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On the n -torsion subgroup of the Brauer group of a number field

par HERSHY KISILEVSKY et JACK SONN

RÉSUMÉ. Pour toute extension galoisienne K de $\mathbb{Q}$ et tout entier positif n premier au nombre de classes de K , il existe une extension abélienne L de K d'exposant n telle que le n -sous-groupe de torsion du groupe de Brauer de K est égal au groupe de Brauer relatif de L/K .

ABSTRACT. Given a number field K Galois over the rational field $\mathbb{Q}$, and a positive integer n prime to the class number of K , there exists an abelian extension L/K (of exponent n) such that the n -torsion subgroup of the Brauer group of K is equal to the relative Brauer group of L/K .

1. Introduction

Let K be a field, $Br(K)$ its Brauer group. If L/K is a field extension, then the relative Brauer group $Br(L/K)$ is the kernel of the restriction map $res_{L/K} : Br(K) \rightarrow Br(L)$. Relative Brauer groups have been studied by Fein and Schacher (see e.g. [2, 3, 4].) Every subgroup of $Br(K)$ is a relative Brauer group $Br(L/K)$ for some extension L/K [2], and the question arises as to which subgroups of $Br(K)$ are *algebraic relative Brauer groups*, i.e. of the form $Br(L/K)$ with L/K an algebraic extension. For example if L/K is a finite extension of number fields, then $Br(L/K)$ is infinite [3], so no finite subgroup of $Br(K)$ is an algebraic relative Brauer group. In [1] the question was raised as to whether or not the n -torsion subgroup $Br_n(K)$ of the Brauer group $Br(K)$ of a field K is an algebraic relative Brauer group. For example, if K is a (p -adic) local field, then $Br(K) \cong \mathbb{Q}/\mathbb{Z}$, so $Br_n(K)$ is an algebraic relative Brauer group for all n . This is not surprising, since this Brauer group is “small”. A counterexample was given in [1] for $n = 2$ and K a formal power series field over a local field. Somewhat surprisingly, $Br_2(\mathbb{Q})$ turned out to be an algebraic relative Brauer group.

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For number fields K , the problem is a purely arithmetic one, because of the fundamental local-global description of the Brauer group of a number field. In [1], it was proved that $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all squarefree n . In this paper, we prove the following affirmative result for number fields: given any number field K Galois over $\mathbb{Q}$ and any n prime to the class number of K , $Br_n(K)$ is an algebraic relative Brauer group, in fact of an abelian extension of K . In particular, $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all n .

2. Algebraic extensions of K with local degree n everywhere

Theorem 2.1. *Let K be a number field Galois over $\mathbb{Q}$ with class number h_K . Let ℓ be a prime number, relatively prime to h_K and let r be a positive integer. There exists an abelian extension L/K of exponent ℓ^r such that the local degree of L/K at every finite prime equals ℓ^r . If $\ell = 2$, then L can be taken to be totally complex.*

Proof. Let k_∞ be the cyclotomic extension of $\mathbb{Q}$ obtained by adjoining all ℓ -power roots of unity to $\mathbb{Q}$. Let s be a positive integer. For ℓ odd, let k_s be the unique subfield of k_∞ of degree ℓ^s over $\mathbb{Q}$. If $\ell = 2$, there are three elements of order 2 in $\text{Gal}(\mathbb{Q}(\mu_{2^{s+2}})/\mathbb{Q})$, one fixing the maximal real subfield, one fixing $\mathbb{Q}(\mu_{2^{s+1}})$, and a third fixing a cyclic totally complex extension of $\mathbb{Q}$ of degree 2^s , which we define to be k_s . (As usual, μ_m denotes the m th roots of unity.) Then ℓ is the unique prime of $\mathbb{Q}$ ramified in k_s and it is totally ramified.

Choose s such that $L_0 = Kk_s$ has degree ℓ^r over K . Then the primes $\mathfrak{l}_1, \dots, \mathfrak{l}_t$ of K dividing ℓ have isomorphic completions, and since ℓ is prime to h_K , they are all totally ramified in L_0/K . In the case $\ell = 2$, the real primes are also ramified in L_0/K .

Let E be the extension of K obtained by adjoining L_0 and the ℓ^r th roots of all the units of K (including the ℓ^r th roots of unity). Let S be the (infinite) set of primes of K which split completely in E . For $\mathfrak{p} \in S$ consider the ℓ -ray class field $R_{\mathfrak{p}}$ with conductor $\mathfrak{p}$, i.e. the ℓ -primary part of the ray class field with conductor $\mathfrak{p}$. Since the class number h_K of K is prime to ℓ , $\mathfrak{p}$ is totally ramified in $R_{\mathfrak{p}}$. Furthermore, the ℓ -ray class group is isomorphic to the ℓ -part of $\overline{K}_{\mathfrak{p}}^* = (\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)^*$ (the multiplicative group of the residue field) modulo the image of the unit group of K . By choice of $\mathfrak{p} \in S$, the absolute norm $\mathcal{N}(\mathfrak{p})$ is congruent to 1 modulo ℓ^r and all units are ℓ^r th powers in $\overline{K}_{\mathfrak{p}}^*$. Hence $\overline{K}_{\mathfrak{p}}^*$ has a (unique cyclic) quotient of order ℓ^r . We define $L^{\mathfrak{p}}$ to be the corresponding cyclic subextension of degree ℓ^r of $R_{\mathfrak{p}}$.

Let $\mathfrak{l} = \mathfrak{l}_i$ be one of the prime divisors of ℓ in K . Consider the condition $\mathfrak{l}$ splits completely in $L^{\mathfrak{p}}$ ($\mathfrak{p} \in S$). This is equivalent to $\mathfrak{l}$ being an ℓ^r th

power in the ray class group mod $\mathfrak{p}$. But since ℓ is prime to $h = h_K$, this is equivalent to the principal ideal $\mathfrak{l}^h = (a)$, $a \in K^*$, being an ℓ^r th power in the ray class group mod $\mathfrak{p}$. Since all the units of K are ℓ^r th powers modulo $\mathfrak{p}$, this is equivalent to a being an ℓ^r th power in $\overline{K}_{\mathfrak{p}}^*$, which for $\mathfrak{p} \in S$ is equivalent to $\mathfrak{p}$ splitting completely in $K(\mu_{\ell^r}, \sqrt[r]{a})$. Denote the a corresponding to $\mathfrak{l}_i$ by a_i .

Let $S' \subset S$ be the set of primes of K that split completely in $E' = E(\sqrt[r]{a_1}, \dots, \sqrt[r]{a_t})$. The prime divisors $\mathfrak{l}_i$ of ℓ in K split completely in $L^{\mathfrak{p}}$ if $\mathfrak{p} \in S'$.

We now define recursively a subsequence S_0 of primes of S' . We begin with any prime $\mathfrak{p}_1 \in S'$ such that $\mathcal{N}(\mathfrak{p}_1) > \mathcal{N}(\mathfrak{l}_i)$ for $\mathfrak{l}_i$ dividing ℓ . (As above, $\mathcal{N}$ denotes the absolute norm.)

We claim there exists a prime $\mathfrak{p}_2 \in S'$ with $\mathcal{N}(\mathfrak{p}_2) > \mathcal{N}(\mathfrak{p}_1)$ satisfying:

- (a) $\mathfrak{p}_2$ splits completely in $L^{\mathfrak{p}_1}$;
- (b) $\mathfrak{p}_1$ splits completely in $L^{\mathfrak{p}_2}$;
- (c) $\mathfrak{q}$ is inert in $L^{\mathfrak{p}_2}$ for all primes $\mathfrak{q} \neq \mathfrak{p}_1, \mathfrak{l}_i$ of K with absolute norm $\mathcal{N}(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{p}_1)$.

To prove the claim, we reduce it to an application of Chebotarev's density theorem. Arguing as above, (b) is equivalent to $\mathfrak{p}_1$ being an ℓ^r th power in the ray class group mod $\mathfrak{p}_2$. But since ℓ is prime to $h = h_K$, this is equivalent to the principal ideal $\mathfrak{p}_1^h = (c)$, $c \in K^*$, being an ℓ^r th power in the ray class group mod $\mathfrak{p}_2$. Since all the units of K are ℓ^r th powers modulo $\mathfrak{p}_2$, this is equivalent to c being an ℓ^r th power in $\overline{K}_{\mathfrak{p}_2}^*$, which is equivalent to $\mathfrak{p}_2$ splitting completely in $K(\mu_{\ell^r}, \sqrt[r]{c})$. Thus (b) is a Chebotarev condition compatible with (a).

We now consider (c). We want all $\mathfrak{q} \neq \mathfrak{p}_1, \mathfrak{l}_1, \dots, \mathfrak{l}_t$ with $\mathcal{N}(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{p}_1)$ (these are finitely many) to be inert in $L^{\mathfrak{p}_2}$. For this it suffices that $\mathfrak{q}$ be inert in $M^{\mathfrak{p}_2}$, where $M^{\mathfrak{p}_2}$ is the subextension of $L^{\mathfrak{p}_2}$ of degree ℓ over K . As above, if $\mathfrak{q}^h = (b)$, $b \in K^*$, this means that (b) is not an ℓ th power in the ray class group mod $\mathfrak{p}_2$, i.e. b is not an ℓ th power in $\overline{K}_{\mathfrak{p}_2}^*$ (again since all units are ℓ^r th powers in $\overline{K}_{\mathfrak{p}_2}^*$), i.e. $\mathfrak{p}_2$ is nonsplit in $K(\mu_{\ell}, \sqrt[r]{b})$. Since $\mathfrak{p}_2$ splits in $K(\mu_{\ell^r})$, this is equivalent to $\mathfrak{p}_2$ being nonsplit in $K(\mu_{\ell^r}, \sqrt[r]{b})$. For this Chebotarev condition to be compatible with (a) and (b), it suffices that the fields $L^{\mathfrak{p}_1} E'$ and $\{K(\mu_{\ell^r}, \sqrt[r]{b}) : \mathfrak{q}^h = (b), \mathfrak{q} \neq \mathfrak{p}_1, \mathfrak{l}_i, \mathcal{N}(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{p}_1)\}$ be linearly disjoint over $K(\mu_{\ell^r})$.

Let $\mathfrak{q}_1, \dots, \mathfrak{q}_u$ be the primes of K distinct from $\mathfrak{p}_1, \mathfrak{l}_1, \dots, \mathfrak{l}_t$ of absolute norm less than or equal to that of $\mathfrak{p}_1$, and let $\mathfrak{q}_i^h = (b_i)$, $i = 1, \dots, u$. Set $K' = K(\mu_{\ell^r})$. We show first that the fields $\{K'(\sqrt[r]{b_i}) : i = 1, \dots, u\}$ are linearly disjoint over K' . If not, then by Kummer theory we have an equation $\prod_1^u b_i^{e_i} = x^{\ell}$ with $x \in K'$, and not all the e_i divisible by

ℓ . Taking ideals in K' , we have $\prod_1^u (b_i)^{e_i} = (x)^\ell$. Since h_K is prime to ℓ , and the primes q_i are unramified in K' , we see that ℓ must divide all the e_i , contradiction. Set $F_1 := L^{p_1} E'(\sqrt[\ell]{c})$, $F_2 := K'(\sqrt[\ell]{b_1}, \dots, \sqrt[\ell]{b_u})$. It remains to show that $F_1 \cap F_2 = K'$. If not, then there is a common cyclic subextension $F_3 \subseteq F_1 \cap F_2$ with $[F_3 : K'] = \ell$. On the one hand, F_3 is of the form $K'(\sqrt[\ell]{\prod_1^u b_i^{e_i}})$ with not all e_i divisible by ℓ . For such an i , the prime divisors of q_i in K' ramify in F_3 . But the only primes ramifying in F_1 are divisors of $p_1, l_1, \dots, l_t$, contradiction. Thus the disjointness assertion is proved.

This shows the existence of p_2 satisfying (a),(b),(c).

We now assume inductively that $n > 2$ and $p_1, \dots, p_{n-1} \in S'$ with $\mathcal{N}(p_i) < \mathcal{N}(p_{i+1})$, $i = 1, \dots, n-2$, have been chosen such that

- (a_i) p_i splits completely in L^{p_j} for all $j < i$, $i = 2, \dots, n-1$
- (b_i) p_j splits completely in L^{p_i} for all $j < i$, $i = 2, \dots, n-1$
- (c_i) q is inert in L^{p_i} for all primes q satisfying $\mathcal{N}(p_{i-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{i-1})$, $q \neq p_{i-1}$, $i = 2, \dots, n-1$ (take $p_0 = 1$)

(Note (a_i) and (b_i) together say p_i splits completely in L^{p_j} for all $i \neq j$, $1 \leq i, j \leq n-1$.)

Claim: There exists a prime $p_n \in S$ satisfying (a_n),(b_n),(c_n).

The argument is similar to that for p_2 : (a_n) is satisfied if and only if p_n splits completely in the composite $L^{p_1} \dots L^{p_{n-1}}$.

(b_n) is satisfied if p_n splits completely in $K'(\sqrt[\ell]{c_1}, \dots, \sqrt[\ell]{c_{n-1}})$, where $p_i^h = (c_i)$, $i = 1, \dots, n-1$

(c_n) is satisfied if p_n remains inert in $K'(\sqrt[\ell]{b})$ for each $q^h = (b)$, $q \neq p_{n-1}$, with $\mathcal{N}(p_{n-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{n-1})$. In order to apply the Chebotarev theorem we need the linear disjointness of $L^{p_1} \dots L^{p_{n-1}} \cdot E'(\sqrt[\ell]{c_1}, \dots, \sqrt[\ell]{c_{n-1}})$ and the $K'(\sqrt[\ell]{b})$ over $K' = K(\mu_{\ell^r})$, for all the above b 's. Since the (c_i) 's and the (b) 's are distinct prime ideals raised to the power h , the previous argument goes through, proving the claim.

We therefore have an infinite sequence $S_0 = \{p_n\}_{n=1}^\infty$ of primes of S' satisfying

- (i) p_i splits completely in L^{p_j} for all $i \neq j$,
and
- (ii) q is inert in L^{p_n} for all $q \neq p_{n-1}$ with $\mathcal{N}(p_{n-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{n-1})$.

Now take L to be the composite of L_0 and all the L^{p_n} . We check the local degrees of L/K :

For $p = p_i \in S_0$, L contains L^{p_i} which is totally ramified of degree ℓ^r at p . p splits completely in L_0 , and by (i), p splits completely in L^{p_j} for $j \neq i$, so $[L_p : K_p] = \ell^r$.

For $\mathfrak{p} \notin S_0$, $\mathfrak{p}$ not dividing ℓ , $\mathfrak{p}$ is unramified in L , so $L_{\mathfrak{p}}/K_{\mathfrak{p}}$ is cyclic of exponent dividing ℓ^r . There exists a positive integer n such that $\mathcal{N}(\mathfrak{p}_{n-1}) < \mathcal{N}(\mathfrak{p}) \leq \mathcal{N}(\mathfrak{p}_n)$. By (ii), $\mathfrak{p}$ is inert in $L^{\mathfrak{p}_{n+1}}$ hence $[L_{\mathfrak{p}} : K_{\mathfrak{p}}] = \ell^r$. For $\mathfrak{p}$ dividing ℓ , $\mathfrak{p}$ is totally ramified in L_0 and splits completely in all the $L^{\mathfrak{p}_i}$, hence $[L_{\mathfrak{p}} : K_{\mathfrak{p}}] = \ell^r$. L_0 is totally complex, hence so is L . This completes the proof of Theorem 2.1. $\square$

Remark 2.2. The hypothesis that K is Galois over $\mathbb{Q}$ guarantees that all primes ℓ_i dividing ℓ in K have isomorphic completions, which is all that is needed in the proof. Also we can have L/K unramified at the infinite primes by choosing the maximal real subfield of $\mathbb{Q}(\mu_{2^s+2})$ in place of k_s .

3. The n -torsion subgroup of the Brauer group of K

Theorem 3.1. *Given a number field K Galois over $\mathbb{Q}$ and a positive integer n prime to the class number of K , there exists an abelian extension L/K (of exponent n) such that the n -torsion subgroup of the Brauer group of K is equal to the relative Brauer group of L/K .*

Proof. Consider the case $n = \ell^r$, ℓ prime. By Theorem 2.1, there exists an abelian ℓ -extension L/K whose local degree at every finite prime is ℓ^r , and is 2 at the real primes if $\ell = 2$. It follows from the fundamental theorem of class field theory on the Brauer group of a number field that L splits every algebra class of order dividing ℓ^r , and conversely, any algebra class split by L has order dividing ℓ^r . For general n , the theorem follows from a straightforward reduction to the prime power case (see [1]). $\square$

Remark 3.2. For $K = \mathbb{Q}$, Theorem 3.1 says that $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all n . The proof of Theorem 2.1 is more concrete in this case because the ray class fields involved are simply the degree ℓ^r subfields of $\mathbb{Q}(\mu_p)$ with $p \equiv 1 \pmod{\ell^r}$. Theorem 3.1 was proved in [1] for the case n squarefree, $K = \mathbb{Q}$. The case $n = 2$ was proved there by constructing $L/\mathbb{Q}$ with local degree 2 everywhere except perhaps at the prime 2. We are grateful to Romyar Sharifi and David Ford (independently) for a construction of $L/\mathbb{Q}$ with local degree 2 everywhere, including 2, the idea of which was instrumental in the proof of Theorem 2.1.

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