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VALENTIN E. VOSKRESENSKII

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# Stably rational algebraic tori

par VALENTIN E. VOSKRESENSKII

RÉSUMÉ. On montre qu'un tore stablement rationnel avec un corps de décomposition cyclique est rationnel.

ABSTRACT. The rationality of a stably rational torus with a cyclic splitting field is proved.

Let $X$ be an irreducible algebraic variety over a field $k$ of characteristic zero. We call $X$ *stably rational* if $X \times_k \mathbf{A}^m$ is rational for some $m$. In 1984 the first and hitherto only known examples of stably rational but non rational varieties were constructed [1]. Such examples exist over nonclosed fields and over the complex field $\mathbf{C}$. Let now $X = T$ be an algebraic torus over a nonclosed field $k$. In the category of algebraic tori we have a criterion of stable rationality which enables us to construct stably rational tori. However, among them one has not yet found a nonrational torus.

**Conjecture.** *Any torus which is stably rational over $k$ is rational over $k$.*

In this paper we suggest a new approach to this conjecture. In particular, we prove the rationality of any stably rational torus with a cyclic splitting field.

First recall some facts and definitions. Let $T$ be an algebraic torus defined over a field $k$, $L/k$ the normal finite splitting field of $T$, $\Pi = Gal(L/k)$, and $\hat{T}$ the $\Pi$-module of rational characters of $T$. An important rôle is played by quasi-split tori. A torus $S$ over $k$ is called quasi-split if the module $\hat{S}$ has a basis permuted by $\Pi$. For example, any maximal $k$-torus of the group $GL_k(n)$ is quasi-split. Dividing the permutation basis into orbits of $\Pi$, we get a representation of the $\Pi$-module $\hat{S}$ as a direct sum of indecomposable permutation modules. This construction gives a representation of the torus $S$ as a direct product

$$S = R_{F_1/k}(G_m) \times \cdots \times R_{F_t/k}(G_m). \tag{1}$$

The group $S$ is a maximal torus in the general linear group $GL_k(n)$, $n = \dim S$, and

$$S(k) = F_1^* \times \cdots \times F_t^*, \ k \subset F_i \subset L.$$

The following conditions are equivalent [2]:
a) $T$ is stably rational over $k$ ;

b) $T$ can be included in an exact sequence of the form

$$1 \to S_1 \xrightarrow{\alpha} S_2 \xrightarrow{\beta} T \to 1,$$

where $S_1$ and $S_2$ are quasi-split $k$-tori.
Since $S_1$ is quasi-split, $H^1(M/F, S_1(M)) = 0$ for any Galois extension $M/F$, $F \supset k$. From here it follows that there exists a rational $k$-section $\gamma : T \to S_2$ of the morphism $\beta$, $\beta \cdot \gamma = Id$. Let $\delta : S_2 \to S_1$ be the rational $k$-map defined by

$$\delta(g) = \alpha^{-1}(g/\gamma\beta(g)),\ g \in S_2(M),\ M \supset k.$$

Clearly

$$\delta(\alpha(h)g) = h\delta(g),\ h \in S_1(M).$$

The map $\delta$ will be called a *covariant* of the representation $\alpha$. (The map $\gamma : S_2 \to S_1$ with the condition $\gamma(\alpha(h)g) = h\gamma(g)$ is called a covariant of the representation $\alpha$.)

We have a rational map

$$\varphi = (\beta, \delta) : S_2 \to T \times_k S_1,$$

which is birational over $k$. Indeed, let $\varphi(g_1) = \varphi(g_2)$. Then $\beta(g_1) = \beta(g_2)$ and we have $g_2 = \alpha(h)g_1$. Hence $\delta(g_1) = \delta(g_2) = h\delta(g_1)$. Since the element $\delta(g_1)$ is invertible, $h = 1$, i.e. $g_1 = g_2$. Since $\dim S_2 = \dim S_1 + \dim T$, all varieties are irreducible and $\varphi$ is injective on an open subset, we conclude that $\varphi$ is birational over $k$. We established the following fact.

**Proposition.** *Let $\delta : S_2 \to S_1$ be a covariant of an exact representation $\alpha : S_1 \to S_2$ and $W = \delta^{-1}(a)$ the fibre of $\delta$ over $a \in S_1(k)$. Then the varieties $T = S_2/\alpha(S_1)$ and $W$ are birationally equivalent over $k$. All the fibres of $\delta$ are stably rational over $k$.*

This proposition allows us to reformulate the question on stably rational tori in terms of linear representations. Consider one component $R_{F/k}(G_m)$ of $S_1$, $R_{F/k}(k) = F^*$ , where $F^*$ is the multiplicative group of the field $F$, $(F : k) = r$. Denote by $V_r$ the vector space of dimension $r$ over $k$. We have the regular exact representation of $F^*$ on $V_r$, and this action extends to the action of $R_{F/k}(G_m)$ on the family $V_r \otimes_k M$, $M \supset k$, i.e. $R_{F/k}(G_m)$ acts on $V_r$ in the sense of algebraic geometry. Let $U$ be a direct sum of the $V_r$'s corresponding to decomposition (1) of $S_1$, and let $V$ be the analogous sum corresponding to $S_2$. The map $\alpha$ defines a linear representation of $S_1$ on $V$. The groups $S_1$ and $S_2$ act faithfully on $U$ and $V$. Each $S_i$ has an open orbit in the corresponding space. We shall sometimes identify this orbit and the group itself. We can now extend the covariant $\delta : S_2 \to S_1$ to a rational covariant

$$\delta : V \to U,\ \delta(\alpha(h)v) = h\delta(v),\ h \in S_1(M),\ v \in V(M) = V \otimes_k M.$$

**Remark.** Since the field $k$ is of characteristic zero, any representation of $S_1$ can be studied at the level of the group $S_1(k)$, i.e. we can consider usual linear representations of groups of the type $F^*$.

Thus the field of rational functions $k(T)$ of a stably rational torus $T$ is the field of invariants of the quasi-split torus $S_1$ acting faithfully on $V$ by the monomorphism $\alpha$ :

$$k(T) = k(V)^{S_1}.$$

Obviously the converse is also true, i.e. if a quasi-split torus $S$ acts faithfully on a linear space $V$, the field $k(V)^S$ is stably rational over $k$. Indeed, the torus $S$ is a subgroup of a maximal $k$-torus $S'$ of $GL(V)$ and the field $k(V)^S$ is the field of rational functions of the stably rational torus $S'/S$.

**Example.** Let $L$ be a finite extension of a field $k$ and $F$ a subfield of $L$, $k \subset F \subset L$. We have an embedding

$$\alpha : S_1 = R_{F/k}(G_m) \to R_{L/k}(G_m) = S_2.$$

Consider the quotient $T = S_2/S_1$. We have an epimorphism of linear spaces over $k$

$$\delta = Tr_{L/F} : L \to F,\ \delta(ax) = a\delta(x),\ a \in F,\ x \in L,$$

i.e. $\delta$ is a covariant of $\alpha$. The fibre $W = \delta^{-1}(1)$ is an affine space over $k$, $\dim W = \dim T$. The varieties $T$ and $W$ are birationally equivalent, hence the torus $T$ is rational over $k$.

We now consider a more complicated example and presents a new approach to solution these problem. Let $V_m$ and $V_n$ be vector $k$-spaces, $S_m$ and $S_n$ maximal $k$-tori in $GL(V_m)$ and $GL(V_n)$. Then we have a representation of $S_m \times_k S_n$ in the tensor space $V_m \otimes_k V_n = V$. Let $N$ be the image of $S_m \times_k S_n$ in $GL(V)$ under this tensor representation. The group $N$ is contained in a maximal $k$-torus $S$ of $GL(V)$. There are two exact sequences of $k$-tori

$$1 \to G_{m,k} \to S_m \times_k S_n \to N \to 1, \tag{2}$$

$$1 \to N \to S \to T \to 1. \tag{3}$$

We ask whether $T$ is $k$-rational. Let us write down the exact sequences of $\Pi$-modules dual to (2) and (3)

$$0 \to \hat{N} \to \hat{S}_m \oplus \hat{S}_n \xrightarrow{\varepsilon} \mathbf{Z} \to 0, \tag{4}$$

$$0 \to \hat{T} \to \hat{S} \to \hat{N} \to 0, \tag{5}$$

where $\Pi = Gal(L/k)$, $L$ is the normal finite splitting field of all tori in sequences (2) and (3). We only consider the case $(m, n) = 1$. From (2) we

obtain the cohomology exact sequence

$$(\hat{S}_m \oplus \hat{S}_n)^{\Pi} \xrightarrow{\varepsilon} \mathbf{Z} \to H^1(\Pi, \hat{N}) \to 0.$$

The group $Im(\varepsilon)$ contains the integers of the form $am + bn$, this implies that $\varepsilon$ is an epimorphism. Hence $H^1(\Pi, \hat{N}) = 0$, it follows that sequence (4) splits, i.e.

$$\hat{N} \oplus \mathbf{Z} \cong \hat{S}_m \oplus \hat{S}_n. \tag{6}$$

This proves that $T$ is stably rational. Note that (6) implies $H^1(k, N) = 0$, i.e. any principal homogeneous space $X$ of $N$ is trivial, $X(k) \cong S_m(k)S_n(k)$ in $S(k)$.

One can view $S_m$ and $S_n$ as subgroups of $GL_k(V)$, $V = V_m \otimes_k V_n$. We denote by $R$ the set of split tensors in $V_m \otimes_k V_n$. The group $S_m \times_k S_n$ acts on $R$, and $R$ contains an open orbit of $S_m \times_k S_n$, $\dim R = m + n - 1$. Let $D_m$ be a maximal $k$-diagonal subgroup of $GL_k(V_m)$ calculated with respect to a certain $k$–basis of $V_m$, $D_n$ is defined analogously. The set $R$ is stable under $D_m \times_k D_n$, and $R$ contains an open orbit of $D_m \times_k D_n$. Let $D$ be the maximal $k$-diagonal torus of $GL(V)$ which contains $Im(D_m \times_k D_n) = D_m D_n \subset GL_k(V)$. The factor–group $D/D_m D_n = D_0$ is a split $k$–torus hence it is $k$–rational. We have a decomposition $D = D_m D_n \times_k D_0$ as a direct product, $\dim D_0 = (m-1)(n-1)$. Considering the variety $D$ as an open orbit in $V$ and taking into account that orbit of the group $D_m D_n$ is open in $R$, we obtain a birational decomposition $V \cong R \times_k D_0$. Because $R$ is invariant set with respect to the action of the group $S_m \times S_n$ on $V$ hence the variety $T = S/S_m S_n$ is birationally equivalent to $D_0$ over $k$, $S_m S_n = Im(S_m \times_k S_n) \subset GL_k(V)$. The result of our discussion can be summed up as follows.

**Theorem 1.** *Let $S_i$ be a maximal $k$-torus of $GL_k(V_i)$. Then the quotient space $(V_m \otimes_k V_n)/(S_m \times_k S_n)$ is rational over $k$ if $(m, n) = 1$.* $\triangle$

This theorem was proved by Klyachko [4] by another method. Now consider the problem of rationality of tori with a cyclic splitting field. The idea used in the proof of Theorem 1 allows us to make a step forward in the problem of rationality of tori.

Let $L/k$ be a cyclic extension, $\Pi = Gal(L/k)$ is the cyclic group of order $n$ with generator $\sigma$. Let $\mathbf{Z}[\zeta_n]$ be the ring of integers in the cyclotomic field $\mathbf{Q}[\zeta_n]$ , where $\zeta_n$ is a primitive $n$-th root of unity. Define a $\Pi$-module structure on $\mathbf{Z}[\zeta_n]$ by putting $\sigma(\alpha) = \zeta_n \alpha$. Let $T_n$ be the $k$-torus with character module $\hat{T}_n = \mathbf{Z}[\zeta_n]$. It is known that all tori $T_n$ are stably rational [2]. Chistov [5] proved that any stably rational $k$-torus $T$ with a cyclic splitting field is birationally equivalent over $k$ to the product of tori of the form $T_n$. Moreover, it suffices to check their rationality in the case when $n$ is square-free [3].

Thus, let $L_n$ be a cyclic extension of $k$ of degree $n$, $\Pi_n = Gal(L_n/k)$, and $T_n$ the $L_n/k$-torus with character module $\hat{T}_n = \mathbf{Z}[\zeta_n]$, $n = p_1 \cdots p_t$ is square-free, $p_i$ is a prime number. If $n = p$ is a prime number there is an exact sequence of $\Pi_p$-modules

$$0 \to \mathbf{Z}[\zeta_p] \to \mathbf{Z}[\Pi_p] \to \mathbf{Z} \to 0, \tag{7}$$

whence by duality one concludes that the torus $T_p$ is isomorphic to the quotient $R_{L_p/k}(G_m)/G_{m,k}$ which, in turn, is an open subset in the projective space $\mathbf{P}^{p-1}$. Thus $T_p$ is $k$-rational. Now let $t \geq 2$. We have an epimorphism $\mathbf{Z}[\Pi_n] \to \mathbf{Z}[\Pi_n/\Pi_p]$ for every $p|n$, whence the following exact sequence of $\Pi_n$-modules

$$0 \to \mathbf{Z}[\zeta_n] \to \mathbf{Z}[\Pi_n] \to \oplus_{p|n} \mathbf{Z}[\Pi_n/\Pi_p]. \tag{8}$$

The dual sequence of $k$-tori is of the form

$$\prod_{i=1}^{t} R_{F_i/k}(G_m) \to R_{L/k}(G_m) \to T_n \to 1,$$

where $F_i$ is the subfield of $L = L_n$, $(F_i : k) = n/p_i$.

Short sequence (8) is a part of the long exact sequence obtained by tensoring resolutions of the form (7). It is convenient to describe this situation in the language of tensor representations. Let $L_p$ be the subfield of $L_n$, $(L_p : k) = p$. Then

$$L_n = L_{p_1} \otimes \cdots \otimes L_{p_t}, \; F_i = \otimes_{m \neq i} L_{p_m}, \; 1 \leq i \leq t.$$

We have embeddings of fields $\psi_i : F_i \to L_n$ which determine natural monomorphisms of groups of linear operators $GL_k(F_i) \to GL_k(L_n)$. The group $F_i^*$ is a maximal $k$-torus of $GL_k(F_i)$, let $D_i$ be the maximal diagonal subgroup of $GL_k(F_i)$ calculated with respect to a certain basis of extension $F_i/k$. Choose a point in general position $v = v_1 \otimes \cdots \otimes v_t$, $v_i \in L_{p_i}$, so that for each $i$ the orbits of $v$ under $D_i$ and $S_i = R_{F_i/k}(G_m)$ are open in $F_i$. Denote by $R$ (resp. $R_1$) the closure of the orbit of $v$ under $S_1 \times \cdots \times S_t$ (resp. $D_1 \times \cdots \times D_t$), we have $R = R_1$. Let $D$ be the maximal $k$-diagonal torus of $GL_k(L_n)$ which contains $Im(D_1 \times \cdots \times D_t) = D_1 \cdots D_t$. The factor-group $D/D_1 \cdots D_t = D_0$ is $k$-rational. The group $D$ is the direct product $D_1 \cdots D_t \times D_0$, $\dim D_0 = (p_1 - 1) \cdots (p_t - 1)$. We have a birational decomposition into direct product

$$D_0 \times R \cong L_n.$$

The group $S_1 \times \cdots \times S_t$ acts on $R \times D_0$ birationally, respecting orbits in $R$. Let $g \in S_1 \times \cdots \times S_t$. If $g(v, w) = (gv, w')$ and $gv = v$, then $g = 1$, hence $w' = w$. Therefore the set $v \times D_0$ parametrizes the quotient $R_{L/k}(G_m)/(S_1 \times \cdots \times S_t) = T_n$. We have the following statement.

**Theorem 2.** *Any stably rational torus with a cyclic splitting field is rational over the ground field of characteristic 0.* △

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Valentin E. VOSKRESENSKII
Dept. of Math.
Samara State University
Acad. Pavlov str.1
Samara, 443011, Russia
*E-mail* : voskres@info.ssu.samara.ru