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EQUATIONS ON THE SEMIDIRECT PRODUCT OF A FINITE SEMILATTICE BY A \mathcal{J} -TRIVIAL MONOID OF HEIGHT k (*)

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Abstract. – Let \mathbf{J}_k denote the k th level of Simon's hierarchy of \mathcal{J} -trivial monoids. The 1st level \mathbf{J}_1 is the \mathbf{M} -variety of finite semilattices. In this paper, we give a complete sequence of equations for the product $\mathbf{J}_1 \star \mathbf{J}_k$ generated by all semidirect products of the form $M \star N$ with $M \in \mathbf{J}_1$ and $N \in \mathbf{J}_k$. Results of Almeida imply that this sequence of equations is complete for the product \mathbf{J}_1^{k+1} or $\mathbf{J}_1 \star \dots \star \mathbf{J}_1$ ($k+1$ times) generated by all semidirect products of $k+1$ finite semilattices and that $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by a finite sequence of equations if and only if $k = 1$. The equality $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ implies that a conjecture of Pin concerning tree hierarchies of \mathbf{M} -varieties is false.

Résumé. – Soit \mathbf{J}_k le niveau k de la hiérarchie de Simon des monoïdes \mathcal{J} -triviaux. Le premier niveau \mathbf{J}_1 est la \mathbf{M} -variété des monoïdes idempotents et commutatifs ou demi-treillis. Dans cet article, nous donnons une suite complète d'équations pour le produit $\mathbf{J}_1 \star \mathbf{J}_k$ engendré par les produits semidirects de la forme $M \star N$ avec $M \in \mathbf{J}_1$ et $N \in \mathbf{J}_k$. Des résultats d'Almeida entraînent que cette suite d'équations est aussi complète pour le produit \mathbf{J}_1^{k+1} ou $\mathbf{J}_1 \star \dots \star \mathbf{J}_1$ ($k+1$ fois) engendré par les produits semidirects de $k+1$ demi-treillis et que $\mathbf{J}_1 \star \mathbf{J}_k$ est défini par une suite finie d'équations si et seulement si $k = 1$. L'égalité $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ entraîne qu'une conjecture de Pin concernant des hiérarchies d'arbres de \mathbf{M} -variétés est fautive.

1. INTRODUCTION

Let \mathbf{J}_k denote the \mathbf{M} -variety of \mathcal{J} -trivial monoids of height k . The first level \mathbf{J}_1 is the \mathbf{M} -variety of finite semilattices. In this paper, we give an equational characterization of the product $\mathbf{J}_1 \star \mathbf{J}_k$ generated by all semidirect products of the form $M \star N$ with $M \in \mathbf{J}_1$ and $N \in \mathbf{J}_k$. A result of Almeida [3] gives an equational characterization of the product

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$\mathbf{J}_1 \star \dots \star \mathbf{J}_1$ ($k + 1$ times) or \mathbf{J}_1^{k+1} , which turns out to be our equational characterization of $\mathbf{J}_1 \star \mathbf{J}_k$. The equality $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ implies that a conjecture of Pin concerning tree hierarchies of \mathbf{M} -varieties is false. Almeida [3] implies that $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by a *finite* sequence of equations if and only if $k = 1$. The methods used in this paper were developed by Almeida [1], [2].

1.1 Preliminaries

The reader is referred to the books of Eilenberg [15], Lallement [19] or Pin [20] for terminology not defined in this paper.

Let A be a finite set called an alphabet, whose elements are called letters. We will denote by A^* the *free monoid* over A . The elements of A^* are the finite sequences of letters called words. The empty word (denoted by 1) corresponds to the empty sequence.

Let L be a subset of A^* (or a *language* over A) and \sim be an equivalence relation on A^* . We say that \sim saturates L if L is a union of classes modulo \sim or for every $u, v \in A^*$, $u \sim v$ and $u \in L$ imply $v \in L$.

The *syntactic congruence* of L is the congruence \sim_L on A^* defined by $u \sim_L v$ if and only if for every $x, y \in A^*$, $xuy \in L$ if and only if $xvy \in L$. We can show that \sim_L is the coarsest congruence saturating L . The *syntactic monoid* of L is the quotient monoid $M(L) = A^* / \sim_L$.

Let S and T be semigroups. We say that S is a *quotient* of T if there exists a surjective morphism $\varphi : T \rightarrow S$ and we say that S *divides* T ($S \prec T$) if S is a quotient of a submonoid of T . The division relation is transitive. The syntactic monoid of a language L is the smallest monoid recognizing L , where smallest is taken in the sense of the division relation.

A *variety* V is a class of semigroups closed under division and products. By the well-known theorem of Birkhoff such a variety is defined by equations that must hold for all elements of semigroups in V . Thus equations give rise to varieties.

An \mathbf{S} -*variety* is a class of finite semigroups closed under division and finite products and an \mathbf{M} -*variety* is a class of finite monoids closed under division and finite products. Equivalently, a class \mathbf{V} of finite monoids is an \mathbf{M} -variety if \mathbf{V} satisfies the following two conditions:

- if $T \in \mathbf{V}$ and $S \prec T$, then $S \in \mathbf{V}$;
- if $S, T \in \mathbf{V}$, then $S \times T \in \mathbf{V}$.

Eilenberg has shown the existence of a bijection between the \mathbf{M} -varieties and some classes of languages called the \star -varieties of languages.

A class \mathcal{V} is a \star -variety of languages if

- for every alphabet A , $A^* \mathcal{V}$ is a set of recognizable languages over A closed under boolean operations;
- if $\varphi : A^* \rightarrow B^*$ is a free monoid morphism, then $L \in B^* \mathcal{V}$ implies $L\varphi^{-1} = \{u \in A^* \mid u\varphi \in L\}$ is in $A^* \mathcal{V}$;
- if $L \in A^* \mathcal{V}$ and $a \in A$, then $a^{-1}L = \{u \in A^* \mid au \in L\}$ and $La^{-1} = \{u \in A^* \mid ua \in L\}$ are in $A^* \mathcal{V}$.

If \mathbf{V} is an \mathbf{M} -variety and A is an alphabet, we denote by $A^* \mathcal{V}$ the set of recognizable languages over A whose syntactic monoid is in \mathbf{V} . Equivalently, $A^* \mathcal{V}$ is the set of languages of A^* recognized by a monoid of \mathbf{V} . If \mathcal{V} is a \star -variety of languages, we denote by \mathbf{V} the \mathbf{M} -variety generated by the monoids of the form $M(L)$ where $L \in A^* \mathcal{V}$ for some alphabet A .

A result of Simon enables us to describe the \star -variety of languages corresponding to the \mathbf{M} -variety of \mathcal{J} -trivial monoids denoted by \mathbf{J} .

A word $a_1 \dots a_i \in A^*$ is a subword of a word u of A^* if there exist words $u_0, u_1, \dots, u_i \in A^*$ such that $u = u_0 a_1 u_1 \dots a_i u_i$. For each integer $k \geq 0$, we define an equivalence relation \sim_k on A^* by $u \sim_k v$ if and only if u and v have the same subwords of length less than or equal to k . We can verify that \sim_k is a congruence on A^* with finite index. Note that $u \sim_1 v$ if and only if u and v have the same letters. The set of letters that occur in a word u will be denoted by $u\alpha$.

A language L over A is called *piecewise testable* if it is a union of classes modulo \sim_k for some integer k , or equivalently if it is in the boolean algebra generated by all languages of the form $A^* a_1 A^* \dots a_i A^*$ where $i \geq 0$ and $a_1, \dots, a_i \in A$. Simon [24] has proved that a language is piecewise testable if and only if its syntactic monoid is \mathcal{J} -trivial. For every alphabet A , we will denote by $A^* \mathcal{J}_k$ the boolean algebra generated by all languages of the form $A^* a_1 A^* \dots a_i A^*$, where $0 \leq i \leq k$ and $a_1, \dots, a_i \in A$. One can show that \mathcal{J}_k is a \star -variety of languages and we will denote by \mathbf{J}_k the corresponding \mathbf{M} -variety. The \mathbf{M} -variety \mathbf{J} is the union of the \mathbf{M} -varieties \mathbf{J}_k .

1.2 Product of varieties of semigroups

Let S and T be semigroups. To simplify the notation we will represent S additively (without necessarily supposing that S is commutative) and T multiplicatively.

An *action* of T on S is a function

$$\begin{aligned} T \times S &\rightarrow S \\ (t, s) &\mapsto ts \end{aligned}$$

satisfying for every $t, t' \in T$ and $s, s' \in S$:

- $t(s + s') = ts + ts'$;
- $t(t' s) = (tt') s$.

Given an action of T on S , the *semidirect product* $S \star T$ is the semigroup defined on $S \times T$ by the multiplication

$$(s, t)(s', t') = (s + ts', tt').$$

The multiplication in $S \star T$ is associative. Thus $S \star T$ is a semigroup.

In this paper, we only consider semidirect products $S \star T$ given by actions of T on S that are described by monoid homomorphisms $\varphi : T^1 \rightarrow \text{End } S$ from T^1 into the monoid of endomorphisms of S . In the terminology adopted by Eilenberg [15], this means that we only consider left unitary actions, that is actions of T on S that satisfy $1s = s$ for every $s \in S$. Here T^1 denotes the semigroup $T \cup \{1\}$ obtained from T by adjoining an identity if T does not have one, and $T^1 = T$ otherwise.

If V and W are varieties of semigroups, the product $V \star W$ is the variety generated by all semigroups of the form $S \star T$ with $S \in V$ and $T \in W$. The product of two **S**-varieties (or **M**-varieties) is defined analogously. The operation \star defined on varieties is associative.

There remain many problems to be solved on products of **S**-varieties (or **M**-varieties). The most important of these is the following. Given two decidable **S**-varieties (or **M**-varieties), is the product decidable? A particular case of this problem is well known in the theory of semigroups. Karnofsky and Rhodes [18] have established the decidability of the **M**-varieties $\mathbf{A} \star \mathbf{G}$ and $\mathbf{G} \star \mathbf{A}$. Here, \mathbf{A} denotes the **M**-variety of aperiodic monoids and \mathbf{G} the **M**-variety of groups.

This paper deals in particular with products of the form \mathbf{J}_1^k . It is known that $\bigcup_{k \geq 0} \mathbf{J}_1^k$ is the **M**-variety \mathbf{R} of all finite \mathcal{R} -trivial monoids (Stiffler [25]) and that \mathbf{J}_1^k is decidable (Pin [21]).

1.3 Equations on products of varieties of semigroups

Let A^+ be the free semigroup over a denumerable alphabet A and let $u, v \in A^+$. We say that a semigroup S satisfies the equation $u = v$ or the equation $u = v$ holds in S (and we write $S \models u = v$) if for every morphism $\varphi : A^+ \rightarrow S$, $u\varphi = v\varphi$. This means that, if we substitute elements of S for the letters in u and v , we reach equalities in S . For example, S is idempotent if it satisfies the equation $x = x^2$ and S is commutative if it satisfies the equation $xy = yx$. For a sequence \mathcal{E} of equations and an equation $u = v$, $\mathcal{E} \vdash u = v$ (and we say $u = v$ is deducible from \mathcal{E}) means that for every semigroup S , if $S \models \mathcal{E}$, then $S \models u = v$.

Let $\mathbf{V}(u, v)$ be the class of finite semigroups S satisfying the equation $u = v$. It is easy to show that $\mathbf{V}(u, v)$ is an \mathbf{S} -variety.

Let $(u_i, v_i)_{i>0}$ be a sequence of pairs of words of A^+ . Consider the following \mathbf{S} -varieties:

$$\mathbf{W} = \bigcap_{i>0} \mathbf{V}(u_i, v_i)$$

$$\mathbf{W}' = \bigcup_{I>0} \bigcap_{i \geq I} \mathbf{V}(u_i, v_i).$$

We say that \mathbf{W} is *defined* by the equations $u_i = v_i$ ($i > 0$). This corresponds to the fact that a finite semigroup is in \mathbf{W} if and only if it satisfies the equations $u_i = v_i$ for every $i > 0$. We say that \mathbf{W}' is *ultimately defined* by the equations $u_i = v_i$ ($i > 0$). This corresponds to the fact that a finite semigroup is in \mathbf{W}' if and only if it satisfies the equations $u_i = v_i$ for every i sufficiently large.

The arguments above apply equally well to \mathbf{M} -varieties. We only need to replace A^+ by A^* throughout.

Eilenberg and Schützenberger [16] have proved the following result. Every nonempty \mathbf{M} -variety is ultimately defined by a sequence of equations, or every \mathbf{S} -variety containing the trivial semigroup is ultimately defined by a sequence of equations. If \mathbf{V} is the \mathbf{S} -variety ultimately defined by the equations $u_i = v_i$, $i > 0$, then the same equations ultimately define the \mathbf{M} -variety consisting of all the monoids in \mathbf{V} . Also every \mathbf{M} -variety generated by a single monoid is defined by a (finite or infinite) sequence of equations.

Equational characterizations of all the \mathbf{M} -varieties \mathbf{J}_k are known [23], [5], [6], [10], [11]. In particular,

- the \mathbf{M} -variety \mathbf{J}_1 is defined by the equations $x = x^2$ and $xy = yx$, so \mathbf{J}_1 is the \mathbf{M} -variety of idempotent and commutative monoids;
- the \mathbf{M} -variety \mathbf{J}_2 is defined by the equations $xyzx = xyxz$ and $(xy)^2 = (yx)^2$;
- the \mathbf{M} -variety \mathbf{J}_3 is defined by the equations $xzyxvxy = xzxyvxy$, $ywxvxyzx = ywvxvxyzx$ and $(xy)^3 = (yx)^3$.

DEFINITION 1.1: Let $k \geq 1$ and let $A = \{x_1, x_2, \dots\}$ be a denumerable alphabet of variables including x ($x = x_1$).

\mathcal{E}_k is the sequence of all equations (over A) of the form

$$u_i \dots u_1 v_1 \dots v_j = u_i \dots u_1 x v_1 \dots v_j$$

where

$$\{x\} \subseteq u_1 \alpha \subseteq \dots \subseteq u_i \alpha$$

$$\{x\} \subseteq v_1 \alpha \subseteq \dots \subseteq v_j \alpha$$

and where $i + j = k$.

THEOREM 1.1 [10]: Let $k \geq 1$. The \mathbf{M} -variety \mathbf{J}_k is defined by \mathcal{E}_k .

These results lead to the following question. Can the \mathbf{M} -varieties \mathbf{J}_k be defined by a *finite* sequence of equations? This question has been answered in [11]. The \mathbf{M} -varieties \mathbf{J}_k can be defined by a *finite* sequence of equations if and only if $k = 1, 2$ or 3 .

Equations are known for the product of the \mathbf{S} -variety of semilattices, groups, and \mathcal{R} -trivial semigroups by the \mathbf{S} -variety of locally trivial semigroups [15]. These results have important applications to language theory [14], [15].

Pin [22] has shown that the \mathbf{M} -variety $\mathbf{J}_1 \star \mathbf{J}_1$ is defined by the equations $xux = xux^2$ and $xuyvxy = xuyvxy$. A result of Irastorza [17] shows that the \mathbf{M} -varieties $\mathbf{J}_1 \star (Z_k)$ are not defined by finite sequences of equations. Here, (Z_k) denotes the \mathbf{M} -variety generated by the cyclic group Z_k of order k which is defined by the equations $x^k = 1$ and $xy = yx$. Almeida [3] has shown that \mathbf{J}_1^k is defined by a finite sequence of equations if and only if $k = 1$ or 2 . Ash [4] has shown that $\mathbf{J}_1 \star \mathbf{G} = \mathbf{Inv}$ is defined by the equation $x^\omega y^\omega = y^\omega x^\omega$. The \mathbf{M} -variety of groups \mathbf{G} is defined by the equation $x^\omega = 1$, and \mathbf{Inv} denotes the \mathbf{M} -variety generated by the inverse semigroups.

2. ON A COMPLETE SEQUENCE OF EQUATIONS FOR $\mathbf{J}_1 \star \mathbf{J}_k$

In this section, in order to simplify the notation, we will denote also by \mathbf{J}_k the \mathbf{S} -variety generated by \mathbf{J}_k . It will be convenient to denote by \mathbf{J}_0 the \mathbf{S} -variety defined by the equation $x = y$. In this section, we work essentially with semigroups.

Our results follow from an approach to the semidirect product that was introduced in Almeida [1].

The free object on the set X in the variety generated by an \mathbf{S} -variety (or \mathbf{M} -variety) \mathbf{V} will be denoted by $F_X \mathbf{V}$. We will also write $F_i \mathbf{V}$ as an abbreviation for $F_{\{x_1, \dots, x_i\}} \mathbf{V}$. For every $i \geq 1$ and $k \geq 1$, the free object $F_i(\mathbf{J}_k)$ can be viewed as a set of representatives of classes modulo \sim_k of words over $\{x_1, \dots, x_i\}$. This set is finite. For $i \geq 1$ and $k \geq 1$, let $p_{i,k} : \{x_1, \dots, x_i\}^+ \rightarrow F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ be the canonical projection that maps the letter x_j onto the generator x_j of $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$, and let $q_{i,k} : \{x_1, \dots, x_i\}^+ \rightarrow F_i(\mathbf{J}_k)$ be the canonical projection that maps the letter x_j onto the generator x_j of $F_i(\mathbf{J}_k)$. If $u \in \{x_1, \dots, x_i\}^+$, then $uq_{i,k}$ can be viewed as a representative of the class modulo \sim_k of u .

DEFINITION 2.1: Let $k \geq 1$ and $u \in \{x_1, \dots, x_i\}^+$.

$u\alpha_{i,k}$ is the set of all pairs of the form

$$(u'q_{i,k}, x) \in (F_i(\mathbf{J}_k))^1 \times \{x_1, \dots, x_i\}$$

where $u = u'xu''$ for some $u', u'' \in \{x_1, \dots, x_i\}^*$.

In the case of $k = 0$, $(F_i(\mathbf{J}_0))^1 = \{1\}$ and so $u\alpha_{i,0} = \{1\} \times u\alpha$.

The following lemmas will help us give an equational characterization of $\mathbf{J}_1 \star \mathbf{J}_k$. Lemma 2.1 provides an algorithm to decide when an equation holds in $\mathbf{J}_1 \star \mathbf{J}_k$.

LEMMA 2.1: Let $k \geq 0$ and $u, v \in \{x_1, \dots, x_i\}^+$. Then

$$\mathbf{J}_1 \star \mathbf{J}_k \models u = v$$

if and only if $u\alpha_{i,k} = v\alpha_{i,k}$.

Proof: For $k = 0$, we have that $\mathbf{J}_1 \models u = v$ if and only if $u\alpha = v\alpha$. Since $F_i(\mathbf{J}_k)$ is finite for every $i \geq 1$ and $k \geq 1$, a representation of free objects for a semidirect product of \mathbf{S} -varieties obtained in [1] implies that $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ is also finite for every $i \geq 1$ and $k \geq 1$. Moreover, there

is an embedding of $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ into $F_Y(\mathbf{J}_1) \star F_i(\mathbf{J}_k)$ that maps x_j into $((1, x_j), x_j)$. Here $Y = (F_i(\mathbf{J}_k))^1 \times \{x_1, \dots, x_i\}$ and the action in the semidirect product of the free objects is given by $x_j(s, x_{j'}) = (x_j s, x_{j'})$ for $s \in (F_i(\mathbf{J}_k))^1$. The word $x_{j_1} \dots x_{j_r}$ is mapped into

$$((1, x_{j_1}) + (x_{j_1}, x_{j_2}) + \dots + (x_{j_1} \dots x_{j_{r-1}}, x_{j_r}), x_{j_1} \dots x_{j_r})$$

Suppose that $\mathbf{J}_1 \star \mathbf{J}_k \models u = v$, or that $u p_{i,k} = v p_{i,k}$. This is equivalent to the two conditions $u \alpha_{i,k} = v \alpha_{i,k}$ and $\mathbf{J}_k \models u = v$. Observe that $\mathbf{J}_k \models u = v$ if and only if $u q_{i,k} = v q_{i,k}$. The result follows since $u \alpha_{i,k} = v \alpha_{i,k}$ implies $u q_{i,k} = v q_{i,k}$. \square

Let $k \geq 1$. Let $u, v \in \{x_1, \dots, x_i\}^+$ be such that $u \alpha_{i,k} = v \alpha_{i,k}$. Let $x \in u \alpha$ and consider the first occurrence of x in u .

Case 1. If x is the last letter occurring for the first time in u , then there is a factorization $u = u_1 x u_2$ with $u_1, u_2 \in \{x_1, \dots, x_i\}^*$, $x \notin u_i \alpha$ and $u_2 \alpha \subseteq (u_1 x) \alpha$. In such a case, since $u \alpha_{i,k} = v \alpha_{i,k}$, there is also a factorization $v = v_1 x v_2$ with $v_1, v_2 \in \{x_1, \dots, x_i\}^*$ and $x \notin v_1 \alpha$.

Case 2. If x is not the last letter occurring for the first time in u , then there is a factorization $u = u_1 x u_2 y u_3$ with $u_1, u_2, u_3 \in \{x_1, \dots, x_i\}^*$, $x \notin u_1 \alpha$, $u_2 \alpha \subseteq (u_1 x) \alpha$ and $y \notin (u_1 x u_2) \alpha$. In such a case, since $u \alpha_{i,k} = v \alpha_{i,k}$, there is also a factorization $v = v_1 x v_2 y v_3$ with $v_1, v_2, v_3 \in \{x_1, \dots, x_i\}^*$, $x \notin v_1 \alpha$ and $y \notin (v_1 x v_2) \alpha$.

LEMMA 2.2: *In Case 1 and Case 2, $u_2 \alpha_{i,k-1} = v_2 \alpha_{i,k-1}$.*

Proof: Let $u_2 = u'_2 z u''_2$ with $z \in \{x_1, \dots, x_i\}$. Consider the pair $(u'_2 q_{i,k-1}, z)$ in $u_2 \alpha_{i,k-1}$. The pair $((u_1 x u'_2) q_{i,k}, z)$ is in $u \alpha_{i,k}$. Since $u \alpha_{i,k} = v \alpha_{i,k}$, there is a factorization $v = v' z v''$ with $(u_1 x u'_2) q_{i,k} = v' q_{i,k}$. It follows that the \sim_k -class of $u_1 x u'_2$ is equal to the \sim_k -class of v' and hence $x \in v' \alpha$ and, in Case 2, $y \notin v' \alpha$. Therefore, the chosen occurrence of z in $v = v' z v''$ must be in v_2 . There is then a factorization $v_2 = v'_2 z v''_2$ such that $v' = v_1 x v'_2$. Hence $(u'_2 q_{i,k-1}, z) = (v'_2 q_{i,k-1}, z)$ and the pair $(u'_2 q_{i,k-1}, z)$ is in $v_2 \alpha_{i,k-1}$. Then inclusion $u_2 \alpha_{i,k-1} \subseteq v_2 \alpha_{i,k-1}$ follows. The reverse inclusion is similar. \square

DEFINITION 2.2: *Let $k \geq 1$ and let $A = \{x_1, x_2, x_3, \dots\}$ be a denumerable alphabet of variables including x and y ($u = x_1$ and $y = x_2$).*

C_k *is the sequence of all equations (over A) of the form*

$$u_k \dots u_1 x = u_k \dots u_1 x^2$$

where

$$\{x\} \subseteq u_1 \alpha \subseteq \dots \subseteq u_k \alpha$$

\mathcal{D}_k is the sequence of all equations (over A) of the form

$$u_k \dots u_1 xy = u_k \dots u_1 yx$$

where

$$\{x, y\} \subseteq u_1 \alpha \subseteq \dots \subseteq u_k \alpha.$$

We define \mathcal{C}_0 as the sequence consisting of the equation $x = x^2$ and \mathcal{D}_0 the sequence consisting of $xy = yx$.

Let J_k denote the variety of all semigroups that satisfy all the equations in \mathcal{E}_k . The variety J_k is locally finite, or every finitely generated semigroup in J_k is finite. For a class \mathcal{C} of semigroups, we denote by \mathcal{C}^F the class of all finite semigroups of \mathcal{C} . The equality $\mathbf{J}_k = (J_k)^F$ holds. By [1], if $k \geq 1$, then the equality $(J_1 \star J_k)^F = \mathbf{J}_1 \star \mathbf{J}_k$ holds and $J_1 \star J_k$ is locally finite. Hence $J_1 \star J_k$ is generated by $\mathbf{J}_1 \star \mathbf{J}_k$ and so $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ is the free object on $\{x_1, \dots, x_i\}$ in the variety $J_1 \star J_k$.

THEOREM 2.1: *Let $k \geq 0$. The variety $J_1 \star J_k$ is defined by $\mathcal{C}_k \cup \mathcal{D}_k$.*

Proof: We first want to show that $J_1 \star J_k \models \mathcal{C}_k \cup \mathcal{D}_k$. Let $u, v \in \{x_1, \dots, x_i\}^+$ be such that $u = v$ is an equation in \mathcal{D}_k (the case of equations in \mathcal{C}_k is similar). By Lemma 2.1, it suffices to show that $u \alpha_{i,k} = v \alpha_{i,k}$. Let $u = u_k \dots u_1 xy$ and $v = u_k \dots u_1 yx$ be such that $\{x, y\} \subseteq u_1 \alpha \subseteq \dots \subseteq u_k \alpha$. Note that

$$((u_k \dots u_1) q_{i,k}, x) = ((u_k \dots u_1 y) q_{i,k}, x)$$

since the words $u_k \dots u_1$ and $u_k \dots u_1 y$ are \sim_k -equivalent. Note also that

$$((u_k \dots u_1 x) q_{i,k}, y) = ((u_k \dots u_1) q_{i,k}, y)$$

The equality $u \alpha_{i,k} = v \alpha_{i,k}$ follows.

Conversely, we want to show that if $u, v \in \{x_1, \dots, x_i\}^+$ are such that $u \alpha_{i,k} = v \alpha_{i,k}$, then $\mathcal{C}_k \cup \mathcal{D}_k \vdash u = v$. So, assume that $u \alpha_{i,k} = v \alpha_{i,k}$. Let $x \in u \alpha$ and consider the first occurrence of x in u and v . As in Lemma 2.2, we denote by u_1 (respectively v_1) the longest prefix of u (respectively v) in which the letter x does not occur, and we denote by u_2 (respectively v_2) the longest segment of u (respectively v) following the first occurrence of x in u (respectively v) that does not involve any new letters. By Lemma 2.2, the equality $u_2 \alpha_{i,k-1} = v_2 \alpha_{i,k-1}$ holds. By the inductive hypothesis on

k , we conclude that the equation $u_2 = v_2$ is deducible from $C_{k-1} \cup D_{k-1}$. By a result of [3] (Proposition 2.3), since $C_{k-1} \cup D_{k-1} \vdash u_2 = v_2$ and $u_2 \alpha \subseteq (u_1 x) \alpha$, then $C_k \cup D_k \vdash u_1 x u_2 = u_1 x v_2$.

Let $z \in \{x_1, \dots, x_i\}$. Let u' (respectively v') be the longest prefix of u (respectively v) before the first occurrence of z . We show that the equation $u' = v'$ is deducible from $C_k \cup D_k$. If z is the first letter in u (and so also the first letter in v), then the equation $u' = v'$ becomes $1 = 1$. We assume that it is true for the first occurrence of $z = x$ (as in Lemma 2.2), or $C_k \cup D_k \vdash u_1 = v_1$. Here $u_1 x u_2 = u_1 x v_2 = v_1 x v_2$ is deducible from $C_k \cup D_k$. If x is the last letter occurring for the first time in u (as in Case 1 of Lemma 2.2), we obtain that the equation $u = v$ is deducible from $C_k \cup D_k$. Otherwise, the induction step allows us to proceed until the first occurrence of another letter, say $z = y$ (as in Case 2 of Lemma 2.2). After every letter of u has been found, we obtain the deducibility of the equation $u = v$ from $C_k \cup D_k$. \square

Since $\mathbf{J}_1 \star \mathbf{J}_k = (J_1 \star J_k)^F$, any sequence of equations for $J_1 \star J_k$ is also a sequence of equations for $\mathbf{J}_1 \star \mathbf{J}_k$.

COROLLARY 2.1: *Let $k \geq 0$. The \mathbf{S} -variety $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by $C_k \cup D_k$.*

Note that if two words u and v form an equation $u = v$ for $\mathbf{J}_1 \star \mathbf{J}_k$, then $u \sim_{k+1} v$. Equations for other \mathbf{S} -varieties generalizing the \mathbf{S} -varieties \mathbf{J}_k have been built from properties of congruences generalizing the congruences \sim_k (see [7], [8], [9], [12]).

Pin has given the equational characterization of $\mathbf{J}_1 \star \mathbf{J}_1$ of Theorem 2.2 and Almeida the characterization of \mathbf{J}_1^k of Theorem 2.3.

THEOREM 2.2. (Pin [22]): *The \mathbf{S} -variety $\mathbf{J}_1 \star \mathbf{J}_1$ is defined by $C_1 \cup D_1$ or equivalently by the two equations $xux = xux^2$ and $xuyvxy = xuyvyx$.*

THEOREM 2.3 (Almeida [3]): *Let $k \geq 0$. The \mathbf{S} -variety \mathbf{J}_1^{k+1} is defined by $C_k \cup D_k$.*

From the preceding results, we deduce the following corollary.

COROLLARY 2.2: *Let $k \geq 0$. The \mathbf{S} -varieties $\mathbf{J}_1 \star \mathbf{J}_k$ and \mathbf{J}_1^{k+1} are equal and hence the \mathbf{S} -variety $\mathbf{J}_1 \star \mathbf{J}_k$ is decidable.*

A result of Almeida [3] implies the following.

COROLLARY 2.3: *The \mathbf{S} -variety $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by a finite sequence of equations if and only if $k = 1$.*

As mentioned at the beginning of this section, we have worked essentially with semigroups in section 2. As explained in [3], since the \mathbf{S} -variety generated by the \mathbf{M} -variety \mathbf{J}_k is monoidal, results such as Theorems 2.2 and 2.3, and Corollaries 2.1, 2.2 and 2.3 can be translated to results on the \mathbf{M} -varieties $\mathbf{J}_1 \star \mathbf{J}_k$ and \mathbf{J}_1^{k+1} .

3. ON A CONJECTURE OF PIN

Theorem 3.1 gives a new proof that a conjecture of Pin concerning tree-hierarchies of \mathbf{M} -varieties is false (another proof was given in [13] using different techniques). Let M_1, \dots, M_k be finite monoids. The Schützenberger product of M_1, \dots, M_k , denoted by $\diamond_k(M_1, \dots, M_k)$, is the submonoid of upper triangular $k \times k$ matrices with the usual multiplication of matrices, of the form $x = (x_{ij})$, $1 \leq i, j \leq k$, in which the (i, j) -entry is a subset of $M_1 \times \dots \times M_k$ and all of whose diagonal entries are singletons, that is

1. $x_{ij} = \emptyset$ if $i > j$;
2. $x_{ii} = \{(1, \dots, 1, m_i, 1, \dots, 1)\}$ for some $m_i \in M_i$ (here, m_i is the i th component in the k -tuple);

3.

$$x_{ij} \subseteq \{(m_1, \dots, m_k) \in M_1 \times \dots \times M_k \mid m_1 = \dots = m_{i-1} = 1 = m_{j+1} = \dots = m_k\}$$

(here, 1 is the identity of M_1, \dots, M_k).

Condition (2) allows to identify x_{ii} with an element of M_i and Condition (3) x_{ij} with a subset of $M_i \times \dots \times M_j$. If

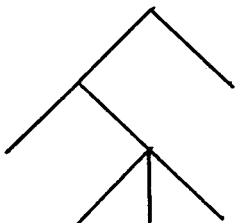
$$\bar{m} = (m_i, \dots, m_j) \in M_i \times \dots \times M_j$$

and

$$\bar{m}' = (m'_{i'}, \dots, m'_{j'}) \in M_{i'} \times \dots \times M_{j'},$$

then $\bar{m}\bar{m}' = (m_i, \dots, m_{j-1}, m_j m'_{i'}, m'_{i'+1}, \dots, m'_{j'})$ if $j = i'$, and is undefined otherwise. This multiplication is extended to sets in the usual fashion; addition is given by set union.

We will denote by \mathcal{T} the set of trees on the alphabet $\{a, \bar{a}\}$. Formally, \mathcal{T} is the set of words in $\{a, \bar{a}\}^*$ congruent to 1 in the congruence generated by the relation $a\bar{a} = 1$. Intuitively, the words in \mathcal{T} are obtained as follows: we draw a tree and starting from the root we code a for going down and \bar{a} for going up. For example,



is coded by $aa\bar{a}aa\bar{a}aa\bar{a}aa\bar{a}aa\bar{a}$. The number of leaves of a word t in $\{a, \bar{a}\}^*$, denoted by $l(t)$ is by definition the number of occurrences of the factor $a\bar{a}$ in t . Each tree t factors uniquely into $t = at_1 \bar{a}at_2 \bar{a} \dots at_k \bar{a}$ where $k \geq 0$ and where the t_i 's are trees. Let t be a tree and let $t = t_1 at_2 \bar{a}t_3$ be a factorization of t . We say that the occurrences of a and \bar{a} defined by this factorization are related if t_2 is a tree. Let t and t' be two trees. We say that t is *extracted* from t' if t is obtained from t' by removing in t' a certain number of related occurrences of a and \bar{a} . We now give Pin's tree hierarchy construction using Schützenberger's product.

To each tree t and to each sequence $\mathbf{V}_1, \dots, \mathbf{V}_{l(t)}$ of \mathbf{M} -varieties is associated an \mathbf{M} -variety $\diamond_t(\mathbf{V}_1, \dots, \mathbf{V}_{l(t)})$ defined recursively by:

1. $\diamond_1(\mathbf{V}) = \mathbf{V}$ for every \mathbf{M} -variety \mathbf{V} ;
2. if $t = at_1 \bar{a}at_2 \bar{a} \dots at_k \bar{a}$ with $k \geq 0$ and $t_1, \dots, t_k \in \mathcal{T}$, $\diamond_t(\mathbf{V}_1, \dots, \mathbf{V}_{l(t)})$ is the \mathbf{M} -variety of monoids that divide some $\diamond_k(M_1, \dots, M_k)$ with $M_1 \in \diamond_{t_1}(\mathbf{V}_1, \dots, \mathbf{V}_{l(t_1)}), \dots, M_k \in \diamond_{t_k}(\mathbf{V}_{l(t_1)+\dots+l(t_{k-1})+1}, \dots, \mathbf{V}_{l(t_1)+\dots+l(t_k)})$.

When $\mathbf{V}_1 = \dots = \mathbf{V}_{l(t)} = \mathbf{V}$, we denote simply by $\diamond_t(\mathbf{V})$ the \mathbf{M} -variety $\diamond_t(\mathbf{V}_1, \dots, \mathbf{V}_{l(t)})$. More generally, if T is a language contained in \mathcal{T} , we denote by $\diamond_T(\mathbf{V})$ the smallest \mathbf{M} -variety containing the \mathbf{M} -varieties $\diamond_t(\mathbf{V})$ with $t \in T$.

Let \mathbf{I} denote the trivial \mathbf{M} -variety. In [21], the following equalities are shown: $\diamond_{(a\bar{a})^{k+1}}(\mathbf{I}) = \mathbf{J}_k$ and $\diamond_{(a\bar{a})^*}(\mathbf{I}) = \mathbf{J}$. Also, it is shown there that if \mathbf{V} is an arbitrary \mathbf{M} -variety, then $\diamond_{(a\bar{a})^2}(\mathbf{V}, \mathbf{I}) = \mathbf{J}_1 \star \mathbf{V}$.

Among the many problems concerning these tree hierarchies, is the comparison between the \mathbf{M} -varieties inside a hierarchy. More precisely, the problem consists in comparing the different \mathbf{M} -varieties $\diamond_t(\mathbf{V})$ (or even $\diamond_T(\mathbf{V})$). A partial result and a conjecture on this problem was given in Pin [21]. It was shown that for every \mathbf{M} -variety \mathbf{V} , if t is extracted

from t' , then $\diamond_t(\mathbf{V}) \subseteq \diamond_{t'}(\mathbf{V})$, and it was conjectured that if $t, t' \in T'$, $\diamond_t(\mathbf{I}) \subseteq \diamond_{t'}(\mathbf{I})$ if and only if t is extracted from t' . Here, T' denotes the set of trees in which each node is of arity different from 1.

THEOREM 3.1: *The above conjecture is false.*

Proof: To see this, let $k > 1$ and let $t = a^{k+1}(\bar{a}a\bar{a})^{k+1}$ and $t' = a(a\bar{a})^{k+1}\bar{a}a\bar{a}$. The equalities $\diamond_t(\mathbf{I}) = \mathbf{J}_1^{k+1}$ and $\diamond_{t'}(\mathbf{I}) = \diamond_{(a\bar{a})^2}(\mathbf{J}_k, \mathbf{I}) = \mathbf{J}_1 * \mathbf{J}_k$ hold. But $\mathbf{J}_1 * \mathbf{J}_k = \mathbf{J}_1^{k+1}$ by Corollary 2.2 (M-variety version), and it is easy to verify that the tree t is not extracted from the tree t' . \square

REFERENCES

1. J. ALMEIDA, Semidirect Products of Pseudovarieties from the Universal Algebraist's Point of View, *J. of Pure and Applied Algebra*, 1989, 60, pp. 113-128.
2. J. ALMEIDA, Semidirectly Closed Pseudovarieties of Locally Trivial Semigroups, *Semigroup Forum*, 1990, 40, pp. 315-323.
3. J. ALMEIDA, On Iterated Semidirect Products of Finite Semilattices, *J. of Algebra*, 1991, 142, pp. 239-254.
4. C. J. ASH, Finite Semigroups with Commuting Idempotents, *J. of Australian Math. Soc., Ser., A*, 1987, 43, pp. 81-90.
5. F. BLANCHET-SADRI, Some Logical Characterizations of the Dot-Depth Hierarchy and Applications, Ph. D. Thesis, McGill University, 1989.
6. F. BLANCHET-SADRI, Games, Equations and the Dot-Depth Hierarchy, *Computers and Mathematics with applications*, 1989, 18, pp. 809-822.
7. F. BLANCHET-SADRI, On Dot-Depth Two, *R.A.I.R.O. Informatique Théorique et applications*, 1990, 24, pp. 521-530.
8. F. BLANCHET-SADRI, Games, Equations and Dot-Depth Two Monoids, *Discrete Applied Mathematics*, 1992, 39, pp. 99-111.
9. F. BLANCHET-SADRI, The Dot-Depth of a Generating Class of Aperiodic Monoids is Computable, *J. Foundations Comput. Sci.*, 1992, 3, pp. 419-442.
10. F. BLANCHET-SADRI, Equations and Dot-Depth One, *Semigroup Forum*, 1993, 47, pp. 305-317.
11. F. BLANCHET-SADRI, Equations and Monoid Varieties of Dot-Depth One and Two, *Theoretical Comput. Sci.*, 1994, 123, pp. 239-258.
12. F. BLANCHET-SADRI, On a Complete Set of Generators for Dot-Depth Two, *Discrete Applied Mathematics*, 1994, 50, pp. 1-25.
13. F. BLANCHET-SADRI, Some Logical Characterizations of the Dot-Depth Hierarchy and Applications, *J. Comp. Syd. Sci.* (à paraître).
14. J. A. BRZOWSKI and I. SIMON, Characterizations of Locally Testable Events, *Discrete Mathematics*, 1973, 4, pp. 243-271.
15. S. EILENBERG, *Automata, Languages and Machines, B*, Academic Press, New York, 1976.
16. S. EILENBERG and M. P. SCHÜTZENBERGER, On Pseudovarieties, *Advances in Mathematics*, 1976, 19, pp. 413-418.

17. C. IRASTORZA, Base Non Finie de Variétés, *Lecture Notes in Comput. Sci.*, Springer Verlag, Berlin, 1985, 182, pp. 180-186.
18. J. KARNOFSKI and J. RHODES, Decidability of Complexity One-Half for Finite Semi-groups, *Semigroup Forum*, 1982, 24, pp. 55-66.
19. G. LALLEMENT, *Semigroups and Combinatorial Applications*, Wiley, New York, 1979.
20. J. E. PIN, *Variétés de langages formels*, Masson, Paris, 1984; *Varieties of Formal Languages*, North Oxford Academic, London, 1986 and Plenum, New York, 1986.
21. J. E. PIN, Hiérarchies de concaténation, *R.A.I.R.O. Informatique Théorique*, 1984, 18, pp. 23-46.
22. J. E. PIN, On Semidirect Products of Two Finite Semilattices, *Semigroup Forum*, 1984, 28, pp. 73-81.
23. I. SIMON, Hierarchies of Events of Dot-Depth One, Ph. D. Thesis, University of Waterloo, 1972.
24. I. SIMON, Piecewise Testable Events, Proc. 2nd GI Conference, *Lecture Notes in Comput. Sci.*, Springer Verlag, Berlin, 1975, 33, pp. 214-222.
25. P. STIFFLER, Extension of the Fundamental Theorem of Finite Semigroups, *Advances in Mathematics*, 1973, 11, pp. 159-209.