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 $D = b^2 - 4ac > 0$

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The Diophantine equation

$$ax^2 + bxy + cy^2 = N, \quad D = b^2 - 4ac > 0$$

par KEITH MATTHEWS

RÉSUMÉ. Nous revisitons un algorithme dû à Lagrange, basé sur le développement en fraction continue, pour résoudre l'équation $ax^2 + bxy + cy^2 = N$ en les entiers x, y premiers entre eux, où $N \neq 0$, $\text{pgcd}(a, b, c) = \text{pgcd}(a, N) = 1$ et $D = b^2 - 4ac > 0$ n'est pas un carré.

ABSTRACT. We make more accessible a neglected simple continued fraction based algorithm due to Lagrange, for deciding the solubility of $ax^2 + bxy + cy^2 = N$ in relatively prime integers x, y , where $N \neq 0$, $\text{gcd}(a, b, c) = \text{gcd}(a, N) = 1$ and $D = b^2 - 4ac > 0$ is not a perfect square. In the case of solubility, solutions with least positive y , from each equivalence class, are also constructed.

Our paper is a generalisation of an earlier paper by the author on the equation $x^2 - Dy^2 = N$. As in that paper, we use a lemma on unimodular matrices that gives a much simpler proof than Lagrange's for the necessity of the existence of a solution.

Lagrange did not discuss an exceptional case which can arise when $D = 5$. This was done by M. Pavone in 1986, when $N = \pm\mu$, where $\mu = \min_{(x,y) \neq (0,0)} |ax^2 + bxy + cy^2|$. We only need the special case $\mu = 1$ of his result and give a self-contained proof, using our unimodular matrix approach.

1. Introduction

The standard approach to solving the equation

$$(1.1) \quad ax^2 + bxy + cy^2 = N$$

in relatively prime integers x, y , is via reduction of quadratic forms, as in Mathews ([6, p 97]). There is a parallel approach in Faisant's book ([2, pp 106–113]) which uses continued fractions.

However, in a memoir of 1770, Lagrange ([11, Oeuvres II, pp 655–726]), gave a more direct method for solving (1.1) when $\text{gcd}(a, b, c) = \text{gcd}(a, N) = 1$ and $D = b^2 - 4ac > 0$ is not a perfect square. This paper seems to have

been largely overlooked. (Admittedly, the necessity part of his proof is long and not easy to follow.)

M. Pavone ([10, p 271]) solved (1.1) when $N = \pm\mu$, where

$$\mu = \min_{(x,y) \neq (0,0)} |ax^2 + bxy + cy^2|.$$

He had essentially solved (1.1) in general, as Lagrange showed how to reduce the problem to the case $N = \pm 1$. (See (4.2) and (4.6)).

Strangely Pavone made no mention of Lagrange's paper, referring instead to Serret ([12, p 80]), who had earlier drawn attention to the possibility of an exceptional case.

A. Nitaj has also discussed the equation in his thesis, ([9, pp 57–88]), using a standard convergent sufficiency condition of Lagrange, which resulted in a restriction $D \geq 16$, thus making rigorous the necessity part of Lagrange's discussion. Nitaj discussed only the case $b = 0$ in detail, along the lines of Cornacchia ([1, pp 66–70]).

Our contribution in this paper is to use the convergent criterion of Lemma 2, which results in no restriction on D , while allowing us to deal with the non-convergent case, without having to appeal to the case $\mu = 1$ of Pavone, whose proof is somewhat complicated.

The continued fractions approach also has the attraction that it produces the solution (x, y) with least positive y from each class, if $\gcd(a, N) = 1$.

Our treatment generalises an earlier paper by the author on the equation $x^2 - Dy^2 = N$ (See Matthews [7]).

The assumption that $\gcd(a, N) = 1$ involves no loss of generality. For as pointed out by Gauss in his *Disquisitiones* (see [3, p 221] (also see Lemma 2 of Hua [5, pp 311–312]), there exist relatively prime integers α, γ such that $a\alpha^2 + b\alpha\gamma + c\gamma^2 = A$, where $\gcd(A, N) = 1$. Then if $\alpha\delta - \beta\gamma = 1$, the unimodular transformation $x = \alpha X + \beta Y, y = \gamma X + \delta Y$ converts $ax^2 + bxy + cy^2$ to $AX^2 + BXY + CY^2$. Also the two forms represent the same integers.

2. The structure of the solutions

We outline the structure of the integer solutions of (1.1) as given in Skolem ([13, pp 42–45]).

The primitive solutions $x + y\sqrt{D}$ of $ax^2 + bxy + cy^2 = N$ (i.e. with $\gcd(x, y) = 1$) fall into equivalence classes, with $x + y\sqrt{D}$ and $x' + y'\sqrt{D}$ being equivalent if and only if

$$(2.1) \quad 2ax + by + y\sqrt{D} = \frac{(u + v\sqrt{D})}{2}(2ax' + by' + y'\sqrt{D}),$$

where u and v are integers satisfying $u^2 - Dv^2 = 4$.

This is equivalent to the equations

$$(2.2) \quad x = \left(\frac{u - bv}{2}\right)x' - cvy', \quad y = avx' + \left(\frac{u + bv}{2}\right)y'.$$

It is easy to verify that (2.1) holds if and only if the following congruences hold:

$$(2.3) \quad 2axx' + b(xy' + x'y) + 2cyy' \equiv 0 \pmod{|N|}$$

$$(2.4) \quad xy' - x'y \equiv 0 \pmod{|N|}.$$

Each primitive solution gives rise to a root n of the congruence

$$n^2 \equiv D \pmod{4|N|}.$$

In fact if (α, γ) is a solution of (1.1) and $\alpha\delta - \beta\gamma = 1$, then

$$(2.5) \quad n = (2a\alpha + b\gamma)\beta + (b\alpha + 2c\gamma)\delta.$$

Equivalent solutions give rise to congruent $n \pmod{2|N|}$.

Conversely, primitive solutions which give rise to congruent $n \pmod{2|N|}$ are equivalent. This follows from the equations

$$\begin{aligned} -\gamma n + 2N\delta &= 2a\alpha + b\gamma \\ \alpha n - 2N\beta &= b\alpha + 2c\gamma \end{aligned}$$

and congruences (2.3) and (2.4).

It is also straightforward to verify that if $ax^2 + bxy + cy^2$ is replaced by an equivalent form $AX^2 + BXY + CY^2$ under a unimodular transformation, then equivalent primitive representations (x, y) and (x', y') of N map into equivalent primitive representations (X, Y) and (X', Y') . In fact the n of equation (2.5) is replaced by Δn , where Δ is the determinant of the transformation. (See Gauss [3, pp 130–131].)

3. Some lemmas

Lemma 1. Assume $D > 0$ is not a perfect square and $Q_0|(P_0^2 - D)$.

If $(P_n + \sqrt{D})/Q_n$ is the n -th complete convergent in the simple continued fraction for $x = (P_0 + \sqrt{D})/Q_0$ and $G_{n-1} = Q_0A_{n-1} - P_0B_{n-1}$, where A_{n-1}/B_{n-1} denotes a convergent to x , then

$$(3.1) \quad G_{n-1}^2 - DB_{n-1}^2 = (-1)^n Q_0 Q_n,$$

or equivalently

$$(3.2) \quad Q_0 A_{n-1}^2 - 2P_0 A_{n-1} B_{n-1} + \frac{P_0^2 - D}{Q_0} B_{n-1}^2 = (-1)^n Q_n.$$

Proof. See Mollin [8, pp 246–248]. □

Lemma 2. If $\omega = \frac{P\zeta+R}{Q\zeta+S}$, where $\zeta > 1$ and P, Q, R, S are integers such that $Q > 0, S > 0$ and $PS - QR = \pm 1$, or $S = 0$ and $Q = 1 = R$, then P/Q is a convergent A_n/B_n to ω . Moreover if $Q \neq S > 0$, then $R/S = (A_{n-1} + kA_n)/(B_{n-1} + kB_n), k \geq 0$. Also $\zeta + k$ is the $(n+1)$ -th complete convergent to ω . Here $k = 0$ if $Q > S$, while $k \geq 1$ if $Q < S$.

Proof. This is an extension of Theorem 172, Hardy and Wright ([4, pp 140–141]), who dealt with the case $Q > S > 0$. See Matthews [7, pp 325–326]. $\square$

The following result is a special case ($\mu(f) = 1$) of a result due to M. Pavone, [10, p 271]. Pavone's proof is rather complicated and we give a self-contained proof using our Lemma 2 as cases (ii) and (iii)(c) of the proof of Lemma 3 below and in the Appendix.

Lemma 3. Suppose $X, y > 0, Q, n, R$ are integers and

$$(3.3) \quad QX^2 + nXy + Ry^2 = 1,$$

where $D = n^2 - 4QR > 0$ is not a perfect square. Also let $\omega = \frac{-n+\sqrt{D}}{2Q}$ and $\omega^* = \frac{-n-\sqrt{D}}{2Q}$ be the roots of $Q\theta^2 + n\theta + R = 0$. Then either

- (i) X/y is a convergent A_{i-1}/B_{i-1} to ω (resp. ω^*) and if $(P_i + \sqrt{D})/Q_i$ denotes the i -th complete convergent to ω (resp. ω^*), then $Q_i = (-1)^{i2}$ (resp. $Q_i = (-1)^{i+12}$), or
- (ii) $D = 5, Q < 0$ and

$$X/y = (A_r - A_{r-1})/(B_r - B_{r-1}) = (A'_s - A'_{s-1})/(B'_s - B'_{s-1}),$$

where A_r/B_r and A'_s/B'_s denote convergents to ω and ω^* , respectively and

$$\omega = [a_0, \dots, a_r, \bar{1}], \quad \omega^* = [b_0, \dots, b_s, \bar{1}],$$

where $a_r > 1$ if $r > 0$ and $b_s > 1$ if $s > 0$.

Moreover X/y is not a convergent to ω or ω^* .

Conversely, if (i) or (ii) hold, then X/y is a solution of (3.3).

Remarks. (a) In the Appendix, we prove that if $D = 5$ and $Q < 0$, then $r - 1 \equiv s \pmod{2}$ and

$$(A_r - A_{r-1})/(B_r - B_{r-1}) = (A'_s - A'_{s-1})/(B'_s - B'_{s-1}),$$

the latter being obtained directly by an appeal to symmetry by Pavone ([10, p 277]).

(b) As Pavone points out, we have

$$\frac{A_{r-2}}{B_{r-2}} < \frac{A_r - A_{r-1}}{B_r - B_{r-1}} < \frac{A_r}{B_r} < \omega \text{ if } r \text{ is even,}$$

$$\omega < \frac{A_r}{B_r} < \frac{A_r - A_{r-1}}{B_r - B_{r-1}} < \frac{A_{r-2}}{B_{r-2}} \text{ if } r \text{ is odd.}$$

The corresponding equations hold if ω is replaced by ω^* , each A_r is replaced by A'_s and each B_r is replaced by B'_s etc.

Consequently, $\frac{A_r - A_{r-1}}{B_r - B_{r-1}}$ is not a convergent to ω or ω^* .

(c) If n is even, say $n = 2P$, then $\omega = (-P + \sqrt{\Delta})/Q$ and $\omega^* = (-P - \sqrt{\Delta})/Q$, where $\Delta = P^2 - QR$. If we then denote the n -th complete convergent to ω (resp. ω^*) by $(P_n + \sqrt{\Delta})/Q_n$, condition (i) becomes $Q_i = (-1)^i$ (resp $Q_i = (-1)^{i+1}$).

Proof. Suppose (3.3) holds. Consider the matrix

$$H = \begin{pmatrix} X & t \\ y & QX + Py \end{pmatrix},$$

where $t = -PX - Ry$ if $n = 2P$, while $t = -(P+1)X - Ry$ if $n = 2P+1$.

Then in both cases,

$$(3.4) \quad \det H = QX^2 + nXy + Ry^2 = 1.$$

Also it is straightforward to verify that

$$\omega = \frac{X\alpha + t}{y\alpha + QX + Py},$$

where

$$\alpha = \begin{cases} \sqrt{\Delta} & \text{if } n = 2P, \\ \frac{\sqrt{D+1}}{2} & \text{if } n = 2P+1. \end{cases}$$

Case (i). Suppose $QX + Py > 0$. Then as $\alpha > 1$, Lemma 2 applies and X/y is a convergent to ω .

Case (ii). Suppose $QX + Py = 0$. On substituting for QX in (3.4), we get

$$(-Py)X + nXy + Ry^2 = 1.$$

Hence $y = 1$ and $-PX + nX + R = 1$. Also $\omega = X - \frac{1}{\alpha}$.

Hence

$$\omega^* = \begin{cases} X + \frac{1}{\sqrt{\Delta}} & \text{if } n = 2P, \\ X + \frac{1}{\frac{\sqrt{D-1}}{2}} & \text{if } n = 2P+1. \end{cases}$$

Hence $X/y = [\omega^*]$ is a convergent to ω^* if $D \neq 5$.

If $D = 5$, we see $\omega^* = [X+1, \bar{1}]$ ($s = 0$) and $\omega = [X-1, 2, \bar{1}]$ ($r = 1$).

Then

$$\begin{aligned}\frac{A_1 - A_0}{B_1 - B_0} &= \frac{(2X - 1) - (X - 1)}{2 - 1} = X, \\ \frac{A'_0 - A'_{-1}}{B'_0 - B'_{-1}} &= \frac{(X + 1) - 1}{1 - 0} = X.\end{aligned}$$

Also $QX^2 + (2P + 1)X + R = 1$ and $P = -QX$ together give

$$-QX^2 + X + R = 1.$$

Hence

$$\begin{aligned}1 = \frac{D - 1}{4} &= P^2 + P - QR = Q^2X^2 - QX - QR \\ &= Q(QX^2 - X - R) = -Q\end{aligned}$$

and hence $Q < 0$.

Case (iii) (a) Suppose $QX + Py < 0$. Then $-(QX + Py) > 0$ and

$$\omega^* = \frac{X(-\alpha^*) - t}{y(-\alpha^*) - (QX + Py)},$$

where $-\alpha^* > 1$ if $D \neq 5$.

Hence X/y is a convergent to ω^* , unless $D = 5$.

(b) If $D = 5$ and $-(QX + Py) \geq y$, then

$$(3.5) \quad \omega^* = \frac{X - t\alpha}{y - (QX + Py)\alpha}$$

and again X/y is a convergent to ω^* by Lemma 2.

In all cases where X/y is the convergent A_{n-1}/B_{n-1} to ω (resp. ω^*), it follows from (3.3) and equation (3.2) of Lemma 1, with $P_0 = -n$, $Q_0 = 2Q$ (resp. $P_0 = n$, $Q_0 = -2Q$), that

$$\begin{aligned}(-1)^n Q_n &= Q_0 A_{n-1}^2 - 2P_0 A_{n-1} B_{n-1} + \frac{P_0^2 - D}{Q_0} B_{n-1}^2 \\ &= \begin{cases} 2QX^2 + 2nXy + 2Ry^2 = 2 & \text{for } \omega, \\ -2QX^2 - 2nXy - 2Ry^2 = -2 & \text{for } \omega^* \end{cases}\end{aligned}$$

and consequently $(-1)^n Q_n = 2$ (resp. -2) in all cases.

(c) Now suppose $D = 5$ and $y > -(QX + Py) > 0$.

Now, from (3.5), Lemma 2 tells us that

$$\begin{aligned}(3.6) \quad \frac{X}{y} &= \frac{A_{i-1} + kA_i}{B_{i-1} + kB_i} \\ (P + 1)X + Ry &= -t = A_i \\ -(QX + Py) &= B_i,\end{aligned}$$

where $k \geq 1$. Moreover $\omega_{i+1}^* = \alpha + k = [k + 1, \bar{1}]$.

Hence $\omega^* = [a_0, \dots, a_s, \bar{1}]$, where $s = i + 1$ and $a_s = k + 1$.

Hence (3.6) gives

$$\begin{aligned} \frac{X}{y} &= \frac{A_{s-2} + (a_s - 1)p_{s-1}}{B_{s-2} + (a_s - 1)B_{s-1}} \\ &= \frac{A_s - A_{s-1}}{B_s - B_{s-1}}. \end{aligned}$$

Next we prove that $Q < 0$. We have from equation (1.1)),

$$\begin{aligned} 1 &= QX^2 + (2P + 1)Xy + Ry^2 \\ Q &= Q^2X^2 + (2P + 1)QXy + QRy^2 \\ &= (QX + Py)^2 + (QX + Py - y)y \\ &= (-B_{s-1})^2 + (-B_{s-1} - (B_s - B_{s-1}))(B_s - B_{s-1}) \\ &= B_{s-1}^2 + B_{s-1}B_s - B_s^2 \\ (3.7) \quad &= -\frac{1}{4}((2B_s - B_{s-1})^2 - 5B_{s-1}^2). \end{aligned}$$

However

$$B_s = a_s B_{s-1} + B_{s-2} \geq 2B_{s-1} > \left(\frac{1 + \sqrt{5}}{2} \right) B_{s-1}$$

so

$$2B_s - B_{s-1} > \sqrt{5}B_{s-1}$$

and hence

$$(2B_s - B_{s-1})^2 - 5B_{s-1}^2 > 0.$$

Then equation (3.7) gives $Q < 0$. □

4. The main result

Theorem 1. Suppose

$$(4.1) \quad ax^2 + bxy + cy^2 = N,$$

where $N \neq 0$, $\gcd(x, y) = 1 = \gcd(a, N)$ and $y > 0$ and $D = b^2 - 4ac > 0$ is not a perfect square.

Let θ satisfy $x \equiv y\theta \pmod{|N|}$, $0 \leq \theta < |N|$. Then

$$a\theta^2 + b\theta + c \equiv 0 \pmod{|N|}.$$

Let $x = y\theta + |N|X$, $n = 2a\theta + b$, $Q = a|N|$, $\omega = \frac{-n + \sqrt{D}}{2Q}$ and $\omega^* = \frac{-n - \sqrt{D}}{2Q}$.

Also let $n = 2P$ or $2P + 1$, according as b is even or odd. Then

- (i) if $QX + Py > 0$, X/y is a convergent to ω ;
- (ii) Suppose $QX + Py \leq 0$. Then

- (a) If $D \neq 5$, or $D = 5$ and $-(QX + Py) \geq y$, then X/y is a convergent to ω^* .
 (b) If $D = 5$ and $y > -(QX + Py) \geq 0$, then

$$\frac{X}{y} = \frac{A_r - A_{r-1}}{B_r - B_{r-1}} = \frac{A'_s - A'_{s-1}}{B'_s - B'_{s-1}}$$

which is not a convergent to ω or ω^* .
 Also $aN < 0$.

Conversely,

- (a) if X/y is a convergent A_{i-1}/B_{i-1} to ω (resp. ω^*)
 and $Q_i = (-1)^i 2N/|N|$ (resp. $(-1)^{i+1} 2N/|N|$), or
 (b) if $D = 5$, $aN < 0$ and $\frac{X}{y} = \frac{A_r - A_{r-1}}{B_r - B_{r-1}}$, where r is defined earlier,
 then (x, y) , with $x = y\theta + |N|X$, will be a solution to (4.1), possibly imprimitive.

Proof. Suppose

$$ax^2 + bxy + cy^2 = N,$$

where $\gcd(x, y) = 1 = \gcd(a, N)$ and $y > 0$. Then clearly $\gcd(y, |N|) = 1$.
 Let $x \equiv y\theta \pmod{|N|}$ and

$$(4.2) \quad x = y\theta + |N|X.$$

Then

$$(4.3) \quad \begin{aligned} a\theta^2 y^2 + b(y\theta)y + cy^2 &\equiv 0 \pmod{|N|} \\ a\theta^2 + b\theta + c &\equiv 0 \pmod{|N|} \end{aligned}$$

$$(4.4) \quad \begin{aligned} 4a^2\theta^2 + 4ab\theta + 4ac &\equiv 0 \pmod{4|N|} \\ (2a\theta + b)^2 &\equiv b^2 - 4ac \pmod{4|N|}, \end{aligned}$$

$$(4.5) \quad \text{or } n^2 \equiv D \pmod{4|N|}.$$

Also

$$(4.6) \quad \begin{aligned} a(y\theta + |N|X)^2 + b(y\theta + |N|X)y + cy^2 &= N \\ |N|^2 aX^2 + (2a\theta + b)|N|Xy + (a\theta^2 + b\theta + c)y^2 &= N \\ QX^2 + nXy + Ry^2 &= \frac{N}{|N|}, \end{aligned}$$

where $Q = a|N|$, $R = (a\theta^2 + b\theta + c)/|N|$ and $n^2 - 4QR = D$.

The conclusions of the theorem then follow from Lemma 3, applied to the equation $Q'X^2 + n'Xy + R'y^2 = 1$, where $Q' = \epsilon Q$, $n' = \epsilon n$, $R' = \epsilon R$ and $\epsilon = |N|/N$. $\square$

5. The algorithm

Let $\Delta = D/4$ if b is even and let the i -th complete convergent to ω or ω^* be denoted by $(P_i + \sqrt{\Delta})/Q_i$ or $(P_i + \sqrt{D})/Q_i$, according as b is even or odd.

If equation (4.1) is soluble with $x \equiv y\theta \pmod{|N|}$, $y > 0$, there will be infinitely many solutions because of equations (2.2). It follows that if ω and ω^* are not purely periodic, we need only examine the first period $m \leq i \leq m + l$ of the continued fractions for ω and ω^* to determine solubility of (4.1). For, with ω (resp. ω^*) being $(-P \pm \sqrt{\Delta})/Q$ (resp. $(-(2P+1) \pm \sqrt{D})/2Q$), the equation $Q_i = \pm 1$ (resp. ± 2) will hold for infinitely many i by periodicity and so there will be at least one such i in the range $m \leq i \leq m + l$. Any such i must have $Q_i = 1$ (resp. 2), as $(P_i + \sqrt{\Delta})/Q_i$ (resp. $(P_i + \sqrt{D})/Q_i$) is reduced for i in this range and so $Q_i > 0$. Moreover if l is even, the sign of $(-1)^i N/|N|$ is preserved from one period to the next. If l is odd, then the first or second period will produce a solution. If ω or ω^* is purely periodic, we must examine Q_2 , which corresponds to the third period.

Moreover there can be at most one i in a period for which $Q_i = 1$ (resp. 2). For if $P_i + \sqrt{\Delta}$ (resp. $(P_i + \sqrt{D})/2$) is reduced, then $P_i = \lfloor \sqrt{\Delta} \rfloor$ (resp. $P_i = 2\lfloor (\sqrt{D} - 1)/2 \rfloor + 1$) and hence two such occurrences of $Q_i = 1$ (resp. 2) within a period would give a smaller period.

Hence we have the following algorithm essentially due to Lagrange, apart from stage 1:

1. If $\gcd(a, N) > 1$, find a unimodular transformation of the given quadratic form into one in which $\gcd(a, N) = 1$. (See the last paragraph of the Introduction.)

2. Find all solutions θ of the congruence (4.3) in the range $0 \leq \theta < |N|$. (This can be done as follows:

First solve $t^2 \equiv b^2 - 4ac \pmod{4|N|}$, $-|N| < t \leq |N|$. (If there are no solutions t , then there is no primitive solution of (4.1) corresponding to t .)

Then solve $a\theta \equiv \frac{t-b}{2} \pmod{|N|}$, $0 \leq \theta < |N|$.)

For each θ , let $n = 2a\theta + b$, $P = \lfloor n/2 \rfloor$ and $Q = a|N|$.

3. For each of the numbers $\omega = \frac{-P+\sqrt{\Delta}}{Q}$ (resp. $\frac{-(2P+1)+\sqrt{D}}{2Q}$), test the first period to see if $Q_i = 1$ (resp. 2) occurs. If l is even, test additionally for $1 = (-1)^i N/|N|$ (resp. $2 = (-1)^i N/|N|$) to hold.

Similarly for each of the numbers $\omega^* = \frac{-P-\sqrt{\Delta}}{Q}$ (resp. $\frac{-(2P+1)-\sqrt{D}}{2Q}$), with i replaced by $i + 1$.

If $D = 5$, test additionally to see if $aN < 0$ holds.

4. For each θ and corresponding ω for which test 3 succeeds, find the least i for which the condition $Q_i = (-1)^i N/|N|$ (resp. $Q_i = (-1)^i 2N/|N|$)

holds. If l is even, this will occur in or before the first period, while if l is odd, this will occur in or before the second period. Similarly for ω^* .

For the corresponding convergent A_{i-1}/B_{i-1} to ω or ω^* , write $X = A_{i-1}$, $y = B_{i-1}$. If $D = 5$ and $aN < 0$, in relation to ω , write $X = A_r - A_{r-1}$, $y = B_r - B_{r-1}$. Then $x = y\theta + |N|X$ produces a solution of (4.1) with $x \equiv y\theta \pmod{|N|}$.

Choose the solution (x, y) with lesser of the y values.

The algorithm will produce a solution (x, y) from each class, with the additional feature that the least positive y is chosen, if the quadratic form satisfies $\gcd(a, N) = 1$.

6. Examples

Example 1 (Gauss, Article 205). [3, p 189]

$$(6.1) \quad 42x^2 + 62xy + 21y^2 = 585.$$

As $\gcd(42, 585) = 3 = \gcd(21, 585)$, we make a suitable transformation

$$x = -x' + y', y = 2x' - y',$$

which gives

$$42x^2 + 62xy + 21y^2 = 2x'^2 + 18x'y' + y'^2.$$

The latter form has $\Delta = 79$ and $\gcd(2, 585) = 1$.

We list the roots of $2\theta^2 + 18\theta + 1 \equiv 0 \pmod{585}$ and corresponding values $P = 2\theta + 9$:

θ	34	47	74	164	412	502	529	542
P	77	103	157	337	833	1013	1067	1093

We find that only $P = 157$ and 1013 give solutions of equation (6.1):

(i) $\omega = (-157 + \sqrt{79})/1170$ gives $Q_3 = 1$, $A_2/B_2 = -1/7$.

Then $y' = 7$ and $x' = 7 \cdot 74 - 585 \cdot 1 = -67$. Hence $(x, y) = (74, -141)$ is a solution of (6.1).

$\omega^* = (-157 - \sqrt{79})/1170$ also gives the solution $(74, -141)$.

(ii) $\omega = (-1013 + \sqrt{79})/1170$ gives $Q_2 = 1$, $A_1/B_1 = -6/7$. Then $y' = 7$ and $x' = 7 \cdot 502 - 585 \cdot 6 = 4$. Hence $(x, y) = (3, 1)$ is a solution of (6.1).

$\omega^* = (-1013 - \sqrt{79})/1170$ also gives the solution $(3, 1)$.

In fact Gauss gave solutions $(83, -87)$ and $(3, 1)$. In the notation of (2.2), the solutions $(x, y) = (83, -87)$ and $(x', y') = (74, -141)$ are related by the solution $(u, v) = (-80, 9)$ of the Pell equation $x^2 - 79y^2 = 1$.

Summarising:

$42x^2 + 62xy + 21y^2 = 585$	
Solution	$n \pmod{1170}$
$(74, -141)$	314
$(3, 1)$	-314

Example 2. $3x^2 - 3xy - 2y^2 = 202$.

Here $D = 33$.

The solutions of $3\theta^2 - 3\theta - 2 \equiv 0 \pmod{202}$ are 39, 63, 140, 164, with corresponding n values 231, 375, 837, 981.

- (i) $\theta = 39$, $\omega = (-231 + \sqrt{33})/1212$, $Q_3 = -2$, $A_2/B_2 = -1/5$.
Then $x = y\theta + |N|X = 5 \cdot 39 + 202 \cdot (-1) = -7$ and $(x, y) = (-7, 5)$.
 $\omega^* = (-231 - \sqrt{33})/1212$ produces the same solution.
- (ii) $\theta = 63$, $\omega = (-375 + \sqrt{33})/1212$, $Q_6 = 2$, $A_5/B_5 = -7/23$.
Then $x = 23 \cdot 63 + 202 \cdot (-7) = 35$ and $(x, y) = (35, 23)$.
 $\omega^* = (-375 - \sqrt{33})/1212$ gives $Q_5 = 2$, $A_4/B_4 = -11/35$.
Then $x = 35 \cdot 63 + 202 \cdot (-11) = -17$ and we get the equivalent solution $(-17, 35)$.
- (iii) $\theta = 140$, $\omega = (-837 + \sqrt{33})/1212$, $Q_4 = 2$, $A_3/B_3 = -24/35$.
Then $x = 35 \cdot 140 + 202 \cdot (-24) = 52$ and $(x, y) = (52, 35)$.
 $\omega^* = (-837 - \sqrt{33})/1212$ gives $Q_5 = 2$, $A_4/B_4 = -16/23$.
Then $x = 23 \cdot 140 + 202 \cdot (-16) = -12$ and we get the equivalent solution $(-12, 23)$.
- (iv) $\theta = 164$, $\omega = (-981 + \sqrt{33})/1212$, $Q_2 = 2$, $A_1/B_1 = -4/5$.
Then $x = 5 \cdot 164 + 202 \cdot (-4) = 12$ and $(x, y) = (12, 5)$.
 $\omega^* = (-981 - \sqrt{33})/1212$ produces the same solution.

Summarising:

$3x^2 - 3xy - 2y^2 = 202$	
Solution	$n \pmod{404}$
$(35, 23)$	29
$(-12, 23)$	-29
$(12, 5)$	231
$(-7, 5)$	-231

There are 4 equivalence classes of solutions.

Example 3. $f(x, y) = 19x^2 - 85xy + 95y^2 = -671$.

Here $D = 5$.

The solutions of $19\theta^2 - 85\theta + 95 \equiv 0 \pmod{671}$ are 443, 454, 504, 515, with corresponding n values 16749, 17167, 19067, 19485.

The exceptional solutions give the solutions with smallest y :

- (i) $\theta = 443$: $\omega = (-16749 + \sqrt{5})/25498 = [-1, 2, 1, 10, 1, 1, 2, \bar{1}]$,
 $Q_7 = 2$, $A_5/B_5 = -44/67$. Also $A_6/B_6 = -111/169$.

Exceptional solution:

$$(X, y) = (A_6 - A_5, B_6 - B_5) = (-67, 102), (x, y) = (229, 102).$$

$$\omega^* = [-1, 2, 1, 10, 3, \bar{1}] \text{ gives } Q_6 = 2 \text{ and correspondingly } (x, y) = (301, 137).$$

$$(ii) \theta = 454: \omega = (-17167 + \sqrt{5})/25498 = [-1, 3, 16, 1, 2, \bar{1}],$$

$$Q_5 = 2, A_3/B_3 = -35/52. \text{ Also } A_4/B_4 = -103/153.$$

Exceptional solution:

$$(X, y) = (A_4 - A_3, B_4 - B_3) = (-68, 101), (x, y) = (226, 101).$$

$$\omega^* = [-1, 3, 16, 3, \bar{1}] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (329, 150).$$

$$(iii) \theta = 504: \omega = (-19067 + \sqrt{5})/25498 = [-1, 3, 1, 26, 2, \bar{1}],$$

$$Q_5 = 2, A_3/B_3 = -80/107 \text{ and } A_4/B_4 = -163/218.$$

Exceptional solution:

$$(X, y) = (A_4 - A_3, B_4 - B_3) = (-83, 111), (x, y) = (251, 111).$$

$$\omega^* = [-1, 3, 1, 28, \bar{1}] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (254, 115).$$

$$(iv) \theta = 515: \omega = (-19485 + \sqrt{5})/25498 = [-1, 4, 4, 5, 2, \bar{1}],$$

$$Q_5 = 2, A_3/B_3 = -68/89. \text{ Also } A_4/B_4 = -149/195.$$

Exceptional solution:

$$(X, y) = (A_4 - A_3, B_4 - B_3) = (-81, 106), (x, y) = (239, 106).$$

$$\omega^* = [-1, 4, 4, 7, \bar{1}] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (271, 123).$$

Summarising:

$19x^2 - 85xy + 95y^2 = -671.$	
Solution	$n \pmod{1342}$
(226, 101)	279
(251, 111)	-279
(239, 106)	645
(229, 102)	-645

There are 4 equivalence classes of solutions.

7. Appendix

Lemma 4. *Let*

$$\omega = \frac{-(2P+1) + \sqrt{5}}{2Q} = [a_0, \dots, a_r, \bar{1}],$$

$$\omega^* = \frac{-(2P+1) - \sqrt{5}}{2Q} = [b_0, \dots, b_s, \bar{1}],$$

where $a_r > 1$ if $r > 0$ and $b_s > 1$ if $s > 0$. Then $r - 1 \equiv s \pmod{2}$ and

$$A_r - A_{r-1} = A'_s - A'_{s-1} \text{ and } B_r - B_{r-1} = B'_s - B'_{s-1}.$$

Proof. We have

$$\omega = \frac{-P + \alpha^{-1}}{Q} = \frac{A_r \alpha + A_{r-1}}{B_r \alpha + B_{r-1}}$$

and hence

$$(7.1) \quad QA_r = -PB_r + B_{r-1}$$

$$(7.2) \quad -QA_{r-1} = -B_r + (P+1)B_{r-1}.$$

Then (7.1) gives

$$(7.3) \quad B_{r-1} = PB_r + QA_r,$$

and (7.2) and (7.3) give

$$\begin{aligned} -QA_{r-1} &= -B_r + (P+1)(PB_r + QA_r) \\ &= -B_r + P(P+1)B_r + Q(P+1)A_r \\ &= QR B_r + Q(P+1)A_r. \end{aligned}$$

Hence

$$(7.4) \quad -A_{r-1} = RB_r + (P+1)A_r.$$

Also (7.2) and (7.3) imply

$$(7.5) \quad -A_r = PA_{r-1} + RB_{r-1}.$$

Now let $X = A_r - A_{r-1}$, $y = B_r - B_{r-1}$.

Hence

$$\begin{aligned} QX^2 + (2P+1)Xy + Ry^2 &= X(QX + Py) + y((P+1)X + Ry) \\ &= (A_r - A_{r-1})(2B_{r-1} - B_r) \\ &\quad + (B_r - B_{r-1})(A_r - 2A_{r-1}) \\ &= A_r B_{r-1} - A_{r-1} B_r = (-1)^{r-1}. \end{aligned}$$

Now $y = B_r - B_{r-1} \geq B_r - 2B_{r-1} = -(QX + Py) \geq 0$.

Similarly with

$$\omega^* = \frac{-P - \alpha}{Q} = \frac{A'_s \alpha + A'_{s-1}}{B'_s \alpha + B'_{s-1}}$$

and with $X = A'_s - A'_{s-1}$, $y = B'_s - B'_{s-1}$, we find

$$\begin{aligned} QX + Py &= -B'_{s-1} \\ (P+1)X + Ry &= A'_{s-1}, \end{aligned}$$

and

$$QX^2 + (2P+1)Xy + Ry^2 = (-1)^s.$$

Also

$$y = B'_s \alpha - B'_{s-1} \geq B'_{s-1} = -(QX + Py) \geq 0.$$

It follows from cases (ii) and (iii)(c) in the proof of Lemma 3, that

$$(-1)^{r-1}Q < 0, \quad (-1)^sQ < 0$$

and

$$A_r - A_{r-1} = A'_s - A'_{s-1} \text{ and } B_r - B_{r-1} = B'_s - B'_{s-1}.$$

□

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References

- [1] G. CORNACCHIA, *Su di un metodo per la risoluzione in numeri interi dell' equazione* $\sum_{h=0}^n C_h x^{n-h} = P$. *Giornale di Matematiche di Battaglini* **46** (1908), 33–90.
- [2] A. FAISANT, *L'equation diophantienne du second degré*. Hermann, Paris, 1991.
- [3] C. F. GAUSS, *Disquisitiones Arithmeticae*. Yale University Press, New Haven, 1966.
- [4] G. H. HARDY, E. M. WRIGHT, *An Introduction to Theory of Numbers*, Oxford University Press, 1962.
- [5] L. K. HUA, *Introduction to Number Theory*. Springer, Berlin, 1982.
- [6] G. B. MATHEWS, *Theory of numbers*, 2nd ed. Chelsea Publishing Co., New York, 1961.
- [7] K. R. MATTHEWS, *The Diophantine equation* $x^2 - Dy^2 = N, D > 0$. *Exposition. Math.* **18** (2000), 323–331.
- [8] R. A. MOLLIN, *Fundamental Number Theory with Applications*. CRC Press, New York, 1998.
- [9] A. NITAJ, *Conséquences et aspects expérimentaux des conjectures abc et de Szpiro*. Thèse, Caen, 1994.
- [10] M. PAVONE, *A Remark on a Theorem of Serret*. *J. Number Theory* **23** (1986), 268–278.
- [11] J. A. SERRET (Ed.), *Oeuvres de Lagrange, I–XIV*, Gauthiers–Villars, Paris, 1877.
- [12] J. A. SERRET, *Cours d'algèbre supérieure*, Vol. I, 4th ed. Gauthiers–Villars, Paris, 1877.
- [13] T. SKOLEM, *Diophantische Gleichungen*, Chelsea Publishing Co., New York, 1950.

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