INFORMATIQUE THÉORIQUE ET APPLICATIONS

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Informatique théorique et applications, tome 26, nº 2 (1992), p. 151-161

<http://www.numdam.org/item?id=ITA_1992__26_2_151_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 26, n° 2, 1992, p. 151 à 161)

CROSSABILITY OF CANCELLATIVE KLEENE SEMIGROUPS (*)

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Communicated by J. BERSTEL

Abstract. – *Every cancellative Kleene semigroup satisfies Eilenberg's theorem.*

Résumé. – *Si S est un semigroupe simplifiable de type Kleene, alors S satisfait le théorème d'Eilenberg.*

INTRODUCTION

A morphism $\varphi : T \to S$ of semigroups is called crossable if every rational subset R of T contains a rational cross-section R_0 for the restriction of φ to R or (in other words) if there exists for each rational subset R of T another rational subset R_0 of T satisfying:

(1) $R_0 \subset R$;

(2) $\varphi(R_0) = \varphi(R)$; and

(3) φ is injective on R_0.

The following classical crossability result is useful in the theory of rational relations.

EILENBERG'S THEOREM [1]: *If* Σ^* *and* Γ^* *are finitely generated free monoids, then every morphism* $\varphi : \Sigma^* \to \Gamma^*$ *is crossable.* ■

We say that a semigroup S satisfies Eilenberg's theorem, or that S is crossable, if every morphism $\varphi : \Sigma^+ \to S$ is crossable for every free semigroup Σ^+.

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Crossability results often have interesting consequences. For example, if S satisfies Eilenberg's theorem then every rational subset of S is unambiguously rational. Moreover, an effective proof that S satisfies Eilenberg's theorem enables us to decide whether a given rational expression over S is unambiguously rational.

Pelletier [3] introduced a technique for constructing congruences from equivalence relations, used it to produce various counter-examples in the theory of Kleene semigroups, and in this way showed that not all Kleene semigroups satisfy Eilenberg's theorem.

Our major result, Theorem 2 below, proves that every cancellative Kleene semigroup satisfies Eilenberg's theorem, by modifying a method used by Sakarovitch [5] (to prove a special case of Eilenberg's theorem) and by Johnson (to show that every deterministic rational equivalence relation has a rational cross-section, *cf*. Theorem 5.3 in [2]). The method produces rational cross-sections of equivalence relations by lexicographic minimalization, a tactic which does not work in general (*cf*. Theorem 8.2 in [2]) but does work here.

I. PRELIMINARIES

Recall some definitions and theorems.

A subset R (of a semigroup S), which is saturated by a congruence $\equiv$ of finite index on S, is called recognizable. $\mathrm{Rec}(S)$ denotes the set of recognizable subsets of S.

NERODE'S THEOREM: *A subset R of a semigroup S is recognizable iff there are only finitely many different quotient sets $s^{-1}R := \{t \in S : st \in R\}$.* ■

LEMMA 1: *Let R be a recognizable subset of a semigroup S; for each $s \in R$, define the set $[s]_R := \{t : t^{-1}R = s^{-1}R\}$. Then there are only finitely many sets $[s]_R$ and each of these sets is recognizable.* ■

Rational subsets of a semigroup S are defined as follows: the empty set $\varnothing$ is rational and so is every singleton $s \in S$; if U and V are rational, then so are the union $U \cup V$, product $UV := \{uv : u \in U, v \in V\}$, and subsemigroup $U^+ \subset S$ generated by U. $\mathrm{Rat}(S)$ denotes the collection of rational subsets of S.

In an arbitrary semigroup S, $\mathrm{Rec}(S)$ and $\mathrm{Rat}(S)$ are not closely related. However, the following result holds.

KLEENE'S THEOREM: *If Σ^+ is a finitely generated free semigroup, then every rational subset of Σ^+ is recognizable and conversely.* ■

Motivated by this result, we call a semigroup S Kleene if $\mathrm{Rat}(S)=\mathrm{Rec}(S)$. Clearly, a Kleene semigroup is finitely generated.

By a regulator $\rho:\Sigma^+\to\Sigma^+$, we mean a rationality-preserving relation: every rational subset $R\subset\Sigma^+$ has rational ρ-image $\rho(R)$.

LEMMA 2 [3]: *A semigroup S is Kleene iff S is isomorphic to the quotient Σ^+/κ of a finitely-generated free semigroup Σ^+ by a congruence κ which is also a regulator.* ■

LEMMA 3: *Any relation $\Sigma^+\to\Sigma^+$ which is rational in $\Sigma^*\times\Sigma^*$ is a regulator.* ■

LEMMA 4: *The set of regulators is closed under finite union and under composition. If ψ is a regulator and if P and Q are rational subsets of Σ^+ then $(P\times Q)\cap\psi$ is also a regulator.*

Proof: Suppose that ψ and θ are regulators; if $R\in\mathrm{Rat}(\Sigma^+)$, then $(\psi\cup\theta)(R)=\psi(R)\cup\theta(R)$ and $\psi\circ\theta(R)=\psi(\theta(R))$; so the first sentence holds. If R is rational in Σ^+, then

$$\Delta_R=\{(r,r):r\in R\}$$

is a rational relation $\Sigma^*\to\Sigma^*$. Now $(P\times Q)\cap\psi$ is simply $\Delta_Q\circ\psi\circ\Delta_p$; if P and Q are rational, this is a composite of regulators; so the second sentence holds. ■

We also use another closure property of regulators. Given any relations $\psi:\Sigma^+\to\Sigma^+$ and $\varphi:\Sigma^+\to\Sigma^+$, define the product relation $\varphi\wedge\psi:\Sigma^+\to\Sigma^+$ by

$$\varphi\wedge\psi:=\{(ac,bd):(a,b)\in\varphi,\ (c,d)\in\psi\}.$$

LEMMA 5: *If $\psi:\Sigma^+\to\Sigma^+$ and $\varphi:\Sigma^+\to\Sigma^+$ are regulators, then the product relation $\varphi\wedge\psi:\Sigma^+\to\Sigma^+$ is also a regulator.*

Proof: We begin with the following claim.

Claim: For $R\subset\Sigma^+$, $\varphi\wedge\psi(\mathrm{R})=\bigcup_{x\in\Sigma^+}\varphi([x]_R\cap R(\Sigma^+)^{-1})\psi(x^{-1}R)$.

Explanation: Suppose $t\in\varphi\wedge\psi(R)$. Choose $(a,b)\in\varphi$, $(c,d)\in\psi$ with $ac=s\in R$ and $bd=t$. Then $b\in\varphi([a]_R\cap R(\Sigma^+)^{-1})$ (since $a\in[a]_R$ and $ac=s\in R$), and $d\in\psi(a^{-1}R)$ (since $c\in a^{-1}R$). So $t=bd\in\varphi([a]_R\cap R(\Sigma^+)^{-1})\psi(a^{-1}R)$

and thus

$$\varphi \wedge \psi(R) \subset \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R).$$

For the opposite inclusion, suppose that

$$t \in \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R).$$

Then $t \in \varphi([p]_R \cap R(\Sigma^+)^{-1}) \psi(p^{-1} R)$ for some $p \in \Sigma^+$. So $t = bd$ for some $a \in [p]_R \cap R(\Sigma^+)^{-1}$, $b \in \varphi(a)$ and $d \in \psi(p^{-1} R)$. Since $a \in [p]_R$, $[a]_R = [p]_R$ and $a^{-1} R = p^{-1} R$; so $a \in [a]_R \cap R(\Sigma^+)^{-1}$ and $d \in \psi(a^{-1} R)$. Choose $c \in a^{-1} R$ with $d \in \psi(c) \subset \psi(a^{-1} R)$. As $(a, b) \in \varphi$, $(c, d) \in \psi$, and $ac \in R$, so $t = bd \in \varphi \wedge \psi(R)$, and therefore

$$\bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R) \subset \varphi \wedge \psi(R),$$

which completes the proof of the claim. □

We now show that $\varphi \wedge \psi$ is a regulator. Suppose $R \in \mathrm{Rat}(\Sigma^+)$. Then the sets $[x]_R$, $R(\Sigma^+)^{-1}$, and $x^{-1} R$ are also rational; hence so is each set $\varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R)$. There are but finitely many distinct sets $x^{-1} R$ and similarly only finitely many sets $[x]_R$. It follows that

$$\varphi \wedge \psi(R) = \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R)$$

actually reduces to a finite union of rational sets. Thus, $\varphi \wedge \psi(R)$ is rational and so $\varphi \wedge \psi$ is a regulator. ■

II. A METHOD OF SAKAROVITCH

By an order on a set X, we understand a binary relation $>$ on X which is asymmetric (no element $x \in X$ satisfies $x > x$) and transitive. A linear order is an order verifying trichotomy:

$$\forall x \in X \quad \forall y \in X \quad x = y \quad \text{or} \quad x > y \quad \text{or} \quad y > x.$$

If $>$ is an order on X and R is a subset of X, then by $a >$-minimal element of $R \subset X$ we mean any $r \in R$ with

$$\{s \in R : r > s\} = \varnothing.$$

When κ is a relation on X, $\Lambda = \Lambda(>, \kappa)$ denotes the relation

$$\kappa \cap >^{-1} = \{(u, v) \in \kappa : v > u\}.$$

If κ is an equivalence relation, $\mathrm{Min}(R) = \mathrm{Min}(>, \kappa, \mathrm{R})$ denotes the set

$$\{r \in R : r \text{ is a} > \text{-minimal element of } [r]_\kappa \cap R\},$$

where $[r]_\kappa$ denotes the κ-class of $r \in X$.

Lexicographic orders on a free semigroup Σ^+ are constructed as follows: fix a linear order $.>$ on the alphabet Σ; for distinct words $u \in \Sigma^+$ and $v \in \Sigma^+$, $v > u$ means either that u is a proper prefix of v or that there exist (possibly empty) words w, x, and y over the alphabet Σ and letters $\sigma .> \tau$ in Σ such that $u = w \tau x$ and $v = w \sigma y$. Any lexicographic order is linear.

LEMMA 6: *If κ is a relation and $>$ a lexicographic order on Σ^+ then $\Lambda(>, \kappa)$ is a union $\Lambda_1 \cup \Lambda_2$, where Λ_1 denotes the relation*

$$\{(x \sigma u, x \tau v) \in \kappa : x \in \Sigma^*, u \in \Sigma^*, v \in \Sigma^*, \sigma \in \Sigma, \tau \in \Sigma, \tau > \sigma\}$$

and Λ_2 denotes the relation

$$\{(x \sigma, x \sigma v) \in \kappa : x \in \Sigma^*, \sigma \in \Sigma, v \in \Sigma^+\}.$$

Proof: Obvious from the definition of lexicographic order. ■

Now the method of Sakarovitch [5] is essentially this: for any lexicographic order $>$ and any morphism $\pi : \Sigma^+ \to \Gamma^*$ from the free semigroup Σ^+ to the free monoid Γ^*, $\Lambda(>, \pi^{-1}\pi)$ is a rational relation $\Sigma^* \to \Sigma^*$, and the set $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is therefore rational whenever $R \subset \Sigma^+$ is; when π is non-erasing, $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is a cross-section for the restriction of $\pi^{-1}\pi$ to R, and Eilenberg's theorem follows easily.

The next lemmas isolate some key ideas of this method.

LEMMA 7: *Let $>$ and κ (respectively) be an order and an equivalence relation on the set X. Then the following are equivalent:*

(1) *For each $r \in R$, there exists at least one $>$-minimal element of the set $[r]_\kappa \cap R$; and*

(2) $\mathrm{Min}(>, \kappa, R)$ *intersects every κ-class intersecting R. If these conditions hold and $>$ is a linear order, then* $\mathrm{Min}(>, \kappa, R)$ *is a cross-section for the restriction of κ to R.*

Proof: (1) $\Leftrightarrow$ (2) is obvious. If in addition $>$ is a linear order then each set $[r]_\kappa \cap R$ has a unique $>$-minimal element, so $\mathrm{Min}(R)$ must be a cross-section for the restriction of κ to R. ■

LEMMA 8: *Suppose that $>$ and κ (respectively) are an order and an equivalence relation on the finitely generated free semigroup Σ^+, and that $\Lambda(>, \kappa)$ is a regulator. Then $\mathrm{Min}(>, \kappa, R)$ is rational for every rational set R.*

Proof: Suppose that Λ is a regulator; let $\Lambda_0 : \Sigma^+ \to \Sigma^+$ denote the relation $\Delta_R \circ \Lambda$, where $\Delta_R = \{(r, r) : r \in R\}$. Then

$$R \setminus \Lambda_0(R) = R \setminus \{ r \in R : \exists s \in R (s, r) \in \Lambda \}$$
$$= \{ r \in R : \forall s (s \in [r]_\kappa \cap R) \Rightarrow \mathrm{not}(r > s) \} = \mathrm{Min}(R).$$

Whenever R is rational, Λ_0 is a regulator by Lemma 4 and so $\mathrm{Min}(R)$ is rational. ■

III. LEXICOGRAPHIC MINIMALIZATION

For the remainder of this article, we fix a finitely generated free semigroup Σ^+ and a lexicographic order $>$ on Σ^+. To generalize Sakarovitch's argument, we show that $\Lambda(>, \kappa)$ is a regulator when κ is a left-cancellative congruence.

A semigroup S is called left-cancellative if

$$xy = xz \quad \Rightarrow \quad y = z;$$

the notion right-cancellative is dually defined; cancellative means left- and right-cancellative.

We can now obtain our first result.

THEOREM 1: *If Σ^+/κ is a left-cancellative Kleene semigroup, and if $>$ is a lexicographic order on Σ^+, then $\Lambda(>, \kappa) : \Sigma^+ \to \Sigma^+$ is a regulator.*

Proof: Express $\Lambda = \Lambda_1 \cup \Lambda_2$ according to Lemma 6. To show that Λ is a regulator, it suffices (according to Lemma 4) to prove that Λ_1 and Λ_2 are both regulators. Now Λ_2 is the relation

$$\bigcup_{\sigma \in \Sigma} \{(x\sigma, x\sigma w) \in \kappa : x \in \Sigma^*, w \in \Sigma^+\}$$

$$= \bigcup_{\sigma \in \Sigma} (\{\sigma, \sigma w) \in \kappa : w \in \Sigma^+\} \cup \Delta \{(\sigma, \sigma w) \in \kappa : w \in \Sigma^+\})$$

$$= \bigcup_{\sigma \in \Sigma} ((\sigma \times (\sigma \Sigma^+ \cap [\sigma]_\kappa)) \cup \Delta (\sigma \times (\sigma \Sigma^+ \cap [\sigma]_\kappa))),$$

where $[\sigma]_\kappa$ denotes the κ-class of $\sigma \in \Sigma$ and Δ denotes the diagonal $\{(x, x) : x \in \Sigma^+\}$; note that we used the left-cancellativity of κ. Since the semigroup is Kleene, each set $\sigma \Sigma^+ \cap [\sigma]_\kappa$ is rational; thus, Λ_2 is actually a rational relation $\Sigma^* \to \Sigma^*$ and hence a regulator.

To show that Λ_1 is a regulator, we first observe that each relation $(\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa$ (where $\sigma \in \Sigma$, $\tau \in \Sigma$, and $\tau > \sigma$) is a regulator by Lemmas 2 and 4. Thus the union of these relations is another regulator Λ_3. If we show that $\Lambda_1 = \Lambda_3 \cup \Delta \wedge \Delta_3$ then Λ_1 will be a regulator by Lemmas 4 and 5.

Now $(s, t) \in \Lambda_1$ means $s \kappa t$, $(s, t) = (x\sigma u, x\tau v)$ where $\tau > \sigma$ are letters in Σ, and x, u, and v lie in Σ^*. If x is actually the empty word, then

$$(s, t) = (\sigma u, \tau v) \in (\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa \subset \Lambda_3.$$

On the other hand, when $x \in \Sigma^+$ we conclude from $x \sigma u \; \kappa \; x \tau v$ (using left-cancellativity) that

$$(\sigma u, \tau v) \in (\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa \subset \Lambda_3,$$

whence it is immediate (by the definition of $\Delta \wedge \Lambda_3$) that

$$(s, t) = (x \sigma u, x \tau v) \in \Delta \wedge \Lambda_3.$$

Thus $\Lambda_1 \subset \Lambda_3 \cup \Delta \wedge \Lambda_3$; for the opposite inclusion, read backwards, using the left-compatibility

$$y \kappa x \quad \Rightarrow \quad xy \; \kappa \; xz$$

of κ instead of left-cancellativity. ■

Corollaries 1 and 2 below generalize results in [5].

COROLLARY 1: *If $\pi : \Sigma^+ \to S$ is a morphism from Σ^+ to a left-cancellative Kleene semigroup S, and if $>$ is a lexicographic order on Σ^+, then* $\text{Min}(>, \pi^{-1}\pi, R)$ *is rational for every rational subset $R \subset \Sigma^+$.*

Proof: Since S is Kleene, $\pi^{-1}\pi$ is a regulator by Lemma 2; moroever, a subsemigroup of a left-cancellative semigroup is left-cancellative; hence, the theorem guarantees that Λ is a regulator. The result now follows from Lemma 8. ■

LEMMA 9: *Let S be a left-cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism $\pi:\Sigma^+\to S$:*

(1) *For each $s\in\pi(\Sigma^+)$, the set $ss^{-1}\cap\pi(\Sigma^+)=\varnothing$; and*

(2) *Each $\pi^{-1}\pi$-class is finite.*

Proof: (1) $\Rightarrow$ (2): Suppose that some set $\pi^{-1}\pi(w)$ is infinite. Then $\pi^{-1}\pi(w)$ is rational because S has recognizable singletons and consequently $\pi^{-1}\pi(w)$ contains an infinite subset xy^+z by the pumping lemma. From $\pi(xyz)=(\pi(xy^2z)$, we conclude (by left-cancellativity) that

$$\pi(yz)=\pi(y^2z)=\pi(y)\pi(yz) \text{ so } \pi(y)$$

belongs to the set $\pi(yz)\pi(yz)^{-1}$.

(2) $\Rightarrow$ (1) : If $\pi(v)\in\pi(u)\pi(u)^{-1}$ *for some words u* and v, then v^+u is an infinite subset of $\pi^{-1}\pi(u)$. ■

COROLLARY 2: *If S is a left-cancellative Kleene semigroup, $>$ a lexicographic order on Σ^+ and $\pi:\Sigma^+\to S$ a morphism such that that $ss^{-1}\cap\pi(\Sigma^+)=\varnothing$ for each $s\in\pi(\Sigma^+)$, then* $\mathrm{Min}(>,\pi^{-1}\pi, R)$ *is a rational cross-section for the restriction of π to R, for each rational $R\subset\Sigma^+$; in particular, π is crossable.*

Proof: By hypothesis, every $\pi^{-1}\pi$-class is finite. Thus $\mathrm{Min}(R)$ is a rational cross-section by Theorem 1 and Lemmas 7 and 8, regardless of the rational set $R\subset\Sigma^+$. ■

For the next application of these ideas, we recall that an equivalence relation κ_1 is called locally-finite thinning of the equivalence relation $\kappa\subset\Sigma^+\times\Sigma^+$ if κ_1 is a restriction of κ, if the domain of κ_1 intersects every κ-class, and if each κ_1-class is finite. The following result is due to Johnson.

JOHNSON'S THEOREM [2]: *Every rational equivalence relation has a rational locally-finite thinning.* ■

We also need the following result, which can be restated in various forms (*cf*. Proposition 1.4.3 in [3]).

CHOFFRUT'S THEOREM: *If the congruence κ on Σ^+ is rational as a subset of $\Sigma^*\times\Sigma^*$ and if κ has a rational cross-section, then the quotient Σ^+/κ satisfies Eilenberg's theorem.* ■

COROLLARY 3: *Suppose* $\kappa \subset \Sigma^+ \times \Sigma^+$ *is a left-cancellative congruence which is rational as a subset of* $\Sigma^* \times \Sigma^*$. *Then* Σ^+/κ *satisfies Eilenberg's theorem.*

Proof: According to the Johnson's Theorem, we can find a rational locally-finite thinning κ_1 for κ, or (in other words) we can find a rational set $D \subset \Sigma^+$ such that $\kappa_1 := \kappa \cap D \times D$ is a locally-finite thinning of κ. Fix any lexicographic order $>$ on Σ^+. Then $\mathrm{Min}(>, \kappa, D)$ is a rational cross-section for κ by Theorem 1 and Lemmas 7 and 8. The result now follows by Choffrut's theorem. ■

IV. CANCELLATIVE KLEENE SEMIGROUPS

In this section, we show that cancellative Kleene semigroups satisfy Eilenberg's theorem.

LEMMA 10 [4]: *Let S be a semigroup, every singleton subset of which is recognizable. Then every subgroup of S is finite. If S has an identity element* 1, *then every divisor of* 1 *actually belongs to the group of units of S.* ■

LEMMA 11: *Let S be a cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism* $\pi : \Sigma^+ \to S$:

(1) *For each* $s \in \pi(\Sigma^+)$, *the set* $ss^{-1} \cap \pi(\Sigma^+) = \varnothing$;

(2) $\pi(\Sigma^+)$ *does not contain an idempotent;*

(3) *If S has an identity element* 1, *then* $1 \notin \pi(\Sigma^+)$; *and*

(4) *For each* $\sigma \in \Sigma$, *the set* $\pi^{-1}\pi(\sigma) \cap \sigma(\sigma^+) = \varnothing$.

Proof: (4) ⇒ (3): Suppose $\pi(w)$ is an identity element for S; let $\sigma \in \Sigma$ be any letter appearing in w; then, as a divisor of the identity $\pi(w)$, $\pi(\sigma)$ belongs to the group of units of S; moreover, this is a finite group; hence for some $n > 1$, $\pi(\sigma)^n = \pi(\sigma^n) = \pi(\sigma)$ and therefore $\pi^{-1}\pi(\sigma) \cap \sigma(\sigma^+) \neq \varnothing$.

(3) ⇒ (2): An idempotent in a cancellative semigroup must be the identity.

(2) ⇒ (1): If $\pi(v) \in \pi(u)\pi(u)^{-1}$, then $\pi(v^2 u) = \pi(vu) = \pi(u)$ and $\pi(v)$ is idempotent by cancellativity.

(1) ⇒ (4): If $\sigma \in \Sigma$ and $n > 1$ satisfy $\sigma^n \in \pi^{-1}\pi(\sigma)$, then

$$\pi(\sigma^{n-1}) \in \pi(\sigma)\pi(\sigma)^{-1}. \quad ■$$

THEOREM 2: *Let* $\pi : \Sigma^+ \to S$ *be a morphism from* Σ^+ *to the cancellative Kleene semigroup S. Then* π *is crossable.*

Proof: Fix a lexicographic order $>$ on Σ^+. According to part (4) of Lemma 11, we can easily test whether $\pi(\Sigma^+)$ contains an identity element for S. Our proof splits according to the outcome of this test; if $\pi(\Sigma^+)$ does not contain an identity element for S, and if R is any rational set, then (by Lemma 11 and Corollary 2) $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is a rational cross-section for the restriction of π to R.

On the other hand, if S is actually a monoid with identity element $1 \in \pi(\Sigma^+)$, and if $R \subset \Sigma^+$ is any rational set, put $G := \pi^{-1}(1)$, and define $\varepsilon : \Sigma^+ \to \Sigma^+$ by

$$\varepsilon := (\Delta \cup \Theta) * \Theta (\Delta \cup \Theta) *$$

where $\Delta := \{ (x, x) : x \in \Sigma^+ \}$ and

$$\Theta := \bigcup_{\sigma \in \Sigma} ((\sigma G \times \sigma) \cup (G \sigma \times \sigma)).$$

Then ε is an order which is also a rational relation $\Sigma^* \to \Sigma^*$. If $(u, v) \in \varepsilon$, then v has length strictly less than the length of u, so there is no infinite chain

$$w_1 \, \varepsilon \, w_2 \, \varepsilon \, w_3 \, \varepsilon \, \ldots;$$

hence each $\pi^{-1}\pi$-class has an ε-minimal element. As $\varepsilon \subset \pi^{-1}\pi$, we have $\Lambda(\varepsilon, \pi^{-1}\pi) = \varepsilon^{-1}$, which is certainly a regulator. By Lemmas 7 and 8, $\mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$ is rational and $\pi(\mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)) = \pi(R)$.

We claim no $\pi^{-1}\pi$-class contains infinitely many elements of $R_1 := \mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$. If indeed $R_1 \cap \pi^{-1}\pi(w)$ were infinite, then according to the pumping lemma this rational set would contain an infinite subset xy^+z with $y \in \Sigma^+$, by cancellativity, $\pi(y)$ is idempotent so $y \in G$, which implies that $(xy^2z, xyz) \in \varepsilon$; but this contradicts the fact that $xy^2z \in R_1 = \mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$. By Lemmas 7 and 8, $\mathrm{Min}(>, \pi^{-1}\pi, R_1)$ is therefore a rational cross-section for the restriction of π to R_1 and even for the restriction of π to R. ■

We remark that Theorem 2 is effective relative to the given Kleene semigroup S: if we have an explicit finite generating set for S, and an algorithm which produces for each $R \in \mathrm{Rat}(S)$ a congruence of finite index saturating R, then we can really produce the cross-sections described.

ACKNOWLEDGEMENTS

M. Pelletier graciously sent a copy of [3]; an anonymous referee made useful observations and suggestions; and North Carolina Central University provided facilities for manuscript production.

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