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Andrea PULITA

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An uncountable Mittag-Leffler condition with an application to non-archimedean locally convex vector spaces

par Andrea PULITA

To the memory of Jean-Pierre Demailly

Résumé. La condition de Mittag-Leffler assure l'exactitude de la limite inverse d'une famille de suites exactes indexée par un ensemble partiellement ordonné admettant un sous-ensemble cofinal *dénombrable*. Nous généralisons la condition de Mittag-Leffler en affaiblissant relativement la condition de dénombrabilité. Comme application nous démontrons une version ultramétrique d'un résultat de V. P. Palamodov en relation avec l'acyclicité des espaces Fréchet par rapport au foncteur de complétion.

Abstract. Mittag-Leffler condition ensures the exactness of the inverse limit of a family of short exact sequences indexed on a partially ordered set admitting a *countable* cofinal subset. We extend Mittag-Leffler condition by relatively relaxing the countability assumption. As an application we prove an ultrametric analogous of a result of V. P. Palamodov in relation with the acyclicity of Fréchet spaces with respect to the completion functor.

1. Introduction

In several mathematical theories one encounters objects defined as inverse limits. Typically this happens in sheaf theory, where the set of global sections of a sheaf is the inverse limit of the local ones. Analogous structures actually largely appear in several theories such as topos theory, linear algebra, algebraic geometry, functional analysis and many others. Limits contain crucial information of the original systems and it is interesting to study what properties are lost in the limit process. One of these is *the exactness of short exact sequences.* The importance of this property is illustrated again by the example of sheaf theory, where there is an entire cohomology theory devoted to "*measure*" the default of exactness of the global section functor. More specifically, we are interested here in a precise

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criterion, originally due to Mittag-Leffler [3, II.19, $N^o 5$, Exemple], ensuring that the exactness of short exact sequences is preserved when passing to the limit. Here is the classical Mittag-Leffler statement[1]

Theorem 1 (Classical Mittag-Leffler)**.** *Let R be a ring with unit element and let $(I,\leq)$ be a directed[2] partially ordered set. Let $(\rho^A_{i,j} : A_i \to A_j)_{i,j\in I}$, $(\rho^B_{i,j} : B_i \to B_j)_{i,j\in I}$ and $(\rho^C_{i,j} : C_i \to C_j)_{i,j\in I}$ be three inverse systems of left (or right) R-modules indexed on I. For all $i \in I$ consider an exact sequence $0 \to A_i \xrightarrow{g_i} B_i \xrightarrow{h_i} C_i \to 0$ compatible with the transition maps of the systems[3]. Assume that*

(i) *There exists a cofinal[4] subset of I which is at most countable;*
(ii) *For all $i \in I$, there exists $j \geq i$ such that for all $r \geq j$ one has*

$$\rho^A_{j,i}(A_j) = \rho^A_{r,i}(A_r)\ . \tag{1.1}$$

Then, the short sequence of limits

$$0 \longrightarrow \varprojlim_{i\in I} A_i \xrightarrow{g} \varprojlim_{i\in I} B_i \xrightarrow{h} \varprojlim_{i\in I} C_i \longrightarrow 0 \tag{1.2}$$

is exact and the first derived functor $\varprojlim^{(1)}_{i\in I}$ of $\varprojlim_{i\in I}$ vanishes at $(A_i)_i$: $\varprojlim^{(1)}_{i\in I} A_i = 0$.

The condition (ii) of the theorem is not a necessary condition for the vanishing of $\varprojlim^{(1)}_{i\in I}$. Actually, if I is the set of natural numbers $\mathbb{N} = \{1,2,3,\ldots\}$, then condition (ii) characterizes inverse systems $(A_i)_i$ satisfying $\lim^{(1)}_{i\in I} A_i \otimes E = 0$ for all R-module E (cf. [4]). On the other hand, condition (i) is quite restrictive. From it, one deduces the existence of a map $\tau : \mathbb{N} \to I$ respecting the order relation whose image is a cofinal subset of I (cf. Lemma 5.1). The existence of τ is a strong condition because it implies that *for all inverse systems $(Q_i)_{i\in I}$* of R-modules and for all $n \geq 0$ we have a canonical isomorphism $\varprojlim^{(n)}_{i\in I} Q_i = \varprojlim^{(n)}_{j\in\mathbb{N}} Q_{\tau(j)}$ between the n-th derived functors of $\varprojlim$ (cf. [10, Theorem B]). Hence, from a cohomological point of view, *inverse systems over I are indistinguishable from those over* $\mathbb{N}$. In particular, the claim implies $\varprojlim^{(n)}_{i\in I} A_i = 0$, for all integer $n \geq 2$, because this is true for every inverse system of modules indexed by $\mathbb{N}$ (cf. [10], see below).

The proof of Mittag-Leffler Theorem deals with the surjectivity of the map h by a quite explicit set-theoretical argument. Namely, if $x = (x_i)_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} C_i$, then the inverse images $h_i^{-1}(x_i) \subset B_i$ form an inverse system of

[1] Following the tradition, we state it for R-modules. However, it holds more generally for inverse systems of topological groups and certain Abelian categories as considered in [15].
[2] The word directed means that for all $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$.
[3] i.e. for all $j \geq i \in I$ one has $g_i \circ \rho^A_{j,i} = \rho^B_{j,i} \circ g_j$ and $h_i \circ \rho^B_{j,i} = \rho^C_{j,i} \circ h_j$
[4] A subset $J \subseteq I$ is cofinal if for all $i \in I$ there exists $j \in J$ such that $j \geq i$.

sets, whose inverse limit verifies $h^{-1}(x) = \varprojlim_{i\in\mathbb{N}} h_i^{-1}(x_i)$. The fact that this limit of *sets* is not empty follows from the fact that this system is "*locally*"[5] isomorphic to $(A_i)_{i\in\mathbb{N}}$ and condition (ii) allows us to replace this system by a system of sets indexed on $\mathbb{N}$ with surjective transition maps, which obviously has a non empty inverse limit.

In this paper we are interested in extending this statement *relaxing the countability condition (i) of Theorem 1.* The situation is indeed more dangerous because, for instance, there are explicit non trivial examples of inverse systems *of sets* indexed on some uncountable poset I with *surjective transition maps* whose inverse limit is empty (cf. [1, III.94, Exercice 4-d)]), so that the last part of the above proof is strongly jeopardized. Indeed, without the countability assumption (i), there are actually few results in literature ensuring the non vanishing of an inverse limit of sets. The more important ones seem due to Bourbaki [1, III.57, §7, N.4, Théorème 1] and [3, TG.17, §3, N.5, Théorème 1] and impose strong conditions on the sets and the maps, conditions that we can qualify as being of a nature related to finiteness. For instance, it applies to inverse systems of finite sets, finite groups, Artinian modules (cf. [1, III.60, §7, N.4, Examples]) or to compact topological spaces [3, I.64, §9, N.6, Proposition 8].[6]

These issues show that without countability assumption on I the first derived functor $\varprojlim^{(1)}_{i\in I} A_i$ possibly not vanishes for an inverse system with surjective transition maps. Therefore, several authors addressed the question of what can be said about the smallest natural number $s \geq 0$ such that for all $m \geq s$ and all inverse systems $(M_i)_{i\in I}$ one has $\varprojlim^{(m)}_{i\in I} M_i = 0$ (this number is called *cohomological dimension of the poset* I). Barry Mitchel proved that if $(I, \leq)$ admits a cofinal subset of cardinal $\aleph_n$, and if n is the smallest natural number with this property, then for all $k \geq n+2$, the k-derived functor $\varprojlim^{(k)}_{i\in I}$ vanishes on *every* inverse system of R-modules (cf. [10], extending previous results of J.-E. Roos [12, 13, 14, 15], Goblot [5], and Jensen [8, Proposition 6.2, p. 53]). On the other hand, it is known that for any given ring R, one can find a partially ordered set $(I, \leq)$ and an inverse system $(M_i)_{i\in I}$ of R-modules indexed by I such that for all $n \geq 0$ the n-th derived limit $\varprojlim^{(n)}_{i\in I} M_i$ is not zero [8, Proposition 6.1, p. 51].

In particular, this last result shows that for the vanishing of $\varprojlim^{(1)}_{i\in I} A_i$ in Theorem 1, some *finiteness* condition is really needed either on the set I, or on the objects, or on the transition maps. For instance, the countability

[5]The word locally here has a precise meaning. It is possible to associate to $(I, \leq)$ a topology on I such that sheaves on I with respect to this topology are exactly inverse systems indexed on I. In this correspondence, the global sections of a sheaf over I is exactly the inverse limit of the associated systems (cf. [8, p. 4], see Section 2).

[6]See also the more general case of linearly compact modules with continuous maps [8, Théorème 7.1, p. 57].

condition (i) in Theorem 1 can be seen as a finiteness assumption on the set I and condition (ii) is a finiteness condition on the transition maps. On the other hand, the quoted statements of Bourbaki, or their consequence for Artinian R-modules, can be considered as finiteness conditions on the nature of the objects A_i.

Surprisingly enough, if I does not contain any cofinal countable subset and if no condition about on R and the modules A_i are made, then in our knowledge no statement ensuring the vanishing of $\varprojlim^{(1)}_{i\in I} A_i$ exists in literature. Nevertheless, in this general context, there are interesting cases of inverse systems behaving very similarly to Mittag-Leffler ones just because much part of the restriction maps $\rho^A_{i,j}$ are isomorphisms and their limit is then *"controlled" by some countable subset of maps.* Situations of this type show up for instance in sheaf theory as pull-back of some sheaf on a Stein space, which actually inspired our approach to this problem. In Section 6 we give an interesting example provided by the theory of ultrametric locally convex topological vector spaces. We prove an ulrametric analogous of a result of V. P. Palamodov [11], in relation with the acyclicity of Fréchet spaces with respect to the completion functor. In that case, a direct set-theoretical attempt as in Bourbaki is unhelpful as one can easily see.

We provide here two generalizations of Theorem 1 to the case of an uncountable I without countable cofinal subsets that only involve a finiteness condition on the transition maps of the system $(A_i)_{i\in I}$ and no conditions on I nor on the objects.

Theorem 2 (cf. Corollary 3.8)**.** *Let R be a ring with unit element and let $(I,\leq)$ be a directed partially ordered set. Let $(\rho^A_{i,j} : A_i \to A_j)_{i,j\in I}$ be an inverse systems of left (or right) R-modules indexed on I.*

Assume that there exists another directed partially ordered set $(J,\leq)$ together with an inverse system of R-modules $(\rho^S_{i,j} : S_i \to S_j)_{i,j\in J}$ such that

(i) *There exists a cofinal directed subset $I' \subseteq I$, a cofinal directed subset $J' \subseteq J$ and a surjective map preserving the order relation*

$$p \;:\; I' \longrightarrow J'\;; \tag{1.3}$$

(ii) *There exists a system of R-linear isomorphisms $(\psi_i : A_i \xrightarrow{\sim} S_{p(i)})_{i\in I'}$ such that for all $i,j \in I'$ with $i \geq j$ one has a commutative diagram*

$$\begin{array}{ccc} A_i & \xrightarrow[\sim]{\psi_i} & S_{p(i)} \\ {\scriptstyle \rho^A_{i,j}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho^S_{p(i),p(j)}} \\ A_j & \xrightarrow[\sim]{\psi_j} & S_{p(j)} \end{array} \tag{1.4}$$

Then, for all integer $n \geq 0$, we have a canonical isomorphism

$$\varprojlim_{i\in I}{}^{(n)} A_i \xrightarrow{\sim} \varprojlim_{j\in J}{}^{(n)} S_j. \tag{1.5}$$

In particular, if the partially ordered set J and the system $(S_j)_{j\in J}$ satisfy the conditions (i) and (ii) of Theorem 1 respectively, then $\varprojlim^{(n)}_{i\in I} A_i = 0$ for all $n \geq 1$.

Theorem 3 (cf. Corollary 4.7)**.** *Let R be a ring with unit element and let $(I, \leq)$ be a partially ordered set. Let $(\rho^A_{i,j} : A_i \to A_j)_{i,j\in I}$ be an inverse systems of left (or right) R-modules indexed on I.*

Assume that there exists a directed *partially ordered set $(J, \leq)$ together with an inverse system of R-modules $(\rho^T_{i,j} : T_i \to T_j)_{i,j\in J}$ such that*

(i) *There exists a cofinal directed subset $I' \subseteq I$, a cofinal directed subset $J' \subseteq J$ and a map preserving the order relation*

$$q \;:\; J' \longrightarrow I' \tag{1.6}$$

such that for all $i \in I'$, the set $U_i := \{j \in J', q(j) \leq i\}$, endowed with the partial order induced by J', satisfies at least one of the following conditions

(a) *U_i is empty;*

(b) *U_i has a unique maximal element $r(i)$;*

(c) *U_i is directed, it has countable cofinal directed poset J'_i and the system $(\rho^T_{j,k} : T_j \to T_k)_{j,k\in J'_i}$ satisfies* (1.1)*.*

(ii) *For all $i \in I'$ there exists an R-linear isomorphisms*[7] *$\phi_i : A_i \xrightarrow{\sim} \varprojlim_{j\in U_i} T_j$ and for all $k, i \in I'$ with $k \geq i$ one has a commutative diagram*

$$\begin{array}{ccc} A_k & \xrightarrow[\sim]{\phi_k} & \varprojlim_{j\in U_k} T_j \\ {\scriptstyle \rho^A_{k,i}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho^{q_*T}_{k,i}} \\ A_i & \xrightarrow[\sim]{\phi_i} & \varprojlim_{j\in U_i} T_j \end{array} \tag{1.7}$$

where the right hand vertical arrow $\rho^{q_*T}_{k,i}$ is deduced by the universal properties of the limits.

Then, for all integer $n \geq 0$, we have a canonical isomorphism

$$\varprojlim_{i\in I}{}^{(n)} A_i \xrightarrow{\sim} \varprojlim_{j\in J}{}^{(n)} T_j. \tag{1.8}$$

[7] Notice that under condition (a) we have $\varprojlim_{j\in U_i} T_j = 0$, and under condition (b) we have $\varprojlim_{j\in U_i} T_j = T_{r(i)}$.

In particular, if the partially ordered set J and the system $(T_j)_{j\in J}$ satisfy the conditions (i) and (ii) of Theorem 1 respectively, then $\varprojlim^{(n)}_{i\in I} A_i = 0$ *for all* $n \geq 1$.

Remark that if $J' = \mathbb{N}$ the assumptions of Theorem 3 are particularly easy.

It is not hard to see that the assumptions of Theorem 1 imply those of Theorems 2 and 3. Therefore, they are both generalizations of Theorem 1. Indeed, if $I' \subseteq I$ is a countable cofinal directed subset in Theorem 1, then the setting ($I' = J$, $p = q = \mathrm{id}$, $S_i = A_i = T_i$, $\rho^S_{i,j} = \rho^T_{i,j} = \rho^A_{i,j}$, $\psi_i = \phi_i = \mathrm{id}$) satisfies the assumptions of Theorems 2 and 3. Besides, it is clear that Theorems 2 and 3 allow the set $(I, \leq)$ to be arbitrarily large, while Theorem 1 artificially forces it to be relatively small.

The proofs of these results rely on the fact that inverse systems indexed on $(I, \leq)$ can be seen as sheaves on a topological space $X(I)$ canonically associated to $(I, \leq)$. In this correspondence, inverse limits and their cohomology functors $\varprojlim^{(n)}_{i\in I}(-)$ coincide with sheaf cohomology groups $H^n(X(I), -)$. This coincidence of theories permits to apply all sheaf theoretic cohomological operations, such as, for instance, pull-back and push-forward. Indeed, as the reader may recognize, condition (ii) of Theorem 2 expresses the idea that the system $(A_i)_{i\in I'}$, interpreted as a sheaf on $X(I')$, is isomorphic to the *pull-back* of the system $(S_j)_{j\in J'}$ by the map $p : X(I') \to X(J')$. While in Theorem 3, the system $(A_i)_{i\in I'}$ is isomorphic to the *push-forward* of $(T_j)_{j\in J}$ by the map $q : X(J') \to X(I')$. Actually, Theorem 3 is a special case of a more general statement that holds for possibly non directed partially ordered sets and which does not assume specific conditions on U_j (cf. Proposition 4.1). The fact that we move the set of indexes I along pull-back and push-forward is in contrast with Theorem 1, where one fixes the set of indexes once for all and there is no cohomological distinction between cohomology over $\mathbb{N}$ and over I. We show indeed that there is no danger in moving I because, in this particular context, the pull-back and the push-forward operations behave much better than in a general topological space. Namely, they preserve cohomology under quite mild assumptions. Informally speaking, even though $X(I)$ is allowed to have an enormous amount of open subsets, from a cohomological point of view it behaves as a relatively tiny space.

Finally, we observe that a set-theoretical attempt to the proof of Theorems 2 and 3 in similarity to the quoted claims of Bourbaki is not enough powerful to imply these results. It is necessary to use *Čech cohomolgy* of sheaf theory.

Although certainly possible, an extension of these results to the context of inverse limit of *non abelian* groups fits in the context of non abelian cohomology of sheaves and goes beyond the scopes of this paper. Indeed, since this is a result that is used by a wide range of mathematicians, we made the choice to maintain this paper as self contained and basic as possible.

2. Notations

Everywhere in the paper "*countable*" set means *at most countable* (i.e. finite or in bijection with the set of natural numbers $\mathbb{N}$). We fix once for all a ring R with unit element and denote by R-Mod the category of left R-modules. We denote by $\mathcal{S}$ the category of sets.

Let $\leq$ be an partial order relation on a set I. For brevity, we use the terminology *poset* for partially ordered set and we may indicate $(I, \leq)$ by I. For all $i \in I$ we set

$$\begin{aligned} \Lambda(i) &:= \{j \in I, j \leq i\} \,, \\ V(i) &:= \{j \in I, j \geq i\} \,, \end{aligned} \tag{2.1}$$

and $D(i) = I - V(i) = \{j \in I, j \notin V(i)\}$. We say that the poset I is *directed* if for all $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$. If I is directed, we say that a subset $I' \subset I$ is *cofinal* if for all $i \in I$ we can find $i' \in I'$ such that $i' \geq i$, in particular I' is a directed poset too. For the moment, we do not assume I directed, this condition will be specified when necessary.

Following a classical construction, we now define a topological space $X(I)$ associated to a poset $(I, \leq)$ (cf. for instance [8, p. 4]). The points of $X(I)$ are the elements of I and open subsets are the subsets $U \subseteq I$ with the property that for all $i \in U$ one has $\Lambda(i) \subseteq U$ (cf. (2.1)). In this topology arbitrary intersections of open subsets are open and therefore every subset S of $X(I)$ admits a minimum open subset $O(S) = \bigcup_{i \in S} \Lambda(i)$ containing it. In particular, $\Lambda(i)$ is the smallest open subset containing i. On the other hand, the closure of a subset $S \subset X(I)$ is given by $\overline{S} = \bigcup_{j \in S} V(j)$. If $(J, \leq)$ is another poset, then a map $f : X(I) \to X(J)$ is continuous if, and only if, *f preserves the order relations*, that is, if $i \leq j$, then $f(i) \leq f(j)$. We also say that f is order preserving. The space $X(I)$ acquires special properties when I is a *directed* poset and we will need the following Lemma

Lemma 2.1. *Let $S \subseteq I$ be a subset. Then, S is a* directed *poset with respect to the order relation induced by I if, and only if, so is $O(S)$.*

Inverse systems and inverse limits. We quote [8] and [1, Chapter III, §7] for the notations and basic facts about inverse systems of sets or R-modules. We quickly recall some notations that will be constantly used in the sequel of the paper. An *inverse system of sets indexed on a poset* $(I, \leq)$ is a covariant functor $S : (I, \leq) \to \mathcal{S}$, where $(I, \leq)$ is interpreted as a category in the usual way. More concretely, S is a collection $(S_i)_{i \in I}$ of sets indexed by I, together with a family of maps $(\rho^S_{i,j} : S_i \to S_j)_{(i,j) \in I^2, i \geq j}$ such that for all $i \in I$ we have $\rho^S_{i,i} = Id_{S_i}$, and for all $i, j, k \in I$ such that

$i \geq j \geq k$ one has $\rho^S_{j,k} \circ \rho^S_{i,j} = \rho^S_{i,k}$. We will use the notation $S = (S_i)_{i\in I}$ or $S = (\rho^S_{i,j} : S_i \to S_j)_{(i,j)\in I^2, i\geq j}$ to indicate an inverse system. A morphism $g : (S_i)_{i\in I} \to (T_i)_{i\in I}$ of inverse systems is a collection of maps $(g_i : S_i \to T_i)_{i\in I}$ such that for all $i, j \in I$ with $i \geq j$ one has $\rho^T_{i,j} \circ g_i = g_j \circ \rho^S_{i,j}$. The *inverse limit* $\widehat{S} = \varprojlim_{i\in I}(\rho^S_{i,j} : S_i \to S_j)$ of an inverse system $S = (S_i)_{i\in I}$ is the set of sequences $(x_i)_{i\in I} \in \prod_{i\in I} S_i$ satisfying for all $i \geq j$ the compatibility condition $\rho^S_{i,j}(x_i) = x_j$. The projections $(\rho^{\widehat{S}}_i : \widehat{S} \to S_i)_{i\in I}$ satisfy for all $i \geq j$ the compatibility relation $\rho^S_{i,j} \circ \rho^{\widehat{S}}_i = \rho^{\widehat{S}}_j$.

If every S_i is an R-module and every $\rho^S_{i,j}$ is an R-module homomorphism, we say that $(S_i)_{i\in I}$ is a inverse system of R-modules. Morphisms between inverse systems of R-modules are morphisms $(g_i)_{i\in I}$ as above where, for all $i \in I$, g_i is an R-module homomorphism. In this case the limit is naturally an R-module. The category $R\text{-Mod}^I$ of inverse systems of R-modules inherits almost all the properties of R-Mod. In particular, it is abelian and it has enough injective elements. The notion of exactness in $R\text{-Mod}^I$ has a particular interest for us. A *short exact sequence* in $(R\text{-Mod})^I$ is a collection of short exact sequences of R-modules $(0 \to A_i \xrightarrow{g_i} B_i \xrightarrow{h_i} C_i \to 0)_{i\in I}$ indexed on I, such that for every $i, j \in I$ with $i \geq j$ one has the compatibility relation $g_j \circ \rho^A_{i,j} = \rho^B_{i,j} \circ g_i$ and $h_j \circ \rho^B_{i,j} = \rho^C_{i,j} \circ h_i$. It is well known [2, II.89, §6, N.1, Proposition 1] that this gives rise to a left exact sequence of limits $0 \to \varprojlim_{i\in I} A_i \xrightarrow{g} \varprojlim_{i\in I} B_i \xrightarrow{h} \varprojlim_{i\in I} C_i$. In other words, the inverse limit functor $\varprojlim_{i\in I} : R\text{-Mod}^I \to R\text{-Mod}$ is left exact. In particular, we can consider its derived functors $\varprojlim^{(n)}_{i\in I}$, $n \geq 0$.

Sheaves and cohomology. We quote [6, 7] for the notions about sheaves of R-modules, or of sets, on a topological space X. Let us just fix some quick notations. Let τ_X be the family of all open subsets of X.[8] A *pre-sheaf of sets* F on X is a contravariant functor from τ_X to the category of sets and morphisms of pre-sheaves are just morphisms of functors. If $U \subset V$ are two opens, we denote by $\rho^F_{V,U} : F(V) \to F(U)$ the restriction map. The elements of $F(U)$ are called *sections* of F over U and are often indicated by $\Gamma(U, F) := F(U)$. We say that a pre-sheaf of sets F is a *sheaf* if for every family of open subsets $(V_i)_{i\in I}$, $V_i \in \tau_X$ which is *closed by finite intersection* we have $F(V) = \varprojlim_{i\in I} V_i$, where $V = \bigcup_{i\in I} V_i$.

We denote the category of sheaves on X by $Sh(X)$. For all $x \in X$, we denote the *stalk* of F at x by $F_x := \varinjlim_{x\in U} F(U)$. If every $F(U)$ is an R-module and every restriction map $\rho^F_{U,V}$ is an R-linear homomorphism,

[8] τ_X is seen as a category whose objects are the opens and the morphisms are the inculsions.

we obtain a *sheaf in R-modules.* We denote the category of sheaves of R-modules on X by R-Mod(X). It is an abelian category with enough injective objects. A sequence of sheaves of R-modules $F \to G \to H$ is *exact* if for every $x \in X$ so is the sequence of stalks $F_x \to G_x \to H_x$. Typically, this does not implies the exactness of $F(U) \to S(U) \to H(U)$ for all open U. The functor $\Gamma(X, -) : R\text{-Mod}(X) \to R\text{-Mod}$ is left exact and its right satellites functors are called the sheaf *cohomology groups* of F, denoted by $H^n(X, F) = R^n\Gamma(X, F)$ (cf. [6, §4] for the definition). Here is a recipe to compute them. When $H^n(X, A) = 0$ for all $n \geq 1$, we say that A is an *acyclic sheaf* of R-module (cf. [7, Definition 7.4]). Then, if $A^\bullet : 0 \to F \to A^0 \to A^1 \to A^2 \to \cdots$ is an acyclic resolution (i.e. a long exact sequence of sheaves where every sheaf A^k is acyclic), then $H^n(X, F)$ can be computed as the cohomology groups of the complex of R-modules $\Gamma(X, A^\bullet) : 0 \to \Gamma(X, A^0) \to \Gamma(X, A^1) \to \cdots$. That is, if we set $A^{-1} = 0$, then for every $n \geq 0$ the composite map $\Gamma(X, A^{n-1}) \to \Gamma(X, A^n) \to \Gamma(X, A^{n+1})$ is zero, and if we call $B^n := B^n(\Gamma(X, A^\bullet)) \subseteq \Gamma(X, A^n)$ the image of the first map and $Z^n := Z^n(\Gamma(X, A^\bullet)) \subseteq \Gamma(X, A^n)$ the kernel of the second map, then $B^n \subset Z^n$ and we have

$$H^n(X, F) \;=\; Z^n/B^n\;. \tag{2.2}$$

A standard and compact notation to indicate this process consist in writing

$$H^n(X, F) \;=\; R^n\Gamma(X, A^\bullet)\;. \tag{2.3}$$

Inverse systems indexed by $(I, \leq)$ and sheaves on $X(I)$. Let $(I, \leq)$ be a poset. In this paragraph we recall the strong link between the notions of inverse systems indexed on I and sheaves on $X(I)$. Let $S := (\rho^S_{i,j} : S_i \to S_j)_{i,j \in I}$ be a inverse system of sets indexed on I. We can define a pre-sheaf S on $X(I)$ by associating to every open subset U of $X(I)$ the set $\Gamma(U, S) := \varprojlim_{i \in U} S_i$, where U has the order relation induced by I. For every inclusion of open subsets $V \subset U$ there is an obvious restriction map $\rho^S_{U,V} : \Gamma(U, S) \to \Gamma(V, S)$ provided by the universal property of the inverse limit. It is not hard to show that S is automatically a *sheaf* of sets on $X(I)$ and that every sheaf on $X(I)$ is of this type (cf. [8, p. 4]). The stalk of a sheaf S at a point $i \in X(I)$ is $S(\Lambda(i))$ and it coincides with the value S_i at i of the associated inverse system. In the sequel we do not distinguish sheaves on $X(I)$ from inverse systems and we will indicate them by the same symbol S, so that we write $S = (S_i)_{i \in I}$, $S(\Lambda(i)) = S_i$, or $S(U) = \Gamma(U, S)$. In this correspondence, the inverse limit of an inverse system $S = (S_i)_i$ corresponds to the global sections of the associated sheaf:

$$\Gamma(X(I), S) \;=\; \varprojlim_{i \in I} S_i\;. \tag{2.4}$$

Moreover, if $(A_i)_i$ is an inverse system of R-modules, then the derived functors $\varprojlim^{(n)}_{i\in I} A_i$ are defined as the sheaf cohomology groups $H^n(X(I),A)$

$$\varprojlim_{i\in I}{}^{(n)} A_i \;:=\; H^n(X(I),A)\,. \tag{2.5}$$

2.1. Pull-back and push-forward operations. Let $(I,\leq)$ and $(J,\leq)$ be two poset. Let $f:I\to J$ be a map preserving the order. Since $f:X(I)\to X(J)$ is continuous, we may consider usual pull-back $f^{-1}:\mathrm{Sh}(X(J))\to \mathrm{Sh}(X(I))$ and push-forward $f_*:\mathrm{Sh}(X(I))\to\mathrm{Sh}(X(J))$ operations. We refer to [6, 7] for their definitions. We bound ourself to describe them for inverse systems.

Push-forward. Let $S:=(\rho^S_{i,j}:S_i\to S_j)_{i,j\in I}$ be an inverse system of sets indexed by I and let $k\in J$. By definition, for all open subset $U\subseteq X(J)$ the push-forward of S is given by $f_*S(U)=S(f^{-1}(U))$ with evident transition maps $\rho^{f_*S}_{U,V}=\rho^S_{f^{-1}(U),f^{-1}(V)}$ deduced by those of S. In particular the stalk at a point $k\in J$ is given by $(f_*S)_k=f_*S(\Lambda(k))=\varprojlim_{j\in f^{-1}(\Lambda(k))}S_j$ with evident transition maps $\rho^{f_*S}_{k,t}$, $k\geq t\in J$, obtained by universal property of the limits.

Pull-back. Let now $T=(\rho^T_{i,j}:T_i\to T_j)_{i,j\in J}$ be an inverse system of sets indexed by J. In usual sheaf theory f^{-1} is the sheaf associated to the pre-sheaf associating to every open $U\subseteq X(I)$ the set $\varinjlim_{f(U)\subset V}T(V)$. However, in our setting, arbitrary intersections of opens are opens, therefore $\varinjlim_{f(U)\subset V}T(V)=T\big(O(f(U))\big)$, where $O(f(U))=\bigcup_{i\in U}\Lambda(f(i))$. It is indeed easier to define $f^{-1}T$ as an inverse system indexed by I. Namely, for every $i\in I$, we have $(f^{-1}T)_i:=T_{f(i)}$ and for all $i,j\in I$, $i\geq j$, we have $\rho^{f^{-1}T}_{i,j}:=\rho^T_{f(i),f(j)}$. If I is a subset of J with the order relation induced by J and if $f:I\to J$ is the inclusion, we use the notation $T_{|I}:=f^{-1}T$.[9]

Lemma 2.2. *Let $f:I\to J$ be a map of directed posets that preserves the order relations. Assume that the image $f(I)$ is a cofinal subset of J. Then*

$$\Gamma(X(J),-)\;\cong\;\Gamma(X(I),-)\circ f^{-1}.$$

In other words, for all inverse system $T:=(T_j)_{j\in J}$ the natural map $\varprojlim_{j\in J}T_j\to\varprojlim_{i\in I}(f^{-1}T)_i$ is bijective.

[9]Notice that, when using this notation, the partial order relation of I has to be *induced by that of* J. The reason is that the injectivity of f is not enough to ensure good relations between $\Gamma(X(I),f^{-1}F)$ and $\Gamma(X(J),F)$. For example, assume that we have the set-theoretic equality $I=\{i_1,i_2\}=J$ but i_1 and i_2 are not comparable in I while $i_1\leq i_2$ in J. Then the identity $\iota:I\to J$ preserves the order relation and it hence continuous, in this case we do not want to write $F_{|I}=\iota^{-1}F$.

Usual properties of f^{-1} and f_*. By the above descriptions, it is not hard to see that the functor $f^{-1} : R\text{-Mod}(X(J)) \to R\text{-Mod}(X(I))$ is *exact* and $f_* : R\text{-Mod}(X(I)) \to R\text{-Mod}(X(J))$ is *left exact*. On the other hand, it is well known that f^{-1} is *left adjoint* to f_*, i.e. for all pair of sheaves $S \in R\text{-Mod}(X(I))$ and $T \in R\text{-Mod}(X(J))$ there is a canonical functorial isomorphism $\text{Hom}_{R\text{-Mod}(X(I))}(f^{-1}T, S) \xrightarrow{\sim} \text{Hom}_{R\text{-Mod}(X(J))}(T, f_*S)$. Moreover, we have canonical unit and counit morphisms $T \to f_*f^{-1}T$ and $f^{-1}f_*S \to S$ respectively. In general, if (F, G) is a pair of adjoint functors such that F is exact and left adjoint to G, then G sends injective into injective. In particular, this is the case of f_* which preserves injective objects. It is not hard to see that f_* also preserve flabbiness (cf. Section 3).

A typical application of this fact is the following interpretation of the cohomology groups $H^n(X(I), -)$. Let us denote by $\bullet$ the poset with an individual element. The category of sheafs in sets (resp. R-modules) over $X(\bullet)$ is identified with the category of sets (resp. R-modules) itself by the global functor $\Gamma(X(\bullet), -) : R\text{-Mod}(X(\bullet)) \xrightarrow{\sim} R\text{-Mod}$. The poset $\bullet$ is the final object of the category of posets and we denote by $\pi_I : X(I) \to X(\bullet)$ the projection. Then one has an equality of functors $\Gamma(X(I), -) = \Gamma(X(\bullet), -) \circ (\pi_I)_*$. By the above identification, usually we drop the notation $\Gamma(X(\bullet), -)$ and we simply write

$$\Gamma(X(I), -) \ = \ \pi_{I,*} \ . \tag{2.6}$$

If F is a sheaf in R-modules over $X(I)$ we can translate (2.3) into the notation

$$H^n(X(I), F) \ = \ R^n\pi_{I,*}(F) \ , \tag{2.7}$$

where $R\pi_{I,*}$ denotes the derived functor of $\pi_{I,*}$.

3. Some acyclicity results

Unfortunately, in general f^{-1} does not preserve injectives nor acyclic objects, for this reason it does not behave well for the computation of the cohomology of sheaves. Similarly, f_* is not exact and this makes difficult its use in the computation of the cohomology because some spectral sequences are needed. However, we provide in the next sections some interesting situations where f^{-1} *and* f_* *preserve the cohomology groups.*

Let $(I, \leq)$ be a poset. In this section we introduce several types of acyclic sheaves that can be used to compute the derived functor of the inverse limit by means of (2.3), (2.5) and (2.7).

Flabby and skyscraper sheaves. A sheaf F of R-modules on $X(I)$ is *flabby* if for every open subset $U \subseteq X(I)$ the restriction $F(X(I)) \to F(U)$ is set theoretically surjective. Flabby sheaves are acyclic (cf. [6, Théorème 4.7.1]). It follows from the definition that if $f : X(I) \to X(J)$

is any continuous map, and if F is a flabby sheaf on $X(I)$, then its push-forward f_*F is flabby. This is a simple way to construct acyclic sheaves. In particular, assume that $I = \bullet$ is a point and consider the map $\sigma_j : X(\bullet) \to X(J)$ whose image is a point $j \in J$, then for all R-module $A \in R\text{-Mod} = R\text{-Mod}(X(\bullet))$, the push-forward $\sigma_{j,*}(A)$ is flabby. The sheaf $\sigma_{j,*}(A)$ is called the *skyscraper* sheaf at j with value A. It is easily seen that for $k \in J$ we have $\sigma_{j,*}(A)_k = A$, if $k \in V(j)$, and $\sigma_{j,*}(A)_k = 0$ otherwise, and the transition maps $\rho_{k,t}^{\sigma_{j,*}(A)}$ are either the identity maps if $k \geq t \in V(j)$, or they equals 0 otherwise. Skyscraper sheaves are acyclic because $\sigma_{j,*}$ preserves flabbiness.

Godement resolution. We now use skyscraper sheaves to define an acyclic resolution of every sheaf of F of R-modules over $X(J)$ called the *Godement resolution of* F (cf. [6, Section 4.2]). For all open U of $X(J)$ we set $\Gamma(U, \mathrm{Gode}(F)) = \prod_{j\in U} F_j$, endowed with the natural projections as transition maps. It is a sheaf indicated by $\mathrm{Gode}(F)$ and it is given by $\mathrm{Gode}(F) := \prod_{j\in J} \sigma_{j,*}\sigma_j^{-1}F$. The sheaf $\mathrm{Gode}(F)$ is flabby because skyscraper sheaves are flabby, and a product of flabby sheaves is flabby. By adjunction, for all $j \in J$, we have a canonical morphism $F \to \sigma_{j,*}\sigma_j^{-1}F$. Therefore, we have a morphism $\sigma^F : F \to \mathrm{Gode}(F)$, which is easily seen to be a mono-morphism (cf. [8, Proposition 1.1]). Now, we may consider the quotient $\mathrm{Gode}(F)/F$ and include it into its $\mathrm{Gode}(\mathrm{Gode}(F)/\sigma^F(F))$ and repeating inductively this operation we obtain a flabby resolution $0 \to F \to G^0 \to G^1 \to \cdots$ of F which is called the *Godement resolution* of F.

3.1. Directed posets and weak flabbiness. Flabbiness is not really a common property because, for instance, if we have two disjoint open subsets U and V of $X(I)$, then $F(U \cup V) = F(U) \times F(V)$ and the surjectivity of $F(X(I)) \to F(U) \times F(V)$ tells us that *any arbitrary pair of sections over U and V have to glue to a global section over $X(I)$*. In particular, a constant sheaf is possibly not flabby (cf. Remark 3.4). This problem related to connectedness is avoided with the introduction of a weaker notion, due to C. U. Jensen, called weak flabbiness in the context of *directed* posets which is satisfied by a larger class of sheaves over $X(I)$ and is more suitable for our purposes.

Definition 3.1. Let $(I, \leq)$ be a poset. We say that a sheaf of R-modules F is *weakly flabby* if for every open and *directed* subset $J \subseteq I$ the restriction $F(X(I)) \to F(X(J))$ is surjective.

This definition is important when I is a *directed* poset because of the following Theorem

Theorem 3.2 ([8, Théorème 1.8, p. 9])**.** *Assume that $(I, \leq)$ is a directed poset. Then any weakly flabby sheaf on $X(I)$ is acyclic.*

Remark 3.3. Let I be a directed poset. It was proved by C. U. Jensen that, if F is regarded as an inverse system, then F is weakly flabby on I if, and only if, for any subset $J \subseteq I$ which is directed with respect to the partial order induced by I, the restriction $F(X(I)) = \varprojlim_{i\in I} F_i \to \varprojlim_{j\in J} F_j$ is surjective (cf. [8, Lemme 1.3, p. 6]). That is, the open condition in Definition 3.1 can be relaxed if needed.

Remark 3.4 (Acyclicity of constant sheaves over directed posets.)**.** Recall that a constant sheaf on $X(I)$ with value $C \in R$-Mod is by definition the sheaf $\pi_I^{-1}(C)$, where $\pi_I : X(I) \to X(\bullet)$ is the constant function considered in (2.7). For general $X(I)$, constant sheaves are not flabby nor acyclic and their cohmology groups contain important information about the topological space $X(I)$. However, if I is a *directed* poset, it is easy to check that any constant sheaf over $X(I)$ is weakly flabby[10], hence acyclic by Theorem 3.2.

3.2. Inverse image and weakly flabbiness. For general topological spaces, the inverse image functor does not preserve flabbiness. However, in our context, *weak* flabbiness is preserved when we have directed posets.

Proposition 3.5. *Let $f : I \to J$ be a map of directed posets that preserves the order relations. If W is a weakly flabby sheaf on J, then $f^{-1}W$ is weakly flabby.*

Proof. Let $I' \subseteq I$ be a directed subset of I. We consider $f(I') \subseteq f(I)$ as subsets of J with the order relation induced by J. They are both directed poset. They are possibly not open in J. However, with an abuse, let us set $W(X(f(I))) = \varprojlim_{j\in f(I)} W_j$ and similarly for $W(X(f(I')))$. Since W is weakly flabby, both restrictions $W(X(J)) \to W(X(f(I)))$ and $W(X(J)) \to W(X(f(I')))$ are surjective by Remark 3.3. Hence, so is the restriction map $W(X(f(I))) \to W(X(f(I')))$ by composition. Now, by Lemma 2.2 the restriction $f^{-1}W(X(I)) \to f^{-1}W(X(I'))$ equals the restriction $W(X(f(I))) \to W(X(f(I')))$. The claim follows. □

In the proof of Proposition 3.5, a key ingredient is Lemma 2.2 in which the fact that the posets are directed is a crucial assumption. The following proposition is a similar statement for J possibly not directed posets.

Proposition 3.6. *Let $f : I \to J$ be a map of posets that preserves the order relations. Assume moreover that I is directed. Then the following hold:*

(i) *Let A be a skyscraper sheaf on $X(J)$, then the inverse image $f^{-1}A$ of A is weakly flabby.*
(ii) *Let F be a sheaf of R-modules over J and let* Gode(F) *be the Godement sheaf associated with F. Then, $f^{-1}(\mathrm{Gode}(F))$ is wealkly flabby.*

[10] Because the value of a constant sheaf with value A on any open which is directed is equal to the group A, and the restriction maps are the identities.

(iii) *The inverse image of the Godement resolution of F is a weakly flabby resolution of $f^{-1}(F)$.*

Proof. Let $j \in X(J)$ and $A \in R$-Mod. Let us denote by $(A_k)_{k\in J} := \sigma_{j,*}A$ the skyscraper sheaf at $j \in X(J)$ with value A (cf. Section 3). Since A is flabby on $\{j\}$, so is $\sigma_{j,*}A$ on $X(J)$. We want to show that $F := f^{-1}(\sigma_{j,*}A)$ is weakly flabby over $X(I)$. Let $U \subset X(I)$ be any open subset which is directed as a poset with the order relation induced by $X(I)$. Then, we need to show that the restriction map $F(X(I)) \to F(U)$ is surjective. Now, by the definition of f^{-1}, this restriction map identifies to the natural restriction map $\rho^{\sigma_{j,*}A}_{O(f(X(I))),O(f(U))} : \sigma_{j,*}A(O(f(X(I)))) \to \sigma_{j,*}A(O(f(U)))$, which is surjective because $\sigma_{j,*}A$ is flabby and I is directed (cf. Remark 3.4).

Let us now prove (ii). By definition $\mathrm{Gode}(F) = \prod_{j\in J}\sigma_{j,*}\sigma_j^{-1}F$. Since f^{-1} commutes with products and since products of weakly flabby is weakly flabby, it is enough to prove that if $S = \sigma_{j,*}A$ is a skyscraper sheaf on $X(J)$, then $f^{-1}S$ is weakly flabby. The claim then follows from (i). The third statement is also an immediate consequence of the exactness of f^{-1} and of (ii). □

Theorem 3.7. *Let $f : I \to J$ be a map of directed posets preserving the order relations and such that $f(I)$ is a cofinal subset of J. Then for all sheaves of R-modules F on $X(J)$ one has*

$$H^n(X(I), f^{-1}F) = H^n(X(J), F)\,. \tag{3.1}$$

In particular, F is acyclic if, and only if, so is $f^{-1}F$.

Proof. The proof is straightforward. We use (2.2) to compute the cohomology of $f^{-1}F$. Let $0 \to F \to G^0 \to G^1 \to \cdots$ be the Godement resolution of F. Its pull-back $0 \to f^{-1}F \to f^{-1}G^0 \to f^{-1}G^1 \to \cdots$ is a resolution of $f^{-1}F$ because f^{-1} is an exact functor. Every term of the sequence is weakly flabby by Proposition 3.6, hence acyclic by Theorem 3.2. Therefore, by (2.2), we know that the complex $0 \to \Gamma(X(I), f^{-1}F) \to \Gamma(X(I), f^{-1}G^0) \to \Gamma(X(I), f^{-1}G^1) \to \cdots$ computes the cohomology of $f^{-1}F$. Now, Lemma 2.2 ensures that this complex equals $0 \to \Gamma(X(J), F) \to \Gamma(X(J), G^0) \to \Gamma(X(J), G^1) \to \cdots$ and the claim follows. □

Theorem 3.7 holds sometimes for non directed posets as we will see in Proposition 5.2 in the case of Galois connections between posets. Let us now show that Theorem 3.7 implies Theorem 2 quite directly, which we translate in term of sheaves.

Corollary 3.8 (Theorem 2)**.** *Let I and J be directed posets and $I' \subseteq I$ and $J' \subseteq J$ be cofinal directed posets. Let $p : I' \to J'$ be a surjective map preserving the order relations. Let A and S be sheaves of R-modules over*

$X(I)$ and $X(J)$ respectively. Assume that the restriction $A_{|I'}$ of A to $X(I')$ is isomorphic to the pull-back $p^{-1}(S_{|J'})$ of the restriction $S_{|J'}$ of S to $X(J')$

$$\psi \ : \ A_{|I'} \xrightarrow{\sim} p^{-1}(S_{|J'}) \ . \tag{3.2}$$

Then, for every integer $n \geq 0$ one has

$$H^n(X(I), A) \ \cong \ H^n(X(J), S) \ . \tag{3.3}$$

Proof. By Theorem 3.7 applied to the inclusions $I' \to I$ and $J' \to J$ we have $H^n(X(I), A) = H^n(X(I'), A_{|I'})$ and $H^n(X(J), S) = H^n(X(J'), S_{|J'})$, for all integer $n \geq 0$. Hence, we can assume $I = I'$ and $J = J'$. Again, Theorem 3.7 then ensures $H^n(X(J), S) = H^n(X(I), f^{-1}S)$ and $H^n(X(I), f^{-1}S) \cong H^n(X(I), A)$ because $A \cong f^{-1}S$. □

4. Direct image and exactness

As mentioned, the direct image functor f_* is not exact in general. However, we now provide conditions ensuring that f_* preserves the cohomology. Notice that the posets are possibly not directed.

Proposition 4.1. *Let $f : I \to J$ be an order preserving function between posets. Let F be a sheaf of R-modules over $X(I)$. Assume that for all $j \in J$ the restriction $F_{|f^{-1}(\Lambda(j))}$ is acyclic as a sheaf over the open $U_j := f^{-1}(\Lambda(j))$. That is, for all integer $n \geq 1$ one has*

$$H^n(U_j, F) \ = \ 0 \ . \tag{4.1}$$

Then

(i) *For all injective (resp. flabby) resolution $0 \to F \to I^1 \to I^2 \to \cdots$ of F, the push-forward $0 \to f_*F \to f_*I^1 \to f_*I^2 \to \cdots$ is an injective (resp. flabby) resolution of f_*F (i.e. it remains exact).*

(ii) *For all integer $n \geq 0$ one has*

$$H^n(X(J), f_*F) \ = \ H^n(X(I), F) \ . \tag{4.2}$$

Proof. Let $0 \to F \to I^1 \to I^2 \to \cdots$ be an injective (resp. flabby) resolution of F. Let us set $F^0 := F$ and, for all $k \geq 0$, let F^{k+1} be the cokernel of the inclusion of F^k into I^k. We then have the classical diagram

$$\begin{array}{ccccccccccccc} & & & & F^1 & & & & F^3 & & & & \\ & & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & & & I^1 \longrightarrow I^2 & \longrightarrow & & & I^3 & \longrightarrow & \cdots \\ & \nearrow & & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & \\ F & & & & & & F^2 & & & & F^4 & & \end{array} \tag{4.3}$$

We now apply the functor f_* to this diagram. We know that f_*I^k remains injective (resp. flabby), hence acyclic. Now we claim that $0 \to f_*F \to f_*I^0 \to f_*I^1 \to \cdots$ is a resolution of f_*F, i.e. this sequence is exact. This condition can be checked on the stalks. Hence, we have to prove that for all $j \in J$ and for all $k \geq 0$, the sequence $0 \to (f_*F^k)_j \to (f_*I^k)_j \to (f_*F^{k+1})_j \to 0$ is exact. As there is a minimal open subset $\Lambda(j)$ containing j, then if we set $U_j := f^{-1}(\Lambda(j))$, this sequence coincides with the sequence $0 \to F^k(U_j) \to I^k(U_j) \to F^{k+1}(U_j) \to 0$. In other words we have to show that, for every $k \geq 0$, $\Gamma(U_j, -)$ sends the short exact sequence $0 \to F^k \to I^k \to F^{k+1} \to 0$ into an exact one. Let us consider the long exact sequence of cohomology groups

$$(4.4) \quad 0 \longrightarrow H^0(U_j, F^k) \longrightarrow H^0(U_j, I^k) \longrightarrow H^0(U_j, F^{k+1}) \longrightarrow H^1(U_j, F^k) \longrightarrow H^1(U_j, I^k) \longrightarrow H^1(U_j, F^{k+1}) \cdots$$

Since I^k is acyclic on U_j, we have $H^n(U_j, I^k) = 0$ for all $k \geq 0$ and all $n \geq 1$. Therefore for all $k \geq 0$ and all $n \geq 1$ we have an isomorphism

$$(4.5) \qquad H^n(U_j, F^{k+1}) \xrightarrow{\sim} H^{n+1}(U_j, F^k) .$$

Now, for $k = 0$, our assumption gives $H^n(U_j, F^0) = 0$ for all $n \geq 1$ because $F = F^0$ is acyclic on U_j. The isomorphism (4.5) ensures by induction that F^k is acyclic on U_j for all $k \geq 0$. Therefore the sequence $0 \to f_*F \to f_*I^0 \to f_*I^1 \to \cdots$ is exact and it is an injective (resp. flabby) resolution of f_*F.

It follows then by (2.3) that $H^n(X(J), f_*F) = R^n\Gamma(X(J), f_*I^\bullet)$. Finally, for all $k \geq 0$, the definition of push-forward gives $\Gamma(X(J), f_*I^k) = \Gamma(X(I), I^k)$. Hence the sequence $0 \to \Gamma(X(J), f_*F) \to \Gamma(X(J), f_*I^0) \to \Gamma(X(J), f_*I^1) \to \cdots$ coincides with $0 \to \Gamma(X(I), F) \to \Gamma(X(I), I^0) \to \Gamma(X(I), I^1) \to \cdots$ which computes the cohomology of F by (2.3). The claim follows. $\square$

Remark 4.2. In Proposition 5.2 we will treat a special situation where f_* preserves also *weakly-flabby* resolutions.

An interesting case where Proposition 4.1 applies is the following

Theorem 4.3. *Let $f : I \to J$ be an order preserving function between posets. Let F be a sheaf of R-modules over $X(I)$. Assume that for every $j \in J$ the set $U_j = f^{-1}(\Lambda(j))$ satisfies at least one among the following conditions :*

(i) *U_j is empty;*
(ii) *U_j has a unique maximal element (i.e. it is of the form $\Lambda(i)$, for some $i \in I$);*
(iii) *U_j is a directed poset admitting a countable cofinal directed poset I'_j and the system $(A_k)_{k \in I'_j} := F_{|I'_j}$ satisfies Mittag-Leffler condition* (1.1)*.*

Then, the conclusions (i) and (ii) of Proposition 4.1 hold.

Proof. If (i) or (iii) hold for U_j, we know by Theorem 1 that $F_{|U_j}$ is acyclic and the condition of Proposition 4.1 is fulfilled. If (ii) holds for U_j, then $U_j = \Lambda(i)$ for some $i \in I$. Now, the functor $\Gamma(\Lambda(i), -)$ is the fiber functor associating to a sheaf F its stalk F_i at i. Therefore, it is an exact functor and it preserves injective resolutions. Hence, for every sheaf F of R-modules over $X(I)$ the restriction $F_{|\Lambda(i)}$ is acyclic on $\Lambda(i)$. Proposition 4.1 then applies. □

Remark 4.4. It was proved by O. Laudal [9] that the only posets U over which every sheaf of R-modules is acyclic are those admitting a maximum element (i.e. $U = \Lambda(i)$ for some $i \in U$). Therefore, any generalization of Theorem 4.3 to more general maps f requires restrictions on the class of sheaves F that we consider, as we did in condition (i). For instance, let us assume that for all $j \in J$ the poset $U_j = f^{-1}(\Lambda(j))$ has only finitely many maximal elements. This mens that U_j is a finite union of open posets of the form $\Lambda(i)$. In this situation it might be interesting to use Mayer–Vietoris long exact sequence to obtain combinatoric conditions on F ensuring (4.1).

From another angle, if we assume that I is directed, then it might be interesting to replace it by a cofinal directed subset I'. This operation preserve the cohomology groups of F and reduces the size of the sets $f^{-1}(\Lambda(j))$ (which makes possibly easier to verify (4.1)). However, it should be taken with some precaution because it does not preserve the push-forward (i.e. $f_*F \neq f_*(F_{|I'})$). The claim is the following.

Corollary 4.5. *Let I be a directed poset and F a sheaf of R-modules over $X(I)$. Let $I' \subseteq I$ be a directed cofinal subset of I and let $f : I' \to J$ be an order preserving function between posets such that, for all $j \in J$, the restriction $F_{|f^{-1}(\Lambda(j))}$ is acyclic as a sheaf over the open subset $U'_j := f^{-1}(\Lambda(j)) \subset X(I')$. That is, for all integer $n \geq 1$, one has $H^n(U'_j, F_{|X(I')}) = 0$. In particular, this condition is automatically satisfied if one of the conditions (i), (ii), or (iii) of Theorem 4.3 holds for $F_{|U'_j}$. Then, for all integer $n \geq 0$ one has*

$$H^n(X(J), f_*(F_{|X(I')})) \;=\; H^n(X(I), F)\,. \tag{4.6}$$

Another interesting case where Corollary 4.5 applies is of course given by the poset of natural numbers $\mathbb{N}$, where *every bounded* open subset has a maximum element. We obtain the following corollary. Notice that *no cofinality condition is required for the inclusion of $f(I)$ in J.*

Corollary 4.6 (Case of a totally ordered countable poset)**.** *Let I be a poset and F a sheaf of R-modules over $X(I)$. Assume that I is directed and has a totally ordered cofinal subset N which is at most countable (i.e. N is finite* *or isomorphic to* $(\mathbb{N},\leq)$*).*[11] *Let* $f : N \to J$ *be an order preserving function between posets such that, for all* $j \in J$ *the following condition holds*

(i) *if* $U_j := f^{-1}(\Lambda(j)) = N$*, then the restriction of* $F_{|N}$ *satisfies Mittag-Leffler condition* (1.1)*.*

Then, for all integer $n \geq 0$ *one has*

$$H^n(X(J), f_*(F_{|X(N)})) \;=\; H^n(X(I), F)\ . \tag{4.7}$$

In particular, (i) is an empty condition if for every $j \in J$*, there exists* $\eta \in N$ *such that* $f(\eta) \not\leq j$ *(i.e.* $f^{-1}(\Lambda(j)) \neq N$*, for all* $j \in J$*).*

For the benefit of the reader we now translate Theorem 3 in the sheaf language. The role of I and J is reversed with respect to the statement in the introduction and, even though it is not necessary, we assume the posets to be directed in order to allow the restriction to a cofinal poset.

Corollary 4.7 (Theorem 3)**.** *Let* $(J,\leq)$ *be a directed poset and* A *a sheaf of* R*-modules over* $X(J)$*. Assume that there exist a* directed *partially ordered set* $(I,\leq)$ *and a sheaf of* R*-modules* T *over* $X(I)$ *such that*

(i) *There exists a cofinal directed subset* $J' \subseteq J$*, a cofinal directed subset* $I' \subseteq I$ *and a map* $q : I' \to J'$ *preserving the order relation such that for all* $j \in J'$*, the set* $U_j = \{i \in I', q(i) \leq j\}$ *is either empty, or it has a unique maximal element, or it has a countable cofinal directed poset* I'_j *and* $T_{|X(I'_j)}$ *satisfies Mittag-Leffler condition* (1.1)*.*

(ii) *We have an* R*-linear isomorphism of sheaves* $\phi : A_{|J'} \cong q_*T_{|I'}$*.*

Then, for all integer $n \geq 0$*, we have a canonical isomorphism*

$$H^n(X(J), A) \;\cong\; H^n(X(I), T)\ . \tag{4.8}$$

In particular, if T *is acyclic then so is* A*.*

Proof. By Theorem 3.7 applied to the inclusions $I' \to I$ and $J' \to J$ we have $H^n(X(J), A) = H^n(X(J'), A_{|J'})$ and $H^n(X(I), T) = H^n(X(I'), T_{|J'})$, for all integer $n \geq 0$. Hence, we can assume $I = I'$ and $J = J'$. The claim then follows from Proposition 4.5. □

5. Galois connections.

In this section we consider Galois connections between posets. This is a particularly lucky situation, because the operations of push-foward f_* and the pull-back g^{-1} coincide and we automatically have the benefits of both operations (cf. Proposition 5.2 below). We begin by the following Lemma 5.1 which says that when we have a countable cofinal subset, we automatically have a Galois connection with a convenient countable totally ordered subset.

[11] By Lemma 5.1, this is equivalent to the simple existence of a cofinal subset in I which is at most countable.

Lemma 5.1. *Let J be a directed poset that admits a countable cofinal subset. Then, there exists a countable cofinal subset $N \subset J$ which is* directed *and totally ordered. The set N is finite if, and only if, J has a maximum element (in this case we can chose N equal to the maximum element of J). Otherwise, N is isomorphic to the poset of natural numbers* $(\mathbb{N}, \leq)$. *Moreover, if $f : N \to J$ denotes the inclusion, then there exists a map $g : J \to N$ preserving the order relations and such that*

(i) *The map $g \circ f : N \to N$ is the identity map.*
(ii) *For all $j \in J$, $f^{-1}(\Lambda(j)) = \Lambda(g(j))$, that is $g(j)$ is the biggest element of $f^{-1}(\Lambda(j))$.*

Proof. Let $S \subseteq I$ be a countable cofinal subset and let $S = \{s_1, s_2, \dots\}$ be an enumeration of S. Set $\eta_1 := s_1$ and, for all integer $n \geq 2$, chose inductively an $\eta_n \in J$ such that $\eta_n \geq \eta_{n-1}$ and $\eta_n \geq s_n$. We now have an increasing sequence $(\eta_n)_n$ in J. Let $N \subset J$ be the set of its values. Then N is cofinal in J because S is. Clearly N is finite and totally ordered if, and only if, the sequence is stationary, and in this case its maximum is also a maximum of J. Otherwise, we may find a subsequence $(\eta_{n_k})_{k \in \mathbb{N}}$ of $(\eta_n)_n$ which is strictly increasing whose underling subset is N and the map $k \to \eta_{n_k}$ provides a bijection between $\mathbb{N}$ and N preserving the order relations.

Now, as N is cofinal, we have $J = \bigcup_{\eta \in N} \Lambda(\eta)$. Since N is discrete and totally ordered, for every $j \in J$ there exists a minimum $\eta_j \in N$ such that $j \in \Lambda(\eta_j)$. Therefore, we can define a map $g : J \to N$ as $g(i) = \min(\eta \in N, i \in \Lambda(\eta))$. The claim follows. □

Recall that if

$$f : I \longrightarrow J \qquad \text{and} \qquad g : J \longrightarrow I \tag{5.1}$$

are two maps between posets that preserve the order relations, then the following conditions are equivalent

(1) For all $i \in I$ and all $j \in J$ one has $f(g(j)) \leq j$ and $g(f(i)) \geq i$;
(2) For all $i \in I$ and all $j \in J$ we have $f(i) \leq j$ if, and only if, $i \leq g(j)$.

In this case, the pair (f, g) is called a *Galois connection* between I and J. If I and J are seen as categories, these conditions express the fact that f is a *left adjoint* of g and g is a *right adjoint* of f. It is not hard to prove that a map $f : I \to J$, respecting the partial order relations, admits a right adjoint $g : J \to I$ if, and only if, the following condition holds

(3) For all $j \in J$, there exists $i_j \in I$ such that $f^{-1}(\Lambda(j)) = \Lambda(i_j)$.

In this case, i_j is the value of g at j, so that for all $j \in J$ we have

$$f^{-1}(\Lambda(j)) = \Lambda(g(j)) . \tag{5.2}$$

In particular, when the right adjoint g exists, it is uniquely determined by (5.2). Symmetrically, $g : J \to I$ admits a left adjoint if, and only if, for all $i \in I$ there exists $j_i \in J$ such that $g^{-1}(V(i)) = V(j_i)$ and in this case $f(i) = j_i$.

Proposition 5.2. *Let (f, g) be a Galois connection as above. Then*

(i) *The functors $f_* : Sh(X(I)) \to Sh(X(J))$ and $g^{-1} : Sh(X(I)) \to Sh(X(J))$ coincide. In particular, for every sheaf F of R-modules over $X(I)$ we have*

$$f_*F \ = \ g^{-1}F \ . \tag{5.3}$$

(ii) *The conditions of Theorem 4.3 are fulfilled and for every sheaf F of R-modules over $X(I)$ the conclusions (i) and (ii) of Proposition 4.1 hold.*

(iii) *If I and J are both directed posets, then f_* preserves weakly flabbiness. In particular, it sends weakly flabby resolutions of F into weakly flabby resolutions of f_*F.*

Proof. Let us see F as an inverse system $(\rho^F_{i,j} : F_i \to F_k)_{i,k\in I}$. Then, by definition, for all $j \in J$ both f_*F and $g^{-1}F$ verify $(f_*F)_j = F_{g(j)} = (g^{-1}F)_j$ and, for all $j' \geq j$, one has $\rho^{f_*F}_{j',j} = \rho^F_{g(j'),g(j)} = \rho^{g^{-1}F}_{j',j}$. Items (i) and (ii) follow immediately. In particular, f_* is exact. To prove (iii), it is then enough to show that if W is a weakly flabby sheaf of R-modules over I, then so is f_*W on J. Since $f_*W = g^{-1}W$, this follows from Proposition 3.5. □

Remark 5.3. Lemma 5.1 admits the following generalization which does not involve any *cofinality condition.* Let J be a directed poset and $f : \mathbb{N} \to J$ be an order preserving map satisfying the following condition:

- For all $j \in J$, $f^{-1}(\Lambda(j)) \neq \mathbb{N}$ (i.e. for all $j \in J$ there exists $n \in \mathbb{N}$ such that $f(n) \not\leq j$).

Then, by item (iii) before (5.2), f admits a right adjoint $g : J \to \mathbb{N}$ and Proposition 5.2 applies.

6. An application to p-adic locally convex spaces

In this section we give an application to ultrametric locally convex spaces. It is an ultrametric analogous of a result of V. P. Palamodov [11].

An ultrametric absolute value on a field K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ verifying $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$, and $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$. From now on we assume that the absolute value is non trivial (i.e. there exists $x \neq 0$ such that $|x| \neq 1$) and that K is complete with respect to the topology defined by $|\cdot|$. We denote by $\mathscr{O}_K = \{x \in K, |x| \leq 1\}$ its ring of integers.

An ultrametric seminorm on a K-vector space V is a function $u : V \to \mathbb{R}_{\geq 0}$ such that for all $r \in K$ and $x, y \in V$ one has $u(rx) = |r|u(x)$ and $u(x+y) \leq \max(u(x), u(y))$. A locally convex space over K is a topological vector space V whose topology is defined by a family of ultrametric seminorms. Recall that V has a basis of open neighborhoods of 0 formed by $\mathscr{O}_K$-submodules, we call them *convex opens.*

A K-linear continuous map $f : V \to W$ between locally convex spaces is *strict* if the topology induced by W on the image of f coincides with the quotient topology of V.

Proposition 6.1. *Let $f : V \to W$ be a K-linear strict map between Hausdorff complete locally convex spaces. If the kernel of f is a Fréchet space, then the image of f is a Hausdorff complete closed subspace of W.*

Proof. Let V' be the kernel of f and V'' its image. Since f is strict, it is enough to show that V'' is Hausdorff and complete with respect to the quotient topology induced by V. For this, we prove that the strict short exact sequence $0 \to V' \to V \to V'' \to 0$ remains strict exact after the Hausdorff-completion operation. Indeed, V' and V are already Hausdorff and complete. Let I be the family of convex neighborhoods of 0 in V. The set I is naturally partially ordered by the inclusion of subsets. For all $D \in I$, set $D' := D \cap V'$ and denote by D'' the image of D in V''. The Hausdorff completion of the sequence $0 \to V' \to V \to V'' \to 0$ is then the inverse limit of the sequences $0 \to V'/D' \to V/D \to V''/D'' \to 0$ for D running in I. Let J be the set of open neighborhoods of V' of the form $p(D) = D \cap V'$ with $D \in I$. The map $p : I \to J$ is surjective and the inverse system $(V'/D')_{D \in I}$ is the pull-back of $(V'/D')_{D' \in J}$ by $p : I \to J$. The conditions of Corollary 3.8 are fulfilled. It follows that for all $n \geq 0$ we have $\varprojlim^{(n)}_{D \in I} V'/D' = \varprojlim^{(n)}_{D' \in J} V'/D'$. Now, since V' is Hausdorff and Fréchet, then J has a countable cofinal subset N. The transaction maps being surjective, Theorem 1 applies and $\varprojlim^{(n)}_{D' \in J} V'/D' = 0$ for all $n \geq 1$. The claim follows. □

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Andrea Pulita
Université Grenoble Alpes, Institut Fourier,
CS 40700, 38058 Grenoble cedex 9
France
E-mail: andrea.pulita@univ-grenoble-alpes.fr
URL: https://www-fourier.ujf-grenoble.fr/~pulitaa/