta_m^*$. First note that, by continuity of $s_m(\beta)$, we have :

$$s_m(\beta_m^*) = (\beta_m^*)^2 V_S^2 .$$

Assume that $\beta_m^* > \beta_m^0$. So $\beta_m^*$ does not belong to the set $\{\beta_m, \forall \beta \leq \beta_m, s_m(\beta) = \beta^2 V_S^2\}$. So, there exists $\beta \leq \beta_m^*$ such that $s_m(\beta) < \beta^2 V_S^2$. But this contradicts the fact that the function $s_m(\beta) - \beta^2 V_S^2$ is decreasing since $s_m(\beta_m^*) - (\beta_m^*)^2 V_S^2 = 0$. □

*Remarks :* i) Let us stress the importance of corollary 4.1. The fact that $\beta_m^*$ and $\beta_m^0$ and identical is not a priori obvious, and is not general : in Bamberger *et al.* [4], it is only shown to hold under additional asumptions on the coefficients.

ii) Corollary 4.1 asserts that, once the curve $\beta \to s_m(\beta)$ becomes distinct from $\beta^2 V_S^2$, it will never be equal to $\beta^2 V_S^2$ again. But Proposition 4.1 gives even more : the difference $s_m(\beta) - \beta^2 V_S^2$ is negative, and increases in absolute value. □

We now wish to examine the behavior of $\beta_m^*$ for large $m$. For this, we will use a comparison result, that links the eigenvalues in $\Omega$ to those outside a circle contains $R^2 - \bar{\Omega}$. Then, we use the known behavior of the thresholds in the circular case (see section 2.2), to derive the general case.

We prove the comparison result in a more general setting : we suppose that $\Omega$ is expressed as :

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 , \quad \Omega_j \text{ open} \quad j = 1, 2$$

and we set :

$$H_j = L^2(\Omega_j)^3 , \quad V_j = H^1(\Omega_j)^3 , \quad j = 1, 2 .$$

We identify an element in $H_j$ or $V_j$ to its extension by 0 in $R^2 - \bar{\Omega}_j$, which is in $H$. Thus, $H_j \subset H$, and $H = H_1 \oplus H_2$. But the inclusion $V_j \subset V$ does not hold, and we only have $V \subset V_1 \oplus V_2$. With obvious notations, we also set :

$$\forall v \in V , \quad v = v_1 + v_2 , \quad a(\beta ; v, v) = a_1(\beta ; v_1, v_1) + a_2(\beta ; v_2, v_2)$$

and

$$s_m^j(\beta) = \sup_{\substack{F \subset H_j^{m-1} \\ \dim F = m-1}} \inf_{v_j \in V_j \cap F^\perp} \frac{a_j(\beta ; v_j, v_j)}{|v_j|^2} \quad j = 1, 2 .$$

Mimicking Courant-Hilbert [10], we prove :

PROPOSITION 4.3 : $\forall (p, q) \in N^2$, $\operatorname{Min}(s_p^1(\beta), s_q^2(\beta)) \leqslant s_{p+q}(\beta)$.

*Proof :* First observe that it is enough to prove that the left hand side is less than $s_{p+q}^*(\beta)$, where :

$$s_{p+q}^*(\beta) = \sup_{\substack{F \subset H_1^p \times H_2^q \\ \dim F = p+q-1}} \inf_{v \in (V_1 \oplus V_2) \cap F^\perp} \left( \frac{a_1(\beta ; v_1, v_1) + a_2(\beta ; v_2, v_2)}{|v_1|^2 + |v_2|^2} \right) .$$

Using an abstract result, proved in Bamberger *et al.* [5], which states that, for any subspace $F \subset H$ :

$$(V_1 \oplus V_2) \cap F^\perp = (V_1 \cap F^\perp) \oplus (V_2 \cap F^\perp)$$

we have (denoting $F = (F_1, F_2)$) :

$$s_{p+q}^*(\beta) = \sup_{\substack{F \subset H_1^p \times H_2^q \\ \dim F = p+q-1}} \inf_{\substack{v_1 \in V_1 \cap F_1^\perp \\ v_2 \in V_2 \cap F_2^\perp}} \left( \frac{a_1(\beta ; v_1, v_1) + a_2(\beta ; v_2, v_2)}{|v_1|^2 + |v_2|^2} \right) .$$

Now, denoting, for $F_j \subset H_j$ :

$$r_j(F_j) = \inf_{v_j \in V_j \cap F_j^{\perp}} \frac{a_j(\beta ; v_j, v_j)}{|v_j|^2},$$

we have

$$a_1(\beta ; v_1, v_1) + a_2(\beta ; v_2, v_2) \geq \text{Min } (r_1(F_1), r_2(F_2))(|v_1|^2 + |v_2|^2)$$

which shows that :

(4.10) $$s^*_{p+q}(\beta) \geq \sup_{F_1, F_2} \text{Min } (r_1(F_1), r_2(F_2)) .$$

To conclude, let us fix $\varepsilon > 0$. There exist subspaces $F_1^*$ and $F_2^*$, such that :

$$s_p^1(\beta) \leq r_1(F_1^*) + \varepsilon , \quad s_q^2(\beta) \leq r_2(F_2^*) + \varepsilon .$$

So :

$$\text{Min } (s_p^1(\beta), s_p^2(\beta)) \leq \text{Min } (r_1(F_1^*), r_2(F_2^*)) + \varepsilon$$

which, joined to (4.10), proves the result. □

We apply this result to the case where $\Omega_1$ is the exterior of a circle, and we combine it with proposition 2.1.

THEOREM 4.2 : *For any* $\beta > 0$, $A(\beta)$ *has a finite number of eigenvalues.*

*Proof :* Let $R$ be such that $B(0, R) \supset \complement\Omega$. Choose, in the preceeding result, $\Omega_1 = R^2 \backslash B(0, R)$ and $\Omega_2 = \Omega \backslash \bar{\Omega}_1$. We have :

$$\forall m, \ \forall \beta, \quad s_{2m}(\beta) \geq \text{Min } (s_m^1(\beta), s_m^2(\beta)) .$$

Following proposition 2.1, there exists $M_1(\beta)$, such that :

$$s_m^1(\beta) = \beta^2 V_S^2, \quad \forall m \geq M_1(\beta) .$$

On the other hand, $\Omega_2$ is bounded, so that $s_m^2(\beta) \to \infty$ when $m \to \infty$ :

$$\exists M_2(\beta) , \quad s_m^2(\beta) > \beta^2 V_S^2, \quad \forall m \geq M_2(\beta) .$$

Then, for $m > \text{Max } (M_1(\beta), M_2(\beta))$, $s_{2m}(\beta) \geq \beta^2 V_S^2$, and cannot be an eigenvalue. □

An equivalent formulation of theorem 4.2 is :

COROLLARY 4 2 $\lim_{m \to \infty} \beta_m^* = +\infty$

*Proof* Since the sequence $(\beta_m^*)_{m \in N}$ is increasing, it must have a limit Let us suppose it is finite, say $\lim_{m \to \infty} \beta_m^* = \bar{\beta}^*$

If $\beta \geqslant \bar{\beta}^*$, for any $m \in N$, $\beta \geqslant \beta_m^*$, and $A(\beta)$ admits $m$ eigenvalues for all $m$, contradicting theorem 4 2 □

The last result we wish to prove in this section is the existence of two guided modes, for any value of β Put another way

THEOREM 4 3 *If the boundary* Γ *is locally* $C^1$, $\beta_1^* = \beta_2^* = 0$

*Proof* It is clearly enough to prove the result for $\beta_2^*$ We shall give the proof when Γ is globally $C^1$ For the general case a simple truncation argument works [5]

Once more, we shall be guided by physical reasoning It is known that the smaller β, the more spread-out the modes In the limit $\beta \to 0$, the eigenfunction must tend to a constant Formally, if we choose $v_1$ and $v_2$ constant in proposition 3 1, there only remains terms depending on $v_3$, plus the boundary term, proportional to the constant If we choose this constant large enough, we can make $a(\beta, v, v) - \beta^2 V_s^2 |v|^2 < 0$

To make this precise, we define a cut-off function as follows let us fix $R_0$ such that 0 $B(0, R_0)$, and define

$$\bar{v}_R(x) \begin{cases} = 1 & |x| < R_0 \\ = \ln \dfrac{|x|/R}{R_0/R} & R_0 < |x| < R \\ = 0 & R < |x| \end{cases}$$

$\bar{v}_R$ is the function with « smallest gradient » (in $H^1$ semi-norm), going smoothly form 1 to 0 on $]R_0, R[$ (*cf* Bamberger-Bonnet [3]) In particular

$$\lim_{R \to +\infty} \int_{R_0}^{R} |\nabla \bar{v}_R|^2 \, dx = 0 \tag{4 11}$$

Next, fix $(\bar{v}_1, \bar{v}_2) \in V^2$, such that $\bar{v}_{i|\Gamma} = n_i$, $i = 1, 2$ This is where we need some smoothness on Γ Consider the following 2 dimensional subspace of $V$

$$V_2 = \{v \in V, \exists (\gamma_1, \gamma_2) \in R^2, v = (t\gamma_1 \bar{v}_R, t\gamma_2 \bar{v}_R, \gamma_1 \bar{v}_1 + \hat{\gamma}_2 \bar{v}_2)\}$$

($t$ is a parameter to be fixed later)

We now make use of proposition 3.1, and rewrite the form $a$ as :

(4.12)

$$a(\beta, v, v) = \beta^2 V_S^2 |v|^2 + p(\beta, v, v) + b_0(\beta, v, v) + b_1(\beta ; v, v)$$

$$b_0(\beta ; v, v) = \lambda \int_\Omega \left| \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right|^2 dx + 2\mu \int_\Omega \left| \frac{\partial v_1}{\partial x_1} \right|^2 + \left| \frac{\partial v_2}{\partial x_2} \right|^2 dx$$
$$+ \mu \int_\Omega \left| \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right|^2 dx - 2\beta(\lambda + \mu) \int_\Omega \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) v_3 \, dx$$

$$b_1(\beta ; v, v) = \beta^2(\lambda + \mu) \int_\Omega |v_3|^2 dx + \mu \int_\Omega |\nabla v_3|^2 .$$

Remember that $b_0 + b_1$ is positive, thus we must play on $p$ ; the main point of (4.12) is that $b_0(\beta)$ depends on $(\nabla v_1, \nabla v_2)$, and $b_1(\beta)$ depends only on $v_3$.

This last remark shows that if we take $v \in V_2$ in (4.12), then $p(\beta, v, v) + b_1(\beta, v, v)$ is independent on $R$. We fix $t$ large enough for the form $(p + b_1)(\beta)$ to be negative (if $v$ corresponds to $(\gamma_1, \gamma_2)$) :

$$(p + b_1)(\beta, v, v) \leqslant -C(\gamma_1^2 + \gamma_2^2) \quad \forall v \in V_2 ,$$

with $C > 0$, independent on $R$.

Next, we bound $b_0$ :

$$|b_0(\beta, v, v)| \leqslant C |\nabla v_R|^2 (\gamma_1^2 + \gamma_2^2) .$$

Thanks to (4.11), it suffices to take $R$ large enough for $p + q_0 + q_1$ to remain negative. This proves that $a(\beta, v, v) - \beta V_s^2 |v|^2 < 0$, $\forall v \in V_2$, $\forall \beta \geqslant 0$, and concludes the proof. □

*Remark :* It may be proved (Bamberger *et al.* [4]), that $\beta_4^* > 0$. The situation regarding $\beta_3^*$ is still open. □

There is one last point we wish to mention before concluding this section. It is shown in Bamberger *et al.* [5] that the thresholds are solutions of a generalized eigenvalue problem (« threshold equation »), which simply corresponds to setting $a(\beta, v, v) = \beta^2 V_S^2 |v|^2$. The difficulty is that solutions to this problem are no longer square integrable. □

### 4.3. High Frequency Behavior of the Eigenvalues

This is a domain where we only possess partial results. Following work by Wilson and Morrison ([19], [29]), we would like to prove (as in the circular case) that, as $\beta \to \infty$, the velocity of the waves approaches the Rayleigh velocity. We are only able to show (theorem 4.4) that if the limit exists, it is lower than $V_R$. In the general case, it seems, that this result is optimal. This stems from numerical results of Lagasse [14] on waves guided by a wedge. In this case, the waves are non-dispersive. Their number and their velocities depend only on the angle. The velocity may become strictly smaller than $V_R$. Our final conjecture is that, in the case of a smooth boundary, the limit exists and is equal to $V_R$, whereas if the domain has an edge, the limit may be smaller than $V_R$. Our precise result is :

THEOREM 4.4 : $\limsup_{\beta \to \infty} \frac{\omega_m(\beta)}{\beta} \leq V_R \quad \forall m \in N^*$.

*Proof :* We use the same method, as in theorem 4.1, and keep the same notations. We start from (4.5) :

$$\frac{a(\beta ; v, v)}{\beta^2 |v|^2} \leq (V_R^2 + CK(a)^2) + \frac{V_S^2}{\beta^2} \lambda_m(a)$$

(since $|v|^2 = |\phi|^2$). The difference with theorem 4.1 is that now we want to bound the rhs by $V_R^2 + \varepsilon$, for large enough $\beta$. Fix $\varepsilon > 0$, and choose $a_0$ so small that :

$$\sqrt{C}\, K(a) \leq \frac{1}{2} \sqrt{\varepsilon} \quad \forall a < a_0 .$$

This is possible since $f'(0) = 0 \Rightarrow \lim_{a \to 0} K(a) = 0$.

Keep $a_0$ fixed, and choose $\tilde{\beta}_m(\varepsilon)$ such that

$$\frac{V_S^2}{\tilde{\beta}_m(\varepsilon)} \lambda_m(a_0) \leq \frac{1}{2} \sqrt{\varepsilon} .$$

Then, for $\beta \geq \tilde{\beta}_m(\varepsilon)$, $\frac{a(\beta ; v, v)}{\beta^2 |v|^2} \leq \varepsilon + V_R^2$, which proves, the result. □

## CONCLUSION

We have developed in this article the theory of the existence of elastic surface waves at the exterior of a cylindrical cavity of arbitrary cross section with the help of the theory of self adjoint operators. This method is also powerful to derive some qualitative properties of these waves as the high frequency behaviour, the study of the thresholds... Nevertheless, many questions remain open as, for instance, the monotonicity of the dispersion curves, the behaviour of the sequence of the thresholds, the study of the limit propagation velocity when the frequency increases Some results, as we mentioned earlier, have already been stated in the physical literature and would deserve to be justified (and may be generalized) mathematically

As the method we have used is very general, it enables us to study other problems of guided waves in linear elasticity In [4], the same techniques have been applied to develop the theory of elastic guided waves in heterogeneous media We are currently working on two other applications

— the study of guided waves in fluid-solid media,

— the study of elastic surface waves guided by one-dimensional geometrical perturbations of a homogeneous half-space

It is also natural to try to develop numerical techniques to complete, from a quantitative point of view, the theoretical results We are now working on a numerical method for the computation of the dispersion curves of the surface waves we have studied here Our approach is based on a boundary integral formulation of the eigenvalue problem With the help of the finite element method, this approach gives rise to an algorithm for an approximation of the dispersion curves as well as for the displacement fields Moreover, this new formulation can also be used to obtain further theoretical results in some particular situations, as the interesting limit case of the crack

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