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## CONTEXT-FREE LANGUAGES WITH RATIONAL INDEX IN $\Theta(n^\lambda)$ FOR ALGEBRAIC NUMBERS $\lambda$ (\*)

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*Abstract.* – The complexity of a non-empty language  $L$  may be estimated by the asymptotic behavior of its rational index, which is a function  $\rho_L: \mathbb{N} - \{0\} \rightarrow \mathbb{N} - \{0\}$ . For any positive integer  $\lambda$ , we knew a context-free language whose rational index is in  $\Theta(n^\lambda)$ . In this paper we show context-free languages, whose rational indexes are in  $\Theta(n^\lambda)$  for other various values of  $\lambda > 1$ , such as the rational numbers or the algebraic numbers or even some transcendental numbers.

*Résumé.* – La complexité d'un langage non vide  $L$  peut être estimée par le comportement asymptotique de son index rationnel, qui est une fonction  $\rho_L: \mathbb{N} - \{0\} \rightarrow \mathbb{N} - \{0\}$ . On connaissait déjà des langages algébriques d'index rationnel en  $\Theta(n^\lambda)$  pour tout entier positif  $\lambda$ . Dans cet article nous montrons qu'il existe des langages algébriques d'index rationnel en  $\Theta(n^\lambda)$  pour d'autres valeurs de  $\lambda > 1$ , telles que les nombres rationnels, plus généralement les nombres algébriques, et même certains nombres transcendants.

### I. INTRODUCTION

There are many ways to measure the complexity of languages. The rational index introduced by L. Boasson, M. Nivat and B. Courcelle [3, 4] is one of them, that behaves well when combined with rational transductions: if  $L \geq L'$  (i.e. there exists a rational transduction  $\tau$ , such that  $\tau(L) = L'$ ), then the rational index  $\rho_L$  of  $L$  provides an upper bound on  $\rho_{L'}$ , since

$$\exists c \in \mathbb{N} - \{0\}, \quad \forall n \in \mathbb{N} - \{0\}, \quad cn(\rho_L(cn) + 1) \geq \rho_{L'}(n).$$

This is why the rational index can prove helpful when studying sets of languages closed under rational transductions like the set of context-free

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languages. We define the extended rational index  $\bar{\rho}_L$  of a language  $L$  to be  $\rho_L \sqcup_{s^*}$  for any letter  $s$ , which occurs in no word of  $L$ . The extended rational index  $\bar{\rho}_L$  of a given language  $L$  is generally not harder to compute than its rational index  $\rho_L$ . Both indexes are related since

$$\forall n \in \mathbb{N} - \{0\}, \quad \rho_L(n) \leq \bar{\rho}_L(n) < n(1 + \rho_L(n)),$$

but the extended one gives more information about the complexity of the language since

$$L' \leq L \Rightarrow \exists c \in \mathbb{N}, \quad \bar{\rho}_{L'}(n) \leq \bar{\rho}_L(cn).$$

We denote by  $\Theta(n^\lambda)$  the set of functions which are the products of  $n \mapsto n^\lambda$  by positive bounded functions. Given two languages  $L_1$  and  $L_2$  and two numbers  $\lambda_1$  and  $\lambda_2$  such that  $\bar{\rho}_{L_1} \in \Theta(n^{\lambda_1})$  and  $\bar{\rho}_{L_2} \in \Theta(n^{\lambda_2})$  and  $1 \leq \lambda_1 < \lambda_2$ , then you can conclude that  $L_2$  does not belong to the rational cone generated by  $L_1$ . Note that this is true even if  $\lambda_2 - \lambda_1 < 1$ , but this case could not be handled with plain rational index. In reference [6] you can find a way to construct a context-free language with a rational index in  $\Theta(n^k)$  for any positive even integer. For a long time the rational index of a context-free language was thought to necessarily behave asymptotically like a simple function, namely an exponential or a polynomial function. In this paper we give methods to construct context-free languages, whose rational indexes are in  $\Theta(n^\lambda)$  for other various values of  $\lambda > 1$ , such as the rational numbers or the algebraic numbers or even some transcendental numbers. The technic used in this paper is strongly related to the one used in [10], where we proved that some context-free languages have rational indexes, which grow faster than any polynomial, but slower than any exponential function  $\exp(\lambda n)$ .

## II. NOTATIONS AND DEFINITIONS

$\mathbb{N}$  will denote the set of non-negative integers, and  $\mathbb{N}_+ = \mathbb{N} - \{0\}$  the set of positive integers.

$A \sqcup B$  will denote the union of the disjoint sets  $A$  and  $B$ .

An alphabet is a finite set of letters.

A language written over an alphabet  $T$  is a subset of  $T^*$ .

$\varepsilon$  denotes the empty word.

$|u|$  is the length of the word  $u$ , *i.e.* the number of its letters. *E.g.*  $|a^3bac^2| = 7$ . The function  $u \mapsto |u|$  will be denoted  $|\cdot|$ .

$|u|_x$  is the number of occurrences of the letter  $x$  in  $u$ . E.g.  $|a^3 bac^2|_a = 4$ . The function  $u \mapsto |u|_x$  will be denoted  $|\cdot|_x$ .

If  $X$  is an alphabet then  $|u|_X$  is the number of occurrences of letters of  $X$  in  $u$ . E.g.  $|a^3 bac^2|_{\{b, c\}} = 3$ . The function  $u \mapsto |u|_X$  will be denoted  $|\cdot|_X$ .

$L(\mathcal{A})$  denotes the regular language recognized by the finite automaton  $\mathcal{A}$ .

A context-free language is a language generated by a context-free grammar. For instance

$$S_{\neq} = \{a^n b^m, n \neq m, n, m \in \mathbb{N}\}$$

is a context-free language, since it is generated by the grammar

$$\langle \{a, b\}, \{S, T, U\}, \{S \rightarrow aSb + T + U, T \rightarrow aT + a, U \rightarrow bU + b\}, S \rangle.$$

Similarly

$$S_{=} = \{a^n b^n, n \in \mathbb{N}\}$$

is a context-free language generated by the grammar

$$\langle \{a, b\}, \{S\}, \{S \rightarrow aSb + \varepsilon\}, S \rangle.$$

We shall use  $S_{\neq}$  a lot in this paper.

Let  $r$  be a binary relation between the two free monoids  $X^*$  and  $Y^*$ . We say that  $r$  is a rational transduction, if its graph is a rational subset of the monoid  $X^* \times Y^*$ ; i.e. it is the value of an expression containing only products, unions, stars (or  $^+$  operation) and finite sets. The rational transductions may be characterised in another way:

**THEOREM (Nivat) [9]:** *For any rational transduction  $r: X^* \rightarrow Y^*$  there exist an alphabet  $Z$ , a regular language  $K \subset Z^*$  and two morphisms  $\varphi: Z^* \rightarrow X^*$  and  $\psi: Z^* \rightarrow Y^*$  such that:*

$$\forall L \subset X^*, \quad r(L) = \psi(K \cap \varphi^{-1}(L)).$$

Furthermore, we may assume the two morphisms to be alphabetic, i.e.  $\varphi(Z) \subset X \cup \{\varepsilon\}$  and  $\psi(Z) \subset Y \cup \{\varepsilon\}$ . We shall write

$$\tau = \psi \circ \cap K \circ \varphi^{-1}.$$

Let  $L$  and  $L'$  be two languages. If  $L'$  is the image of  $L$  under a rational transduction, then we denote it  $L \geq L'$  and we say that  $L$  rationally dominates  $L'$ . For instance  $S_{=} \geq S_{\neq}$  since  $S_{\neq} = a^+ S_{=} \cup S_{=} b^+$ .

The transformation  $\tau: L \mapsto a^+ L \cup L b^+$  accords with the definition of a rational transduction, since its graph is

$$(\varepsilon, a)^+ \{ (a, a), (b, b) \}^* \cup \{ (a, a), (b, b) \}^* (\varepsilon, b)^+.$$

As an example of Nivat's theorem we can decompose it  $\tau = \psi \circ \cap K \circ \varphi^{-1}$ , where  $X = \{ a, b \}$ ,  $Z = \{ a, b, a', b' \}$

$$\begin{aligned} \varphi: Z^* &\rightarrow X^*, & \psi: Z^* &\rightarrow X^* \\ a &\mapsto a, & a &\mapsto a \\ b &\mapsto b, & b &\mapsto b \\ a' &\mapsto \varepsilon, & a' &\mapsto a \\ b' &\mapsto \varepsilon, & b' &\mapsto b \\ K &= a'^+ X^* \cup X^* b'^+ \end{aligned}$$

If  $L \geq L'$  and  $L' \not\geq L$  then we say that  $L$  dominates strictly  $L'$  and we write  $L > L'$ . E.g.  $S_{=} > S_{\neq}$ .

Reference [1] holds the above definitions.

Every regular language is recognised by a finite automaton.  $\mathcal{R}_n$  is the family of the regular languages recognized by a finite automaton.  $\mathcal{R}_n$  is the family of the regular languages recognized by finite automata with at most  $n$  states.

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  will be said increasing if

$$\forall x, y \in \mathbb{R}, \quad x < y \Rightarrow f(x) \leq f(y).$$

You may notice that, according to this definition, a constant function is increasing.

Let  $f$  be a function  $\mathbb{N} \rightarrow \mathbb{R}$ . We shall use the Landau's notations  $o$  and  $O$  [8], § IV. 7, and the Knuth's notations  $\Omega$  and  $\Theta$  [7]:

$$\begin{aligned} o(f) &= \{ g: \mathbb{N} \rightarrow \mathbb{R}, \forall c \in \mathbb{R}_+^*, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |g(n)| \leq c |f(n)| \} \\ O(f) &= \{ g: \mathbb{N} \rightarrow \mathbb{R}, \exists c \in \mathbb{R}_+^*, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |g(n)| \leq c |f(n)| \} \\ \Omega(f) &= \{ g: \mathbb{N} \rightarrow \mathbb{R}, \exists c \in \mathbb{R}_+^*, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |g(n)| \geq c |f(n)| \} \\ \Theta(f) &= O(f) \cap \Omega(f) \end{aligned}$$

$g \sim f$  will stand for  $g - f \in o(f)$ .

*Remark:* If  $f$  does not take the value 0 then

$$g \sim f \Leftrightarrow \lim g/f = 1,$$

$$g \in o(f) \Leftrightarrow \lim g/f = 0,$$

$$g \in O(f) \Leftrightarrow \limsup |g/f| < \infty$$

and

$$g \in \Theta(f) \Leftrightarrow (\liminf |g/f| > 0 \text{ and } \limsup |g/f| < \infty).$$

$\lfloor x \rfloor$  is the floor of the real number  $x$  i.e. the greatest integer  $k$  such that  $k \leq x$ .

$\lceil x \rceil$  is the ceiling of the real number  $x$  i.e. the lowest integer  $k$  such that  $k \geq x$ .

If  $T$  is a sub-alphabet of an alphabet  $U$ , then  $\pi_T$  will denote the morphism  $U^* \rightarrow (U-T)^*$ , which erases the letters of  $T$  and keeps the letters of  $U-T$ . E.g.

$$\pi_{\{a, \bar{a}\}}(axayz\bar{x}\bar{a}) = xyzx.$$

$|\pi_X|$  will stand for the morphism  $|\cdot| \circ \pi_X$ , so that  $|\pi_X| = |\cdot| - |\cdot|_X$ .

$A \sqcup B$  will denote the shuffle of the languages  $A$  and  $B$ , i.e. the set of the words produced when interspersing words of  $A$  in words of  $B$ . E.g.

$$a^* b^* \sqcup c^* = c^* (ac^*)^* (bc^*)^* = \{a, c\}^* \{b, c\}^*.$$

### III. DEFINITION AND BASIC PROPERTIES OF RATIONAL INDEX

#### 1. Definition of $\rho$ and $\bar{\rho}$

DEFINITION 1: If  $L$  is a non-empty language then its rational index is the function  $\rho_L: \mathbb{N}_+ \rightarrow \mathbb{N}$  defined by

$$\rho_L(n) = \max_{\substack{K \in \mathcal{Q}_n \\ K \cap L \neq \emptyset}} \min_{w \in K \cap L} |w|.$$

DEFINITION 2: Let  $L \subset X^*$  be a non-empty language. Let  $s$  be a letter which does not belong to  $X$ . We define the extended rational index of  $L$  to be the rational index of  $L \sqcup s^*$ , and we denote it by  $\bar{\rho}_L$ .

## 2. Basic properties

A morphism of free monoids  $\varphi: X^* \rightarrow Y^*$  is said to be alphabetic if  $\varphi(X) \subset Y \cup \{\varepsilon\}$ , and strictly alphabetic if  $\varphi(X) \subset Y$ . In [2] Boasson *et al.* give the five following lemmas.

LEMMA 1: *If  $L$  and  $L'$  are two languages then  $\rho_{L \cup L'} \leq \max(\rho_L, \rho_{L'})$ .*

LEMMA 2: *If  $L$  and  $L'$  are two languages then  $\rho_{LL'} \leq \rho_L + \rho_{L'}$ .*

LEMMA 3: *Let  $\varphi: X^* \rightarrow Y^*$  be an alphabetic morphism, and  $L \subset X^*$ . Then  $\rho_{\varphi(L)} \leq \rho_L$ .*

LEMMA 4: *Let  $K$  be a regular language recognised by an  $m$  state automaton. Let  $L$  be a language. Then*

$$\forall n \in \mathbb{N}_+ : \rho_{L \cap K}(n) \leq \rho_L(nm).$$

LEMMA 5: *Let  $\varphi$  be an alphabetic morphism from  $X^*$  to  $Y^*$ . Let  $L$  be a subset of  $Y^*$ . Then*

$$\forall n \in \mathbb{N}_+, \quad \rho_{\varphi^{-1}(L)}(n) < n(\rho_L(n) + 1).$$

Using the last three lemmas and Nivat's theorem they derive the theorem.

THEOREM 1: *If  $L' \leq L$ , then there exists an integer  $c$  such that*

$$\forall n \in \mathbb{N}_+ : \rho_{L'}(n) < cn(\rho_L(cn) + 1).$$

*Proof:* According to Nivat's theorem there exist two alphabetic morphisms  $\varphi$  and  $\psi$  and a regular language  $K$  such that  $L' = \varphi(K \cap \psi^{-1}(L))$ . Let  $c$  be the number of states of an automaton recognising  $K$ . Then

$$\rho_{L'}(n) = \rho_{\varphi(K \cap \psi^{-1}(L))}(n) \leq \rho_{K \cap \psi^{-1}(L)}(n) \leq \rho_{\psi^{-1}(L)}(cn) < cn(1 + \rho_L(cn)). \quad \square$$

We can make a variation on lemma 5:

LEMMA 6: *Let  $\varphi$  be a strictly alphabetic morphism from  $X^*$  to  $Y^*$ . Let  $L$  be a subset of  $Y^*$ . Then  $\rho_{\varphi^{-1}(L)} \leq \rho_L$ .*

The proof is left to the reader. This leads to the following theorem.

THEOREM 2: *If  $L' \leq L$ , then there exists an integer  $c$  such that*

$$\forall n \in \mathbb{N}_+ : \rho_{L'}(n) \leq \bar{\rho}_L(cn).$$

*Proof:* According to Nivat's theorem there exist two alphabetic morphisms  $\varphi$  and  $\psi$  and a regular language  $K$  such that  $L' = \varphi(K \cap \psi^{-1}(L))$ .

Let  $\psi'$  be the strictly alphabetic morphism defined by:

$$\psi'(a) = \psi(a) \quad \text{if } \psi(a) \neq \varepsilon$$

and

$$\psi'(a) = s \quad \text{if } \psi(a) = \varepsilon.$$

Then  $\psi^{-1}(L) = \psi^{-1}(L \sqcup s^*)$ . Let  $c$  be the number of states of an automaton recognizing  $K$ . As in the proof of theorem 1 we have

$$\rho_{L'}(n) = \rho_{\psi(K \cap \psi^{-1}(L))}(n) \leq \rho_{K \cap \psi^{-1}(L)}(n) \leq \rho_{\psi^{-1}(L)}(cn)$$

Hence

$$\rho_{L'}(n) \leq \rho_{\psi^{-1}(L \sqcup s^*)}(cn) \leq \rho_{L \sqcup s^*}(cn) = \bar{\rho}_L(cn). \quad \square$$

This theorem has the corollary:

**THEOREM 3:** *If  $L' \leq L$  then there exists an integer  $c$  such that*

$$\forall n \in \mathbb{N}_+, \quad \bar{\rho}_{L'}(n) \leq \bar{\rho}_L(cn).$$

*Proof:* We have  $L' \sqcup s^* \leq L' \leq L$ . Hence theorem 2 yields that

$$\forall n \in \mathbb{N}_+, \quad \rho_{L' \sqcup s^*}(n) \leq \bar{\rho}_L(cn)$$

for some integer  $c$ .  $\square$

$\pi_{\{s\}}$  is an alphabetic morphism verifying  $\pi_{\{s\}}(L \sqcup s^*) = L$  and  $\pi_{\{s\}}^{-1}(L) = L \sqcup s^*$ . Hence lemmas 3 and 5 yield the theorem:

**THEOREM 4:** *If  $L$  is a language then*

$$\forall n \in \mathbb{N}_+ \quad \rho_L(n) \leq \bar{\rho}_L(n) < n(\rho_L(n) + 1).$$

*Remark:* In this paper, the rational index of a language and its extended rational index will be referred to as its rational indexes.

### 3. The rational come generated by $S_\#$

In order to evaluate the rational indexes of  $S_\#$ , we first give two lemmas.

**LEMMA 7:**  $\forall n \in \mathbb{N}_+ \rho_{S_\#}(n) \geq 2n - 1$ .



*Proof:* Let  $n$  be a positive integer. The shortest word in  $S_{\neq}$  recognised by the  $n$  state automaton drawn in figure 1 is  $a^{n-1}b^n$ . Its length is  $2n-1$ . Hence  $\rho_{S_{\neq}}(n) \geq 2n-1$ .  $\square$

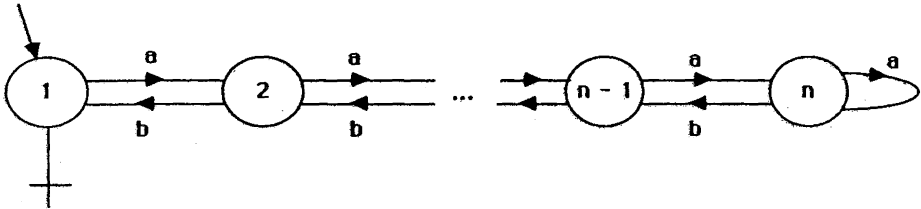


Figure 1.

LEMMA 8:  $\forall n \in \mathbb{N}_+, \bar{\rho}_{S_{\neq}}(n) \leq 2n-1$ .

*Proof:* Let  $n$  be a positive integer. Let  $\mathcal{A}$  be an  $n$  state automaton recognising at least one word in  $S_{\neq} \sqcup s^*$ . Let  $w$  be a shortest word in  $L(\mathcal{A}) \cap (S_{\neq} \sqcup s^*)$ . Let us assume that  $|w| \geq 2n$ . Then a successful path in  $\mathcal{A}$  labeled by  $w$  holds at least two disjoint loops. Hence  $w = \alpha u \beta v \gamma$  for some words  $\alpha, \beta, \gamma, u$  and  $v$  such that  $u$  and  $v$  are non-empty and  $\mathcal{A}$  recognises  $\alpha \beta v \gamma, \alpha u \beta \gamma$  and  $\alpha \beta \gamma$ . These three words belong obviously to  $a^* b^* \sqcup s^*$  but they do not belong to  $S_{\neq} \sqcup s^*$ , since they are shorter than  $w$ . Hence they belong to  $S_{=} \sqcup s^*$ . *I.e.* they hold as many  $a$  as  $b$ , and so do  $u, v$  and  $w$ . This is a contradiction to  $w \in S_{\neq} \sqcup s^*$ . Hence we have proved that  $|w| < 2n$ .  $\square$

THEOREM 5:  $\forall n \in \mathbb{N}_+, \bar{\rho}_{S_{\neq}}(n) = \rho_{S_{\neq}}(n) = 2n-1$ .

*Proof:* Lemmas 7, 8 and theorem 4 yield

$$\forall n \in \mathbb{N}_+, 2n-1 \leq \rho_{S_{\neq}}(n) \leq \bar{\rho}_{S_{\neq}}(n) \leq 2n-1. \quad \square$$

Theorems 2 and 5 yield the proposition:

PROPOSITION 1: *If  $L \leq S_{\neq}$ , then  $\exists c \in \mathbb{N}, \forall n \in \mathbb{N}_+, \rho_L(n) < cn$ .*

We shall handle in this paper a lot of languages dominated by  $S_{\neq}$ . This is why we introduce a new notation:

DEFINITION 3: *Let  $K_1, K_2$ , and  $K_3$  be three languages over the alphabet  $X$ . Let  $\varphi_1$ , and  $\varphi_3$  be two morphisms  $X^* \rightarrow \mathbb{N}$ . Then we shall denote*

$$\nabla_{\neq}(K_1, \varphi_1, K_2, \varphi_3, K_3)$$

the language

$$\{w_1 w_2 w_3 \mid w_1 \in K_1, w_2 \in K_2, w_3 \in K_3, \varphi_1(w_1) \neq \varphi_3(w_3)\}.$$

E.g.  $S_{\neq} = \nabla_{\neq}(a^*, | \cdot |, \varepsilon, | \cdot |, b^*)$ .

LEMMA 9: Let  $K_1, K_2$  and  $K_3$  be three regular languages over the alphabet  $X$ . Let  $\varphi_1$  and  $\varphi_3$  be two morphisms  $X^* \rightarrow \mathbb{N}$ . Then  $\nabla_{\neq}(K_1, \varphi_1, K_2, \varphi_3, K_3) \leq S_{\neq}$ .

Proof: Let  $\varphi'_1: X^* \rightarrow a^*$  be the morphism such that  $\varphi'_1(x) = a^{\varphi_1(x)}$  for every  $x \in X$ . Let  $\varphi'_3: X^* \rightarrow b^*$  be the morphism such that  $\varphi'_3(x) = b^{\varphi_3(x)}$  for every  $x \in X$ . Let  $\sigma$  be the rational transduction, whose graph is the set of the couples  $(w_1 w_2 w_3, \varphi'_1(w_1) \varphi'_3(w_3))$ , when  $w_1 w_2$  and  $w_3$  range over  $K_1 K_2$  and  $K_3$ . Then  $\nabla_{\neq}(K_1, \varphi_1, K_2, \varphi_3, K_3) = \sigma^{-1}(S_{\neq})$ .  $\square$

For instance this lemma proves that  $S_{\neq}$  dominates the language

$$\begin{aligned} \{a^\alpha cb^\beta ca^\gamma cb^\delta \mid \alpha + 2\beta \neq 2\gamma + 5\delta\} \\ = \nabla_{\neq}(a^* cb^*, | \cdot |_a + 2 | \cdot |_b, c, 2 | \cdot |_a + 5 | \cdot |_b, a^* cb^*). \end{aligned}$$

#### IV. STRUCTURE FUNCTIONS

##### 1. Definitions of structure functions

We first define  $S_{\neq}$ -functions.

DEFINITION 4: A  $S_{\neq}$ -function will be a partial function  $g: \mathbb{N}_+ \rightarrow X^*$ , where  $X$  is a finite alphabet, and

$$X^* - g(\mathbb{N}_+) \leq S_{\neq}.$$

Remarks:

- $f$  is a partial function, i.e.  $f(i)$  may not exist for some  $i \in \mathbb{N}_+$ .
- $X^* - g(\mathbb{N}_+)$  is a context-free language, since it is dominated by another context-free language.

- The choice of  $X$  does not matter. Indeed if  $Y$  is a superset of  $X$ , then  $g$  may be considered to be a partial function from  $\mathbb{N}_+$  to  $Y^*$ . And, since

$$X^* - g(\mathbb{N}_+) = (Y^* - g(\mathbb{N}_+)) \cap X^*$$

and conversely

$$Y^* - g(\mathbb{N}_+) = (X^* - g(\mathbb{N}_+)) \cup (Y^* - X^*),$$

it is obvious that  $X^* - g(\mathbb{N}_+) \leq S_{\neq}$  if and only if  $Y^* - g(\mathbb{N}_+) \leq S_{\neq}$ .

**DEFINITION 5:** We define a structure function to be a  $S_{\neq}$ -function  $g: \mathbb{N}_+ \rightarrow X^*$  verifying also the three following properties:

- for some unique letter  $x \in X$ , that we shall denote  $x_g$ , we have  $|g(i)|_x + 1 = i$  for every  $i \in \mathbb{N}_+$ , for which  $g(i)$  exists.
- $g(\mathbb{N}_+)$  does not contain any infinite regular language.
- $g(i)$  is defined for infinitely many  $i$ .

*Remark:* In the first property uniqueness is supposed only for convenience: in order to specify a structure function  $g$ , we only have to give the value of  $g(i)$  whenever it exists; we need not specify which letter is  $x_g$ .

The second property is easily checked by means of the following lemma:

**LEMMA 10:** Let  $g: \mathbb{N}_+ \rightarrow X^*$  be a partial function such that

$$\lim_{i \rightarrow \infty} |g(i)|/i = \infty.$$

Then  $g(\mathbb{N}_+)$  does not contain any infinite regular language.

*Proof:* Let assume  $g(\mathbb{N}_+)$  to contain an infinite regular language. Then we can find three words  $\alpha$ ,  $u$  and  $\beta$  such that  $u$  is not empty and  $\alpha u^+ \beta \subset g(\mathbb{N}_+)$ . Hence for any positive integer  $i$ , there exists a positive integer  $j_i$  such that  $\alpha u^{j_i} \beta = g(j_i)$ . Let  $n$  be a positive integer. Then  $j_1, \dots, j_n$  are  $n$  pairwise distinct positive integers. So that

$$\prod_{i=1}^n j_i \geq n!$$

Thus

$$\prod_{i=1}^n \frac{|g(j_i)|}{j_i} \leq \left( \prod_{i=1}^n |\alpha\beta| + i|u| \right) / n! = \prod_{i=1}^n \frac{|\alpha\beta| + i|u|}{i} \leq |\alpha u \beta|^n$$

hence  $\liminf |g(j_i)|/j_i \leq |\alpha u \beta|$  and thus  $\liminf |g(i)|/i \leq |\alpha u \beta|$  which is not compatible with:

$$\lim_{i \rightarrow \infty} |g(i)|/i = \infty. \quad \square$$

For instance we shall prove later that

$$f_2: \mathbb{N}_+ \rightarrow \{x_1, x_2\}^*, \quad i \mapsto x_1^{i-1} (x_2 x_1^{i-1})^{i-1}$$

is a structure function.

DEFINITION 6: For any structure function  $g$  we define  $\tilde{g}$  to be the partial function  $\mathbb{N}_+ \rightarrow \mathbb{N}_+$  such that  $\tilde{g}(n)$  is the largest integer  $p$  such that  $|g(p)| \leq n - 1$ :

$$\tilde{g}(n) = \max \{ p \mid |g(p)| \leq n - 1 \}.$$

LEMMA 11: If  $g$  is a structure function then:

- there exists an integer  $n_0$  such that  $\tilde{g}(n)$  is defined if and only if  $n \geq n_0$ ;
- $\tilde{g}$  is increasing;
- for any  $n \geq n_0$  we have  $\tilde{g}(n) \leq n$ ;
- $\lim_{n \rightarrow \infty} \tilde{g}(n) = \infty$ .

*Proof:*  $g(\mathbb{N}_+)$  is not empty, since it is infinite. So we can consider the integer  $n_0 = 1 + \min |g(\mathbb{N}_+)|$ . Let us define  $\tilde{G}(n)$  to be the set of numbers  $p$  such that  $g(p)$  exists and  $|g(p)| \leq n - 1$ . Then obviously  $\tilde{G}(n)$  is a increasing sequence of sets, which are non-empty if and only if  $n \geq n_0$ . Furthermore, when  $g(p)$  exists, we have  $|g(p)|_{x_g} = p - 1$ , so that  $|g(p)| \geq p - 1$ . Hence, if  $|g(p)| \leq n - 1$ , then  $p \leq n$ . This proves that  $\tilde{G}(n) \subset [1, n]$ . This completes the proof of the first three assertions of the lemma, since we may notice, that  $\tilde{g}(n)$  is defined if and only if  $\tilde{G}(n)$  is not empty, and then  $\tilde{g}(n) = \max \tilde{G}(n)$ .

Since  $g(i)$  is defined for infinitely many  $i$ , for any integer  $j$  we can find a integer  $p$  such that  $p \geq j$  and  $g(p)$  is defined. Then  $p \in \tilde{G}(|g(p)| + 1)$ , so that

$$p \leq \tilde{g}(|g(p)| + 1).$$

Let  $n$  be an integer such that  $n > |g(p)|$ . Since  $\tilde{g}$  is increasing, we have  $\tilde{g}(n) \geq \tilde{g}(|g(p)| + 1)$  and thus

$$\tilde{g}(n) \geq \tilde{g}(|g(p)| + 1) \geq p \geq j.$$

We have proved that

$$\forall j, \exists p, \forall n, n > |g(p)| \Rightarrow \tilde{g}(n) \geq j.$$

Thus  $\lim_{n \rightarrow \infty} \tilde{g}(n) = \infty$ .  $\square$

DEFINITION 7: Let  $f$  and  $g$  be two structure functions. We shall say that  $f$  dominates  $g$  and we shall write  $f \geq g$ , if there exist two finite alphabets  $X$  and  $Y$  and a rational transduction  $\phi_{f,g}: X^* \rightarrow Y^*$  such that  $f(\mathbb{N}_+) \subset X^*$ ,

$$g(\mathbb{N}_+) \subset Y^*,$$

$$\varphi_{f,g}(X^* - f(\mathbb{N}_+)) = Y^* - g(\mathbb{N}_+),$$

$$\varphi_{f,g}(X^*) = Y^*$$

and

$$\forall u \in X^*, \quad \forall v \in \varphi_{f,g}(u), \quad |u|_{x_f} = |v|_{x_g}.$$

Obviously the domination between structure functions is a pre-order, *i.e.* it is reflexive and transitive.

**DEFINITION 8:** *Let  $f$  and  $g$  be two structure functions. If  $f \geq g$  and  $\tilde{g}(n) \in o(\tilde{f}(n))$ , then we shall say that  $f$  dominates strictly  $g$  and we shall write  $f > g$ .*

Obviously the strict domination between structure functions is transitive.

## 2. Main example of structure function

**DEFINITION 9:** *We define  $X_k = \{x_1, \dots, x_k\}$ , with  $X_0 = \emptyset$ .*

**DEFINITION 10:** *We inductively define the sequence of functions  $f_k: \mathbb{N}_+ \rightarrow X_k^*$  by:*

$$f_0(i) = \varepsilon$$

$$f_k(i) = (f_{k-1}(i) x_k)^{i-1} f_{k-1}(i) \quad \text{if } k > 0.$$

In other words  $f_k(i)$  is the word in  $X_k^{i^k-1}$ , whose  $l$ -th letter is  $x_j$  if  $i^{j-1}$  is the greatest power of  $i$  dividing  $l$ .

So we have

$$|f_k(i)| = i^k - 1$$

and

$$|f_k(i)|_{x_j} = i^{k-j}(i-1).$$

*E.g.*

$$\begin{array}{lll}
 f_0(1) = \varepsilon, & f_0(2) = \varepsilon, & f_0(3) = \varepsilon \\
 f_1(1) = \varepsilon, & f_1(2) = x_1, & f_1(3) = x_1 x_1 \\
 f_2(1) = \varepsilon, & f_2(2) = x_1 x_2 x_1, & f_2(3) = x_1 x_1 x_2 x_1 x_1 x_2 x_1 x_1
 \end{array}$$

$$f_3(1) = \varepsilon, \quad f_3(2) = x_1 x_2 x_1 x_3 x_1 x_2 x_1, \\ f_3(3) = x_1^2 x_2 x_1^2 x_2 x_1^2 x_3 x_1^2 x_2 x_1^2 x_2 x_1^2 x_3 x_1^2 x_2 x_1^2 x_2 x_1^2.$$

DEFINITION 11: Let  $i$  and  $k$  be two positive integers, such that  $i \leq k$ . Let  $w$  be a word of  $X_k^*$ . Then  $\pi_{x_{i-1}}(w)$  can be written in a unique way

$$\pi_{x_{i-1}}(w) = x_i^{\alpha_0} z_1 x_i^{\alpha_1} z_2 x_i^{\alpha_2} \dots z_j x_i^{\alpha_j}$$

where  $\alpha_0, \alpha_1 \dots \alpha_j$  are non-negative integers and  $z_1, z_2 \dots z_j$  are letters of  $X_k - X_i$ . Then  $z_1 z_2 \dots z_j = \pi_{X_i}(w)$  and  $j = |\pi_{X_i}(w)|$ . Let us define the sequence of the groups of  $x_i$  in  $w$  to be the finite sequence

$$(x_i^{\alpha_0}, x_i^{\alpha_1}, \dots, x_i^{\alpha_j}).$$

There are exactly  $|\pi_{X_i}(w)| + 1$  groups of  $x_i$ 's in  $w$ . Some of them may be empty. The length of the group of  $x_i$ 's of rank  $p$  is the number of occurrences of  $x_i$ , which are preceded by exactly  $p$  occurrences of letters of  $X_k - X_i$ . E. g. Let  $k = 3$  and

$$w = x_1 x_2 x_2 x_1 x_1 x_3 x_1 x_1 x_3 x_1 x_2 x_1 x_2 x_2 x_1 x_1 x_3 x_1 x_1 x_1.$$

For  $i = 1$  we have

$$\pi_{x_0}(w) = w = x_1^1 x_2 x_1^2 x_3 x_1^2 x_3 x_1^1 x_2 x_1^1 x_2 x_1^0 x_2 x_1^2 x_3 x_1^3.$$

Note that there is an empty group of  $x_1$  in the middle of the factor  $x_2^2$ . The lengths of the 8 groups of  $x_1$  are 1 2 2 1 1 0 2 and 3. For  $i = 2$ , we have

$$\pi_{x_1}(w) = x_2 x_3^2 x_2^3 x_3 = x_2^1 x_3 x_2^0 x_3 x_2^3 x_3 x_2^0,$$

hence there are 4 groups of  $x_2$ , whose lengths are 1 0 3 and 0. At last

$$\pi_{x_2}(w) = x_3^3,$$

hence  $w$  has 1 group of  $x_3$ , whose length is 3.

$f_k(n)$  is the only word of  $X_k^*$  such that for every  $i \in [1, k]$  the lengths of all its groups of  $x_i$  are equal to  $n - 1$ . And a word of  $X_k^*$  belongs to  $f_k(\mathbb{N}_+)$  if and only if all its groups have the same length.

DEFINITION 12: Let  $A_k = X_k^* - f_k(\mathbb{N}_+)$ .

So a word belongs to  $A_k$  if and only if a group of  $x_i$  and the (only) group of  $x_k$  have different lengths for some  $i$  such that  $1 \leq i < k$ .

LEMMA 12: For every  $k \geq 2$ ,

- $f_k$  is a structure function;
- $\tilde{f}_k(n) = \lfloor \sqrt[k]{n} \rfloor$  and
- $f_k > f_{k+1}$ .

The remaining of this section will be the proof of this lemma. For this we first prove two lemmas.

LEMMA 13: *Let  $k \geq 2$ . There exists a rational transduction  $\sigma_{f_k, f_{k+1}} : X_k^* \rightarrow X_{k+1}^*$  such that*

$$\text{If } w' \in \sigma_{f_k, f_{k+1}}(w) \text{ then } |w'|_{x_{k+1}} = |w|_{w_k} \tag{1}$$

$$\sigma_{f_k, f_{k+1}}(X_k^*) = X_{k+1}^*. \tag{2}$$

$$\sigma_{f_k, f_{k+1}}(A_k) = A_{k+1} \tag{3}$$

*Proof:* Let  $\varphi : X_{k+1}^* \rightarrow X_k^*$  be the morphism defined by:  $\varphi(x_1) = \varepsilon$  and  $\varphi(x_{i+1}) = x_i$  for  $i \geq 1$ . Let  $\varphi' : X_k^* \rightarrow X_{k+1}^*$  be the substitution defined by:  $\varphi'(x_1) = x_1$  and  $\varphi'(x_i) = (x_2 x_1^*)^* x_{i+1} (x_1^* x_2)^*$  for  $i \geq 2$ . We define  $\sigma_{f_k, f_{k+1}}$  by

$$\sigma_{f_k, f_{k+1}}(A) = \varphi^{-1}(A) \cup (x_1^* x_2)^* \varphi'(A) (x_2 x_1^*)^*.$$

(1) holds obviously, and (2) too, since  $\varphi^{-1}(X_k^*) = X_{k+1}^*$ .

DEFINITION 13: *If  $0 < i < k$ , we shall denote  $A_{k,i}$  the set of the words  $w$  belonging to  $X_k^*$  holding a group of  $x_i$  whose length is not  $|w|_{x_i}$ .*

We have

$$A_k = A_{k,1} \cup \dots \cup A_{k,k-1}.$$

If  $w \in X_k^*$  then the groups of  $x_{i+1}$  in a word  $w' \in \varphi^{-1}(w)$  have the lengths of the groups of  $x_i$  in  $w$  for every  $i \in \{1, \dots, k\}$ . Its groups of  $x_1$  have any lengths. Hence  $\varphi^{-1}(A_k)$  is the set of the words of  $X_{k+1}^*$ , in which for some  $i$  such that  $2 \leq i < k+1$  a group of  $x_i$  and the group of  $x_{k+1}$  have different lengths. I.e.  $\varphi^{-1}(A_{k,i}) = A_{k+1,i+1}$  and

$$\varphi^{-1}(A_k) = A_{k+1,2} \cup \dots \cup A_{k+1,k}. \tag{4}$$

Similarly let  $w$  be a word in  $X_k^*$ . Let us consider the groups of  $x_1$  in  $w$ :

$$w = x_1^{\alpha_1} x_{i_1} x_1^{\alpha_2} x_{i_2} \dots x_1^{\alpha_k} x_{i_k} x_1^{\alpha_{k+1}}$$

where  $k = |\pi_{x_1}(w)|$  and  $\forall j, i_j > 1$ . Then

$$(x_1^* x_2^*)^* \varphi'(w) (x_2 x_1^*)^* = (x_1^* x_2^*)^* x_1^{\alpha_1} (x_2 x_1^*)^* x_{i_1} (x_1^* x_2^*)^* x_1^{\alpha_2} (x_2 x_1^*)^* x_{i_2} \dots \\ \dots (x_1^* x_2^*)^* x_1^{\alpha_k} (x_2 x_1^*)^* x_{i_k} (x_1^* x_2^*)^* x_1^{\alpha_{k+1}} (x_2 x_1^*)^* x_{i_{k+1}}.$$

Let  $w'$  be a word in  $(x_1^* x_2^*)^* \varphi'(w) (x_2 x_1^*)^*$ . The groups of  $x_{i+1}$  in  $w'$  have the lengths of the groups of  $x_i$  in  $w$  for every  $i \in \{2, \dots, k\}$ . The groups of  $x_2$  in  $w'$  have any lengths. And the groups of  $x_1$  of  $w$  appear among those of  $w'$ . More precisely every group  $x_1^j$  of  $x_1$  in  $w$  becomes in  $w'$  a factor belonging to  $(x_1^* x_2^*)^* x_1^j (x_2 x_1^*)^*$ , i.e. a group of  $x_2$  of any length  $\lambda$ , whose members alternate with  $\lambda+1$  groups of  $x_1$ , among which one is  $x_1^j$ . Hence  $(x_1^* x_2^*)^* \varphi'(A_k) (x_2 x_1^*)^*$  is the set of the words of  $X_{k+1}$ , in which for some  $i \in \{1, 3, \dots, k\}$  a group of  $x_i$  and the group of  $x_{k+1}$  have different lengths. I.e.

$$(x_1^* x_2^*)^* \varphi'(A_k) (x_2 x_1^*)^* = A_{k+1,1} \cup A_{k+1,3} \cup \dots \cup A_{k+1,k}. \tag{5}$$

(4) and (5) add and yield

$$\varphi^{-1}(A_k) \cup (x_1^* x_2^*)^* \varphi'(A_k) (x_2 x_1^*)^* = A_{k+1,1} \cup \dots \cup A_{k+1,k},$$

i.e.  $\sigma_{f_k, f_{k+1}}(A_k) = A_{k+1}$ .  $\square$

*Remark:* This proof works only if  $k \geq 2$ . For instance in a word of  $A_3$  either a group of  $x_2$  and the group of  $x_3$  have different lengths and then it belongs to  $\varphi^{-1}(A_2)$ , or a group of  $x_1$  and the group of  $x_3$  have different lengths and then it belongs to  $(x_1^* x_2^*)^* \varphi'(A_2) (x_2 x_1^*)^*$ . On the other hand  $A_1 = \emptyset$ . Hence  $\sigma_{f_1, f_2}(A_1) = \emptyset \neq A_2$ .

LEMMA 14:  $A_k \leq S_{\neq}$  for any  $k \geq 2$ .

Proof: We shall prove it inductively.

•  $A_2$  is the set of the words in  $\{x_1, x_2\}^*$  in which two consecutive groups of  $x_1$  have different lengths or the number of  $x_2$  is not the length of the last group of  $x_1$ . I.e.

$$A_2 = (x_1^* x_2^*)^* \nabla_{\neq} (x_1^* | \cdot |, x_2 | \cdot |, x_1^*) (x_2 x_1^*)^* \\ \cup \nabla_{\neq} ((x_1^* x_2^*)^* | \cdot |_{x_2}, \varepsilon | \cdot |, x_1^*).$$

This proves that  $A_2 \leq S_{\neq}$ .

• Let  $k$  be an integer greater than 2. Let us assume that  $A_{k-1} \leq S_{\neq}$ . Lemma 13 yields that  $A_k = \sigma_{f_{k-1}, f_k}(A_{k-1})$ . Hence  $A_k \leq A_{k-1}$ . This proves that  $A_k \leq S_{\neq}$ .  $\square$



*Proof of lemma 12* Let  $k$  be an integer such that  $k \geq 2$ . According to lemma 14,  $f_k$  is a  $S_{\neq}$ -function. For any  $j \in [1, k]$  and any  $i \in \mathbb{N}_+$  we have

$$|f_k(i)|_{x_j} = i^{k-j}(i-1)$$

so that  $x_k$  is the only letter occuring  $i-1$  times in  $f_k(i)$  for every  $i$ . Hence  $x_{f_k} = x_k$ . Since

$$|f_k(i)| = i^k - 1, \tag{6}$$

we have

$$\lim_{i \rightarrow \infty} |f_k(i)|/i = \infty,$$

proving thereby that  $f_k(\mathbb{N}_+)$  holds no infinite regular language. We have shown that  $f_k$  is a structure function. (6) results in the second assertion of lemma 12. So

$$\tilde{f}_k(n) \sim n^{1/k}.$$

This proves that  $\tilde{f}_{k+1}(n) \in o(\tilde{f}_k(n))$ , while lemma 13 proves that  $f_k \geq f_{k+1}$ . So the third assertion of lemma 12 holds.  $\square$

## V. THE LANGUAGE RELATED TO A STRUCTURE FUNCTION

### 1. Definition of $L_g$

Let  $g : \mathbb{N}_+ \rightarrow X^*$  be a structure function. Let  $b_1, a_\infty$  and  $b_\infty$  be three letters not belonging to  $X$ . We shall define a language  $L_g \subset (X \cup \{b_1, a_\infty, b_\infty\})^*$ .  $L_g$  is a subset of the regular language

$$F_g = (b_1^* \sqcup X^*)(a_\infty b_\infty^*)^*,$$

that we shall call its frame. We define the structured part of  $L_g$  to be

$$S_g = \bigcup_{i \in \mathbb{N}_+} (b_1^* \sqcup g(i))(a_\infty b_\infty^*)^i,$$

the unstructured part of  $L_g$  to be

$$U_g = (b_1^* \sqcup (X^* - g(\mathbb{N}_+)))(a_\infty b_\infty^*)^*,$$

and the extended structured part of  $L_g$  to be

$$E_g = \{ w \in F_g, |w|_{x_g} + 1 = |w|_{a_\infty} \}.$$

These three languages are subsets of  $F_g$ . Since  $|g(i)|_{x_g} + 1 = i$ , we notice that  $S_g = E_g - U_g$ .

DEFINITION 14: *The above definitions of  $S_g$ ,  $U_g$  and  $E_g$  allow us to define  $L_g$  as the union of  $E_g$  and  $U_g$ . It is also the disjoint union of  $S_g$  and  $U_g$ .*

$$L_g = E_g \cup U_g = S_g \sqcup U_g.$$

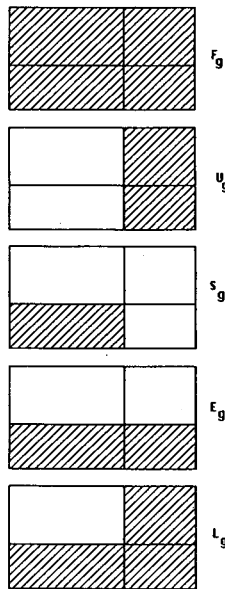


Figure 2.

Figure 2 represents the various languages, we just defined.

$S_g$  is not a context-free language. (We shall not prove it.) But since  $g$  is a  $S_\neq$ -function,  $U_g \leq S_\neq$  and it is obvious that  $E_g \leq S_ =$ . Hence  $U_g$  and  $E_g$  are context-free languages, and so is  $L_g$ .

**2. Lower bound on  $\rho_{L_g}$ .**

Let  $n \in \mathbb{N}_+$ . Let us get a lower bound on  $\rho_{L_g}(n)$ . Let  $p = \tilde{g}(n)$ . Let  $\mathcal{A}$  be the automaton depicted in figure 3.

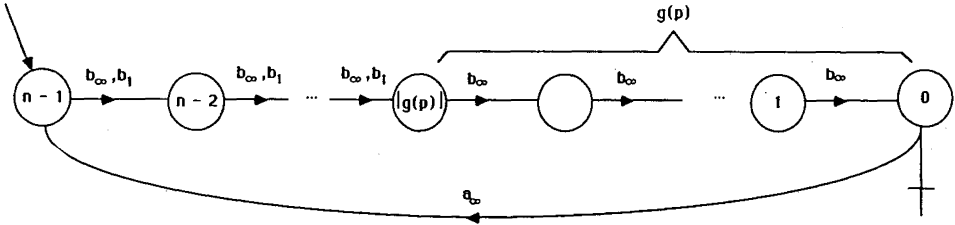


Figure 3.

In this figure



stands for



where  $w = y_1 \dots y_l$ .

This automaton has  $n$  states. It is made of a simple path of length  $n-1$  leading from the only initial state to the only final state. Every arc of this path is labeled by two letters in such a way that the whole path is labeled by  $b_1^{n-1-g(p)} g(p)$  and by  $b_\infty^{n-1}$ . There is also an arc leading from the final state to the initial state labeled by  $a_\infty$ . So  $\mathcal{A}$  recognises a word of  $(b_1^* \sqcup X^*)(a_\infty b_\infty^*)^*$  if and only if it is

$$b_1^{n-1-g(p)} g(p) (a_\infty b_\infty^{n-1})^m$$

for some  $m \in \mathbb{N}$ . This word belongs to  $L_g$  only if  $m = p$  and then it belongs to  $S_g$ . Thus the shortest (and only) word in  $L(\mathcal{A}) \cap L_g$  is

$$w = b_1^{n-1-g(p)} g(p) (a_\infty b_\infty^{n-1})^p.$$

Hence

$$\rho_{L_g}(n) \geq |w| = n-1 + \tilde{g}(n)n. \tag{7}$$

*Remark:*  $|b_1^{n-1-|g(p)|}g(p)|=n-1$  and the letter  $b_1$  is used to ensure that the path labeled by  $b_1^{n-1-|g(p)|}g(p)$  is a simple path (*i.e.* a path holding no loops) of maximal length  $(n-1)$  in an  $n$  state automaton. Similarly  $b_\infty$  is used to ensure that the loop labeled by  $a_\infty b_\infty^{n-1}$  is a simple loop of maximal length.

**3. Upper bound on  $\bar{\rho}_{L_g}$ .**

Let  $n \in \mathbb{N}_+$ . Let  $\mathcal{A}$  be any automaton with  $n$  states recognising at least one word in  $L_g \sqcup s^*$ . Let  $w$  be a shortest word in  $(L_g \sqcup s^*) \cap L(\mathcal{A})$ . We shall give an upper bound on  $|w|$ , that depends only on  $n$  and not on  $\mathcal{A}$  so that it will be also an upper bound on  $\bar{\rho}_{L_g}(n)$ . Let us consider a successful path  $\gamma$  in  $\mathcal{A}$  labeled by  $w$ .

- First let us assume that  $(U_g \sqcup s^*) \cap L(\mathcal{A}) \neq \emptyset$ .

Let  $w'$  be a shortest word in  $(U_g \sqcup s^*) \cap L(\mathcal{A})$ . Then  $|w'| \leq \bar{\rho}_{U_g}(n)$  because of the definition of rational index.  $w'$  belongs to  $(L_g \sqcup s^*) \cap L(\mathcal{A})$ , whose shortest word is  $w$ . Hence  $|w| \leq |w'|$ . Thus  $|w| \leq \bar{\rho}_{U_g}(n)$ .

- Let us assume now that  $U_g \sqcup s^*$  and  $L(\mathcal{A})$  are disjoint.

Then every word in  $(L_g \sqcup s^*) \cap L(\mathcal{A})$  belongs to  $S_g \sqcup s^*$ . Thus  $w$  belongs to  $S_g \sqcup s^*$  and

$$w \in \underbrace{(b_1^* \sqcup g(p) \sqcup s^*)}_{|\cdot| < n} \left( \underbrace{a_\infty (b_\infty^* \sqcup s^*)}_{|\cdot| < n} \right)^p$$

$|\cdot| \leq pn + n - 1$

for some positive interger  $p$ . Braces show upper bounds on the lengths of parts of  $w$ , that we shall prove.

First let us prove that there are at most  $n-1$  letters in  $w$  before the first  $a_\infty$ . Let us assume that this part of  $w$  holds a loop. If the label of this loop belongs to  $b_1^* \sqcup s^*$  then it can be removed yielding a shorter word than  $w$  belonging to  $S_g \sqcup s^*$ . This is a contradiction. Hence the label of this loop does not belong to  $b_1^* \sqcup s^*$ . Since  $g(\mathbb{N}_+)$  holds no infinite regular language, we can change  $g(p)$  into a word of  $X^* - g(\mathbb{N}_+)$  by iterating this loop. This transforms  $w$  into a word of  $(U_g \sqcup s^*) \cap L(\mathcal{A})$ . This is a contradiction. Hence the prefix of  $w$  belonging to  $b_1^* \sqcup g(p) \sqcup s^*$  holds no loop.

If we remove loops from the part of  $w$  belonging to  $b_\infty^* \sqcup s^*$ , then  $w$  changes into a shorter word of  $L(\mathcal{A}) \cap (S_g \sqcup s^*)$ . This is a contradiction. We have proved that the overbraced parts of  $w$  contain no loops. Hence their lengths are smaller than  $n$ .  $w$  is made of  $p+1$  parts, whose lengths are

at most  $n-1$ , and  $p$  times the letter  $a_\infty$ . Hence its length is at most  $pn+n-1$ . We have  $|g(p)| \leq n-1$ . Hence  $p \leq \tilde{g}(n)$ . Thus in this case we have

$$|w| \leq n-1 + \tilde{g}(n)n.$$

The results in the two cases, we have looked at, can be summarized by

$$|w| \leq \max(\bar{\rho}_{U_g}(n), n-1 + \tilde{g}(n)n).$$

Hence

$$\bar{\rho}_{L_g}(n) \leq \max(\bar{\rho}_{U_g}(n), n-1 + \tilde{g}(n)n). \quad (8)$$

#### 4. Value of $\rho_{L_g}$

Since  $U_g \leq S_\#$  proposition 1 yields

$$\bar{\rho}_{U_g}(n) \in O(n),$$

while lemma 11 states  $\lim_{n \rightarrow \infty} \tilde{g}(n) = \infty$ . Hence

$$\bar{\rho}_{U_g}(n) \in o(n-1 + \tilde{g}(n)n).$$

Hence for large enough  $n$  we have

$$\bar{\rho}_{U_g}(n) < n-1 + \tilde{g}(n)n.$$

Hence (7) and (8) and theorem 4 yield

$$\rho_{L_g}(n) = \bar{\rho}_{L_g}(n) = n-1 + \tilde{g}(n)n \quad \text{for large enough } n.$$

We have proved the theorem:

**THEOREM 6:** *If  $g$  is a structure function, then  $L_g$  is a context-free language, whose rational index is*

$$\rho_{L_g}(n) = \bar{\rho}_{L_g}(n) = n-1 + \tilde{g}(n)n \quad \text{for large enough } n.$$

**DEFINITION 15:** *If  $k$  is a integer greater than 1, then  $L_{f_k}$  will be denoted by  $L_k$  for simplicity.*

According to theorem 6, the language  $L_k$  is a context-free language, whose rational index is

$$\rho_{L_k}(n) = \bar{\rho}_{L_k}(n) = n-1 + \lfloor \sqrt[k]{n} \rfloor n \quad \text{for large enough } n.$$

$$\rho_{L_k}(n) \sim n^{1+1/k}.$$

The following section is concerned with relationship between domination of structure functions and domination of their related languages.

**5. Comparison of the various  $L_g$ .**

**THEOREM 7:** *Let  $f$  and  $g$  be two structure functions. If  $f \geq g$  then  $L_f \geq L_g$ .*

*Proof:* Using the rational transduction  $\varphi_{f, g}: X^* \rightarrow Y^*$ , we shall build a rational transduction  $\varphi'$  such that

$$\varphi'(L_f) = L_g. \tag{9}$$

If  $w \in F_f$  then it belongs to  $(b_1^* w_1) w_2$  for some unique  $w_1 \in X^*$  and  $w_2 \in (a_\infty b_\infty^*)^*$  and we define  $\varphi'(w)$  to be  $(b_1^* \varphi_{f, g}(w_1)) w_2$ .

If  $w \notin F_f$  then we define  $\varphi'(w)$  to be  $\emptyset$ . Since  $\varphi_{f, g}$  is a rational transduction and  $F_f$  is a regular language, it follows that  $\varphi'$  is a rational transduction. The properties of  $\varphi_{f, g}$  yield properties of  $\varphi'$ :

- $\varphi_{f, g}(X^*) = Y^*$  hence  $\varphi'(F_f) = F_g$ .
- If  $w_1 \in X^*$  and  $w'_1 \in \varphi_{f, g}(w_1)$  then  $|w_1|_{x_f} = |w'_1|_{x_g}$  hence  $\varphi'(E_f) = E_g$ .
- $\varphi_{f, g}(X^* - f(\mathbb{N}_+)) = Y^* - g(\mathbb{N}_+)$  hence  $\varphi'(U_f) = U_g$ .
- These last two points prove (9).  $\square$

We shall use the notation  $\tilde{f}(n) \in O(\tilde{g}(O(n)))$ . It means that  $\tilde{f}(n) \in O(\tilde{g}(h(n)))$  for some function  $h \in O(n)$ . In other words

$$\exists h: \mathbb{N}_+ \rightarrow \mathbb{N}_+, \exists c > 0, \exists n_0, \forall n > n_0, h(n) \leq cn \text{ and } \tilde{f}(n) \leq c\tilde{g}(h(n)).$$

Eliminating  $h$  yields

$$\exists c > 0, \exists n_0, \forall n > n_0, \tilde{f}(n) \leq c \max_{i \in [0, cn]} \tilde{g}(i).$$

Since  $\tilde{g}$  is increasing, it becomes

$$\exists c > 0, \exists n_0, \forall n > n_0, \tilde{f}(n) \leq c\tilde{g}(cn),$$

or in other words, for some positive  $c$  and large enough  $n$  we have  $\tilde{f}(n) \leq c\tilde{g}(cn)$ . We can also write

$$\exists c > 0, \limsup_{n \rightarrow \infty} \tilde{f}(n)/\tilde{g}(cn) < \infty.$$

Anyway, it is simpler to write  $\tilde{f}(n) \in O(\tilde{g}(O(n)))$  since it saves quantificators.

Similarly  $\tilde{f}(n) \in o(\tilde{g}(O(n)))$  means

$$\exists c > 0, \lim_{n \rightarrow \infty} \tilde{f}(n)/\tilde{g}(cn) = 0,$$

or

$$\exists c > 0, \forall c' > 0, \exists n_0, \forall n > n_0, \tilde{f}(n) \leq c' \tilde{g}(cn).$$

LEMMA 15: *Let  $f$  and  $g$  be two structure functions. If  $L_f \leq L_g$ , then  $\tilde{f}(n) \in O(\tilde{g}(O(n)))$  [i. e. for some  $c$  and for large enough  $n$  we have  $\tilde{f}(n) \leq c\tilde{g}(cn)$ ].*

*Proof:* According to theorem 6,

$$\bar{\rho}_{L_g}(n) = n - 1 + \tilde{g}(n)n \quad \text{and} \quad \bar{\rho}_{L_f}(n) = n - 1 + \tilde{f}(n)n$$

for large enough  $n$ . Since  $L_f \leq L_g$ , theorem 3 proves that for some integer  $c$  we have

$$\forall n \in \mathbb{N}_+, \quad \bar{\rho}_{L_f}(n) \leq \bar{\rho}_{L_g}(cn).$$

So that for large enough  $n$  we have  $n - 1 + \tilde{f}(n)n \leq cn - 1 + \tilde{g}(cn)cn$  i. e.  $\tilde{f}(n) \leq c - 1 + \tilde{g}(cn)c$ , which proves that  $\tilde{f}(n) < 2c\tilde{g}(cn)$ , since  $\tilde{g}(cn) \geq 1$ .  $\square$

Theorem 7 and lemma 15 combine immediatly into the lemma:

LEMMA 16: *Let  $f$  and  $g$  be two structure functions. If  $f \leq g$  then  $\tilde{f}(n) \in O(\tilde{g}(O(n)))$ .*

LEMMA 17: *Let  $f$  and  $g$  be two partial increasing functions from  $\mathbb{N}_+$  to  $\mathbb{N}_+$ . The three following properties cannot all be true.*

- For some integer  $d$ ,  $f(n) \in O(n^d)$ .
- $g(n) \in o(f(O(n)))$ .
- $f(n) \in O(g(O(n)))$ .

*Proof:* Let assume all the three properties to be true. The last two properties result in  $f(n) \in O(o(f(O(O(n)))) = o(f(O(n)))$ . Since  $f$  is increasing, this means that for some positive integer  $c$  we have  $\lim_{n \rightarrow \infty} f(cn)/f(n) = \infty$ . So that

we can find an integer  $n_0$  such that for any  $n \geq n_0$ , we have  $f(cn)/f(n) \geq 2c^d$ . Then we can inductively prove that for any positive integer  $l$  we have  $f(c^l n_0) \geq 2^l c^{ld} f(n_0)$ , so that

$$\lim_{l \rightarrow \infty} f(c^l n_0)/(c^l n_0)^d = \infty,$$

and thus  $\limsup_{n \rightarrow \infty} f(n)/n^d = \infty$ . This is contrary to the first property.  $\square$

Theorem 7 has the corollary:

**THEOREM 8:** *Let  $f$  and  $g$  be two structure functions. If  $f > g$  then  $L_f > L_g$ .*

*Proof:*  $f \geq g$ , hence  $L_f \geq L_g$ .  $\tilde{f}$  and  $\tilde{g}$  are two increasing positive partial functions, verifying  $\tilde{g} \in o(\tilde{f})$  and  $\tilde{f}(n) \leq n$ . So that according to lemma 17, we cannot have  $\tilde{f}(n) \in O(\tilde{g}(O(n)))$ . Lemma 15 yields then that  $L_g \not\geq L_f$ .  $\square$

For instance if  $k \geq 2$  then  $L_{k+1} < L_k$ .

**VI. THE LANGUAGE RELATED TO A FINITE SEQUENCE OF STRUCTURE FUNCTIONS**

The purpose of this section is to build for every finite sequence of structure functions  $g_1, \dots, g_e$  a context-free language whose rational index is  $\Theta\left(n \prod_{i=1}^e \tilde{g}_i(n)\right)$ . Hence it will follow that for every sequence  $k_1, \dots, k_e$  of integers greater than 1, the sequence of structure functions  $f_{k_1}, \dots, f_{k_e}$  yields a context-free language, whose rational index is  $\Theta(n^{1+1/k_1+\dots+1/k_e})$ , so that for every rational number  $\lambda$  greater than 1, we can find a context-free language whose rational index is  $\Theta(n^\lambda)$ .

In order to avoid a lot of subscripts and ellipses (« ... ») and to make the proofs clearer, we shall first handle a sequence  $f, g, h$  of three structure functions, and then we shall generalize the results to any sequence of structure functions.

**1. Definition of  $L_{f, g, h}$**

Let  $f: \mathbb{N}_+ \rightarrow X^*$ ,  $g: \mathbb{N}_+ \rightarrow Y^*$  and  $h: \mathbb{N}_+ \rightarrow Z^*$  be three structure functions. We assume that  $X, Y, Z$  and  $\{b_1, a_2, b_2, a_3, b_3, a_\infty, b_\infty, \#\}$  are four disjoint alphabets.  $L_{f, g, h}$  will be a language on the alphabet

$$X \cup Y \cup Z \cup \{b_1, a_2, b_2, a_3, b_3, a_\infty, b_\infty\},$$

but to define it we shall use the larger alphabet

$$\Omega = X \cup Y \cup Z \cup \{b_1, a_2, b_2, a_3, b_3, a_\infty, b_\infty, \#\}.$$

Let  $A \subset \Omega^*$  and  $B \subset \Omega^*$  be two languages and  $i$  be an integer greater than 1. We define  $A \uparrow_i B$  to be the set of the words of  $A$  in which every factor  $a_\infty b_\infty^*$  is replaced by a word of  $a_i B$ , in which every occurrence of  $b_1$  is replaced by



an occurrence of  $b_i$ . More precisely  $A \uparrow_i B = \tau_{\uparrow_i B}(A)$  where  $\tau_{\uparrow_i B}$  is the substitution defined by:

$$\begin{aligned}\tau_{\uparrow_i B}(b_\infty) &= \varepsilon \\ \tau_{\uparrow_i B}(a_\infty) &= a_i \varphi_{b_1, b_i}(B) \\ \tau_{\uparrow_i B}(x) &= x \quad \text{for any other letter}\end{aligned}$$

where  $\varphi_{b_1, b_i}$  is the strictly alphabetic morphism, which replaces  $b_1$  with  $b_i$  and keeps the other letters unchanged.  $\uparrow$  has interesting obvious properties:

•  $\uparrow$  is associative: For any languages  $A$ ,  $B$  and  $C$  and any integers  $i$  and  $j$  greater than 1, the two languages  $(A \uparrow_i B) \uparrow_j C$  and  $A \uparrow_i (B \uparrow_j C)$  are equal, so that we can denote them  $A \uparrow_i B \uparrow_j C$ .

• If  $A$  and  $B$  are context-free languages, then so is  $A \uparrow_i B$ .

• If  $B$  is a regular language, then  $A \uparrow_i B \leq A$ .

• If  $A$  and  $B$  are both regular languages, then so is  $A \uparrow_i B$ .

At last we define  $\tau_\#$  to be the rational transduction, which keeps words containing at least one  $\#$  and then erases all the  $\#$  in the kept words. *I.e.* if  $A \subset \Omega^*$  then  $\tau_\#(A) = \tau_{\{\#\}}(A \cap \Omega^* \# \Omega^*)$ . For instance

$$\tau_\#(\{dbc, dbb\#c, \#cb\#b\}) = \{dbbc, cbb\}.$$

We can now define  $L_{f, g, h}$ . As  $L_g$  is a subset of its frame  $F_g = (b_1^* \sqcup X^*)(a_\infty b_\infty^*)^*$ , similarly  $L_{f, g, h}$  will be a subset of its frame, which is to be the regular language

$$F_{f, g, h} = F_f \uparrow_2 F_g \uparrow_3 F_h = (b_1^* \sqcup X^*)(a_2 (b_2^* \sqcup Y^*) a_3 (b_3 \sqcup Z^*)(a_\infty b_\infty^*)^*)^*.$$

We define the structured part of  $L_{f, g, h}$  to be

$$S_{f, g, h} = S_f \uparrow_2 S_g \uparrow_3 S_h$$

and the extended structured part of  $L_{f, g, h}$  to be

$$E_{f, g, h} = E_f \uparrow_2 E_g \uparrow_3 E_h.$$

$S_{f, g, h}$  is not a context-free language, but  $E_{f, g, h}$  is.

We define  $U_{f, g, h}$ , the unstructured part of  $L_{f, g, h}$ , to be the set of the words  $w$  in  $F_f \uparrow_2 F_g \uparrow_3 F_h$  such that at least one of the words of  $F_f$ ,  $F_g$  and

$F_h$  involved in the construction of  $w$  is unstructured, *i. e.*

$$\begin{aligned}
 U_{f, g, h} &= \tau_{\#}((F_f \cup \# U_f) \uparrow_2 (F_g \cup \# U_g) \uparrow_3 (F_h \cup \# U_h)) \\
 &= \tau_{\#}(((F_f \cup \# U_f) \uparrow_2 F_g \uparrow_3 F_h) \cup (F_f \uparrow_2 (F_g \cup \# U_g) \uparrow_3 F_h) \quad (10) \\
 &\quad \cup (F_f \uparrow_2 F_g \uparrow_3 (F_h \cup \# U_h))) \\
 &= (U_f \uparrow_2 F_g \uparrow_3 F_h) \\
 &\quad \cup \tau_{\#}(F_f \uparrow_2 (F_g \cup \# U_g) \uparrow_3 F_h) \\
 &\quad \cup \tau_{\#}(F_f \uparrow_2 F_g \uparrow_3 (F_h \cup \# U_h))
 \end{aligned}$$

Conversely  $F_{f, g, h} - U_{f, g, h}$  is made of the words  $w$  belonging to  $F_f \uparrow_2 F_g \uparrow_3 F_h$  such that none of the words of  $F_f, F_g$  and  $F_h$  involved in the construction of  $w$  is unstructured. *I. e.*

$$E_{f, g, h} - U_{f, g, h} = (F_f - U_f) \uparrow_2 (F_g - U_g) \uparrow_3 (F_h - U_h).$$

Hence

$$\begin{aligned}
 E_{f, g, h} - U_{f, g, h} &= E_{f, g, h} \cap (F_{f, g, h} - U_{f, g, h}) \\
 &= (E_f \uparrow_2 E_g \uparrow_3 E_h) \cap ((F_f - U_f) \uparrow_2 (F_g - U_g) \uparrow_3 (F_h - U_h)) \\
 &= (E_f \cap (F_f - U_f)) \uparrow_2 (E_g \cap (F_g - U_g)) \uparrow_3 (E_h \cap (F_h - U_h)) \\
 &= S_f \uparrow_2 S_g \uparrow_3 S_h \\
 &= S_{f, g, h}.
 \end{aligned}$$

DEFINITION 16: *The above definitions of  $S_{f, g, h}, E_{f, g, h}$  and  $U_{f, g, h}$  allow us to define  $L_{f, g, h}$  as the union of its extended structured part and its unstructured part, and it is also the disjoint union of its structured part and its unstructured part.*

$$L_{f, g, h} = E_{f, g, h} \cup U_{f, g, h} = S_{f, g, h} \sqcup U_{f, g, h}.$$

Figure 2 still holds.  $U_f, U_g$  and  $U_h$  are dominated by  $S_{\#}$  and  $F_f, F_g$  and  $F_h$  are regular languages, hence (10) proves that  $U_{f, g, h} \leq S_{\#}$ . Hence  $L_{f, g, h}$  is a context-free language.

We can express  $L_{f, g, h}$  in an another way.  $F_{f, g, h}$  is the union of the sets

$$(b_1^* \sqcup \alpha) \prod_{i=1}^p \left( a_2 (b_2^* \sqcup \beta_i) \prod_{j=1}^{q_i} (a_3 (b_3^* \sqcup \gamma_{i, j}) (a_{\infty} b_{\infty}^*)^{r_i, j}) \right)$$

where

$$\begin{aligned}
 & p \in \mathbb{N}, \quad \alpha \in X^*, \\
 & q_i \in \mathbb{N}, \quad \beta_i \in Y^* \quad \text{for } 1 \leq i \leq p, \\
 & r_{i,j} \in \mathbb{N}, \quad \gamma_{i,j} \in Z^* \quad \text{for } 1 \leq i \leq p \text{ and } 1 \leq j \leq q_i.
 \end{aligned}$$

$U_{f,g,h}$  is made of those sets verifying the condition

$$\begin{aligned}
 & \alpha \in X^* - f(\mathbb{N}_+) \\
 \text{or} & \\
 & \exists i, \beta_i \in Y^* - g(\mathbb{N}_+) \tag{C_u} \\
 \text{or} & \\
 & \exists i, \exists j, \gamma_{i,j} \in Z^* - h(\mathbb{N}_+)
 \end{aligned}$$

$E_{f,g,h}$  is made of the sets verifying the condition

$$\begin{aligned}
 & |\alpha|_{x_f} + 1 = p \\
 \text{and} & \\
 & \forall i, |\beta_i|_{x_g} + 1 = q_i \tag{C_e} \\
 \text{and} & \\
 & \forall i, \forall j, |\gamma_{i,j}|_{x_h} + 1 = r_{i,j}
 \end{aligned}$$

$L_{f,g,h}$  is made of the sets verifying at least one of the two conditions  $(C_e)$  and  $(C_u)$ .  $S_{f,g,h}$  is made of the sets verifying  $(C_e)$  but not  $(C_u)$  *i. e.*

$$\begin{aligned}
 & \alpha = f(r) \\
 \text{and} & \\
 & \forall i, \beta_i = g(q_i) \tag{C_s} \\
 \text{and} & \\
 & \forall i, \forall j, \gamma_{i,j} = h(r_{i,j})
 \end{aligned}$$

Hence

$$\begin{aligned}
 S_{f,g,h} = \bigcup_{p \in \mathbb{N}_+} (b_1^* \sqcup f(p)) \prod_{i=1}^p \left( a_2 \cup_{q_i \in \mathbb{N}_+} (b_2^* \sqcup g(q_i)) \right. \\
 \left. \prod_{j=1}^{q_i} \left( a_3 \cup_{r_{i,j} \in \mathbb{N}_+} (b_3^* \sqcup h(r_{i,j})) (a_\infty b_\infty^*)^{r_{i,j}} \right) \right)
 \end{aligned}$$

**2. Lower bound on  $\rho_{L_{f, g, h}}$**

Let  $n$  be a large enough integer such that the three integers  $p = \tilde{f}(n)$ ,  $q = \tilde{g}(n)$  and  $r = \tilde{h}(n)$  exist. We want to obtain a lower bound on  $\rho_{L_{f, g, h}}(n)$ . Let  $\mathcal{A}$  be the automaton depicted in figure 4.

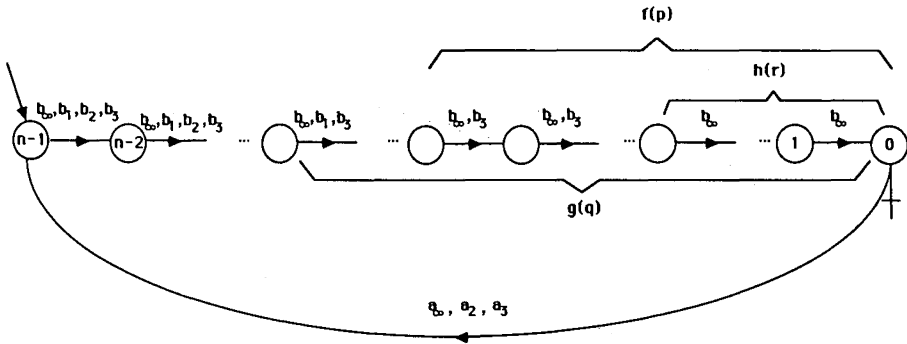


Figure 4.

This automaton has  $n$  states. It is made of a simple path of length  $n - 1$  leading from the only initial state to the only final state. Every arc of this path is labeled by four letters in such a way that the path is labeled by each of the four words  $b_1^{n-1-|f(p)|} f(p)$ ,  $b_2^{n-1-|g(q)|} g(q)$ ,  $b_3^{n-1-|h(r)|} h(r)$  and  $b_\infty^{n-1}$ . There is also an arc leading from the final state to the initial state labeled by the three letters  $a_2$ ,  $a_3$  and  $a_\infty$ . So the set of the words of  $F_{f, g, h}$  that  $\mathcal{A}$  recognizes is

$$b_1^{n-1-|f(p)|} f(p) (a_2 b_2^{n-1-|g(q)|} g(q) (a_3 b_3^{n-1-|h(r)|} h(r) (a_\infty b_\infty^{n-1})^*)^*)^*$$

It is disjoint with  $U_{f, g, h}$ , but it has exactly one element of  $S_{f, g, h}$ , which is

$$b_1^{n-1-|f(p)|} f(p) (a_2 b_2^{n-1-|g(q)|} g(q) (a_3 b_3^{n-1-|h(r)|} h(r) (a_\infty b_\infty^{n-1})^r)^q)^p,$$

whose length is  $n - 1 + p(n + q(n + rn))$ . Hence

$$\rho_{L_{f, g, h}}(n) \geq n - 1 + \tilde{f}(n)(n + \tilde{g}(n)(n + \tilde{h}(n)n)). \tag{11}$$

**3. Upper bound on  $\rho_{L_{f, g, h}}$**

Let  $n \in \mathbb{N}_+$ . Let  $\mathcal{A}$  be any automaton with  $n$  states recognizing at least one word in  $L_{f, g, h} \sqcup s^*$ . Let  $w$  be a shortest word in  $(L_{f, g, h} \sqcup s^*) \cap L(\mathcal{A})$ . We

shall give an upper bound on  $|w|$ , that depends only on  $n$  and not on  $\mathcal{A}$  so that it will be also an upper bound on  $\bar{\rho}_{L_{f,g,h}}(n)$ . Let us consider a successful path  $\gamma$  in  $\mathcal{A}$  labeled by  $w$ .

- First let us assume that  $(U_{f,g,h} \sqcup s^*) \cap L(\mathcal{A}) \neq \emptyset$ .

As in the previous section, we can conclude that  $|w| \leq \bar{\rho}_{U_{f,g,h}}(n)$ .

- Let us assume now that  $U_{f,g,h} \sqcup s^*$  and  $L(\mathcal{A})$  are disjoint. Then every word in  $(L_{f,g,h} \sqcup s^*) \cap L(\mathcal{A})$  belongs to  $S_{f,g,h} \sqcup s^*$ . Thus  $w$  belongs to  $S_{f,g,h} \sqcup s^*$  and

$$w \in \underbrace{(b_1^* \sqcup f(p))}_{|\cdot| < n} \times \prod_{i=1}^p \underbrace{\left( a_2 \underbrace{(b_2^* \sqcup g(q_i))}_{|\cdot| < n} \prod_{j=1}^{q_i} \underbrace{(a_3 \underbrace{(b_3^* \sqcup h(r_{i,j}))}_{|\cdot| < n})}_{|\cdot| \leq r_{i,j} n} (a_\infty \underbrace{(b_\infty^* \sqcup s^*)}_{|\cdot| \leq r_{i,j} n})^{r_{i,j}} \right)}_{|\cdot| \leq q_i(n + \tilde{h}(n)n)}$$

$$\underbrace{\hspace{15em}}_{|\cdot| \leq p(n + \tilde{g}(n)(n + \tilde{h}(n)n))}$$

for some non negative integers  $p, q_1, \dots, q_p, r_{i,1}, \dots, r_{i,q_i}$  for  $1 \leq i \leq p$ . As in the previous section overbraced parts of  $w$  hold no loops. Hence their lengths are smaller than  $n$ . As in the previous section we have  $|f(p)| \leq n - 1$ . Hence  $p \leq \tilde{f}(n)$ . Similarly for every  $i$  in  $\{1, \dots, r\}$  we have  $q_i \leq \tilde{g}(n)$ . And for every  $i$  and  $j$  we have  $r_{i,j} \leq \tilde{h}(n)$ . All of this allows us to compute an upper bound on  $|w|$ . Indeed:

$$|w| \leq n - 1 + \tilde{f}(n)(n + \tilde{g}(n)(n + \tilde{h}(n)n)).$$

The results in the two cases, we have looked at, can be summarized by

$$|w| \leq \max(\bar{\rho}_{U_{f,g,h}}(n), n - 1 + \tilde{f}(n)(n + \tilde{g}(n)(n + \tilde{h}(n)n))).$$

This upper bound on  $|w|$  is also an upper bound on  $\rho_{L_{f,g,h}}(n)$ .

#### 4. Value of $\rho_{L_{f,g,h}}$

As in the previous section we can conclude that

$$\bar{\rho}_{L_{f,g,h}}(n) = \rho_{L_{f,g,h}}(n) = n - 1 + \tilde{f}(n)(n + \tilde{g}(n)(n + \tilde{h}(n)n)) \quad \text{for large enough } n.$$

**5. Generalization to more than three levels**

In the same way we built  $L_{f, g, h}$ , we can define the language  $L_{g_1, \dots, g_e}$  for any sequence  $g_1, \dots, g_e$  of structure functions. In order to describe precisely this language we must change slightly the notations used so far. We assume that  $g_i: \mathbb{N}_+ \rightarrow Y_i^*$  for any  $i \in [1, e]$ , and that  $Y_1 \dots Y_e$  and  $\{b_1, a_2, b_2, \dots, a_e, b_e, a_\infty, b_\infty, \#\}$  are disjoint. We define

$$\Omega = Y_1 \cup \dots \cup Y_e \cup \{b_1, a_2, b_2, \dots, a_e, b_e, a_\infty, b_\infty, \#\}.$$

Indeed these are the notations used so far except for  $Y_1, Y_2$  and  $Y_3$ , which were called  $X, Y$  and  $Z$ .

We define

$$\begin{aligned} F_{g_1, \dots, g_e} &= F_{g_1} \uparrow_2 \dots \uparrow_e F_{g_e} \\ S_{g_1, \dots, g_e} &= S_{g_1} \uparrow_2 \dots \uparrow_e S_{g_e} \\ E_{g_1, \dots, g_e} &= E_{g_1} \uparrow_2 \dots \uparrow_e E_{g_e} \\ U_{g_1, \dots, g_e} &= \tau_{\#}((F_{g_1} \cup \# U_{g_1}) \uparrow_2 \dots \uparrow_e (F_{g_e} \cup \# U_{g_e})) \\ L_{g_1, \dots, g_e} &= E_{g_1, \dots, g_e} \cup U_{g_1, \dots, g_e} = S_{g_1, \dots, g_e} \sqcup U_{g_1, \dots, g_e}. \end{aligned}$$

Obviously the previous results generalize:

**THEOREM 9:** *If  $g_1, \dots, g_e$  are structure functions on disjoint alphabets, then  $F_{g_1, \dots, g_e}$  is a regular language,  $E_{g_1, \dots, g_e}$  and  $L_{g_1, \dots, g_e}$  are context-free languages,  $U_{g_1, \dots, g_e} \leq S_{\#}$  and for large enough  $n$  we have*

$$\bar{\rho}_{L_{g_1, \dots, g_e}}(n) = \rho_{L_{g_1, \dots, g_e}}(n) = n - 1 + \tilde{g}_1(n)(n + \tilde{g}_2(n)(n + \dots \tilde{g}_e(n)n) \dots).$$

**6. Main example**

**DEFINITION 17:** *For any positive integers  $i$  and  $j$  we define the alphabet*

$$X_{i, j} = \{x_{1, j}, x_{2, j}, \dots, x_{i, j}\}.$$

**DEFINITION 18:** *We define  $\iota_{i, j}: X_i^* \rightarrow X_{i, j}^*$  to be the strictly alphabetic isomorphism, which adds the second subscript  $j$  to every letter. I. e.  $\iota_{i, j}(x_l) = x_{l, j}$  for every  $l \in [1, i]$ .*

DEFINITION 19: Let  $k_1, \dots, k_e$  be a finite sequence of integers greater than 1. Then  $L_{k_1, \dots, k_e}$  will be a short notation for

$$L_{(i_{k_1, 1} \circ f_{k_1}), (i_{k_2, 2} \circ f_{k_2}), \dots, (i_{k_e, e} \circ f_{k_e})}$$

Remarks: This notation is compatible with the notation  $L_k$  defined in the previous section to mean  $L_{f_k}$  for an integer  $k > 1$ , if we identify  $X_k$  and  $X_{k, 1}$ .

– The functions  $i$ 's are needed only to ensure, that the structure functions  $i_{k_1, 1} \circ f_{k_1}, i_{k_2, 2} \circ f_{k_2}, \dots, i_{k_e, e} \circ f_{k_e}$  use disjoint alphabets  $(X_{k_1, 1}, \dots, X_{k_e, e})$ .

Theorem 9 yields that  $L_{k_1, \dots, k_e}$  is a context-free language, whose rational index is

$$\bar{\rho}_{L_{k_1, \dots, k_e}}(n) = \rho_{L_{k_1, \dots, k_2}}(n) = n - 1 + \lfloor \sqrt[k_1]{n} \rfloor (n + \lfloor \sqrt[k_2]{n} \rfloor (n + \dots \lfloor \sqrt[k_e]{n} \rfloor n) \dots)$$

for large enough  $n$ . So that

$$\bar{\rho}_{L_{k_1, \dots, k_e}}(n) = \rho_{L_{k_1, \dots, k_e}}(n) \sim n^{1 + 1/k_1 + \dots + 1/k_e}$$

THEOREM 10: Let  $r \in \mathbb{Q} \cap [1, +\infty[$ . Then there exists a context-free language  $L$  such that  $\rho_L(n) = \bar{\rho}_L(n) \in \Theta(n^r)$ .

Proof: If  $r = 1$  then  $L = S_{\neq}$  works, since  $\rho_{S_{\neq}}(n) = \bar{\rho}_{S_{\neq}}(n) = 2n - 1 \in \Theta(n)$ .

• Let us assume  $r > 1$ . Then  $r = p/q$  for some integers  $p$  and  $q$  such that  $0 < q < p$ . Hence  $r = 1 + (p - q) / q$  and we can choose  $L = L_{\underbrace{q, \dots, q}_{p - q \text{ times}}}$ .  $\square$

We study now the domination between the various  $L_{g_1, \dots, g_e}$ . The three following theorems will provide an easy way to build infinite strictly increasing or strictly decreasing sequences of context-free languages.

THEOREM 11: Let  $g_1, \dots, g_e$  and  $h_1, \dots, h_e$  be two sequences of structure functions. If  $g_i \geq h_i$  for all  $i$ , then  $L_{g_1, \dots, g_e} \geq L_{h_1, \dots, h_e}$ , if these two languages exist.

Proof: Let us assume that  $g_i: \mathbb{N}_+ \rightarrow Y_i^*$  and  $h_i: \mathbb{N}_+ \rightarrow Z_i^*$  for  $i = 1, \dots, e$ . The existence of  $L_{g_1, \dots, g_e}$  means, that the  $e + 1$  alphabets  $\{b_1, a_2, b_2, \dots, a_e, b_e, a_{\infty}, b_{\infty}, \#\}$  and  $Y_1, \dots, Y_e$  are disjoint. Similarly, the existence of  $L_{h_1, \dots, h_e}$  means, that the  $e + 1$  alphabets  $Z_1, \dots, Z_e$  and  $\{b_1, a_2, b_2, \dots, a_e, b_e, a_{\infty}, b_{\infty}, \#\}$  are disjoint.

For every  $i$  in  $\{1, \dots, e\}$ , we have  $g_i \geq h_i$ . This means, by definition, the existence of a rational transduction  $\sigma_{g_i, h_i}: Y_i^* \rightarrow Z_i^*$  with some properties. We

define the rational transduction  $\sigma_i: b_i^* \sqcup Y_i^* \rightarrow b_i^* \sqcup Z_i^*$  such that

$$\sigma_i = \pi_{\{b_i\}}^{-1} \circ \sigma_{g_i, h_i} \circ \pi_{\{b_i\}}.$$

It is the rational transduction which maps every word in  $b_i^* \sqcup w$  onto  $b_i^* \sqcup \sigma_{g_i, h_i}(w)$  for every word  $w \in Y_i^*$ .

We define

$$\Omega_g = Y_1 \sqcup \dots \sqcup Y_e \sqcup \{ b_1, a_2, b_2, \dots, a_e, b_e, a_\infty, b_\infty, \# \}$$

and

$$\Omega_h = Z_1 \sqcup \dots \sqcup Z_e \sqcup \{ b_1, a_2, b_2, \dots, a_e, b_e, a_\infty, b_\infty, \# \}.$$

We are now ready to define the rational transduction  $\sigma'': \Omega_g^* \rightarrow \Omega_h^*$  such that

$$\sigma''(L_{g_1, \dots, g_e}) = L_{h_1, \dots, h_e}.$$

- If  $w \in \Omega_g^* - F_{g_1, \dots, g_e}$  then  $\sigma''(w) = \emptyset$ .
- Let us assume now that  $w \in F_{g_1, \dots, g_e}$ . Then we have

$$w \in \alpha \prod_{i_2=1}^p \left( a_2 \alpha_{i_2} \prod_{i_3=1}^{p_{i_2}} \left( a_3 \alpha_{i_2, i_3} \prod_{i_4=1}^{p_{i_2, i_3}} \right. \right. \\ \left. \left. \times \left( \dots \prod_{i_e=1}^{p_{i_2, \dots, i_{e-1}}} (a_e \alpha_{i_2, \dots, i_e} (a_\infty b_\infty^*)^{p_{i_2, \dots, i_e}}) \dots \right) \right) \right)$$

where

$$p \in \mathbb{N}, \quad \alpha \in b_1^* \sqcup Y_1^* \\ p_{i_2} \in \mathbb{N}, \quad \alpha_{i_2} \in b_2^* \sqcup Y_2^* \quad \text{for } 1 \leq i_2 \leq p, \\ p_{i_2, i_3} \in \mathbb{N}, \quad \alpha_{i_2, i_3} \in b_3^* \sqcup Y_3^* \quad \text{for } 1 \leq i_2 \leq p \text{ and } 1 \leq i_3 \leq p_{i_2}, \\ \vdots \\ p_{i_2, \dots, i_e} \in \mathbb{N}, \quad \alpha_{i_2, \dots, i_e} \in b_e^* \sqcup Y_e^* \quad \text{for } 1 \leq i_2 \leq p, \\ 1 \leq i_3 \leq p_{i_2}, \dots, 1 \leq i_{e+1} \leq p_{i_2, \dots, i_e}.$$



Then we define

$$\sigma''(w) = \sigma_1(\alpha) \prod_{i_2=1}^p \left( a_2 \sigma_2(\alpha_{i_2}) \prod_{i_3=1}^{p_{i_2}} \left( a_3 \sigma_3(\alpha_{i_2, i_3}) \prod_{i_4=1}^{p_{i_2, i_3}} \right. \right. \\ \left. \left. \times \left( \dots \prod_{i_e=1}^{p_{i_2, \dots, i_{e-1}}} (a_e \sigma_e(\alpha_{i_2, \dots, i_e}) (a_\infty b_\infty^*)^{p_{i_2, \dots, i_e}}) \dots \right) \right) \right).$$

The graph of the transduction  $\sigma''$  is

$$\Sigma'' = \Sigma_1((a_2, a_2) \Sigma_2((a_3, a_3) \Sigma_3(\dots ((a_e, a_e) \Sigma_e(a_\infty b_\infty^* \times a_\infty b_\infty^*)^* \dots)^*)^*).$$

where  $\Sigma_i$  denotes the graph of the rational transduction  $\sigma_i$ . The product of the two regular sets  $a_\infty b_\infty^* \times a_\infty b_\infty^* = (a_\infty, \epsilon)(b_\infty, \epsilon)^*(\epsilon, a_\infty)(\epsilon, b_\infty)^*$  and the graphs of rational transductions  $\Sigma_1, \dots, \Sigma_i$  are rational subsets of  $\Omega_g^* \times \Omega_h^*$  and so  $\Sigma''$  too. This proves that  $\sigma''$  is a rational transduction.

As in the proof of theorem 7 the properties of the  $\sigma_i$ 's result in  $\sigma''(U_{g_1, \dots, g_e}) = U_{h_1, \dots, h_e}$  and  $\sigma''(E_{g_1, \dots, g_2}) = E_{h_1, \dots, h_e}$  hence  $\sigma''(L_{g_1, \dots, g_e}) = L_{h_1, \dots, h_e}$  and  $L_{g_1, \dots, g_e} \geq L_{h_1, \dots, h_e}$ .  $\square$

Theorem 11 has the corollary:

**THEOREM 12:** *Let  $g_1, \dots, g_e$  and  $h_1, \dots, h_e$  be two sequences of structure functions on disjoint alphabets such that  $g_i \leq h_i$  for all  $i$ , and  $g_{i_0} < h_{i_0}$  for some  $i_0$ . Then  $L_{g_1, \dots, g_e} < L_{h_1, \dots, h_e}$ .*

*Proof:* This theorem can be proved in the same way as theorem 8:

$$\bar{\rho}_{L_{g_1, \dots, g_e}}(n) \sim n \prod_{i=1}^e \tilde{g}_i(n) \\ \bar{\rho}_{L_{h_1, \dots, h_e}}(n) \sim n \prod_{i=1}^e \tilde{h}_i(n).$$

For all  $i$ , since  $g_i \leq h_i$ , lemma 16 yields

$$\tilde{g}_i(n) \in O(\tilde{h}_i(O(n)))$$

For  $i_0$  we have

$$\tilde{g}_{i_0}(n) \in o(\tilde{h}_{i_0}(n)).$$

These facts result in

$$\bar{\rho}_{L_{g_1, \dots, g_e}}(n) \in o(\bar{\rho}_{L_{h_1, \dots, h_e}}(O(n))).$$

On the other hand we have  $\bar{\rho}_{L_{h_1, \dots, h_e}}(n) \in O(n^{e+1})$ .

Lemma 17 yields then that  $\bar{\rho}_{L_{h_1, \dots, h_e}}(n) \notin O(\bar{\rho}_{L_{g_1, \dots, g_e}}(O(n)))$  so that lemma 15 yields that  $L_{g_1, \dots, g_e} \not\leq L_{h_1, \dots, h_e}$ .  $\square$

Hence, if  $k_1, \dots, k_e$  and  $l_1, \dots, l_e$  are two different sequences of integers, such that for all  $i$  we have  $2 \leq k_i \leq l_i$ , then  $L_{k_1, \dots, k_e} > L_{l_1, \dots, l_e}$ .

NOTATION: Let  $(g_1, \dots, g_e)$  be a finite sequence of length  $e$ . We shall denote by  $(g_1, \dots, \hat{g}_{e'}, \dots, g_e)$  the finite sequence of length  $e-1$  obtained by the removal of  $g_{e'}$ .

THEOREM 13: *Let  $e$  be an integer greater than 1. Let  $g_1, \dots, g_e$  be a sequence of structure functions. Let  $e' \in \{1, \dots, e\}$ . Then*

$$L_{g_1, \dots, g_e} > L_{g_1, \dots, \hat{g}_{e'}, \dots, g_e}$$

*Proof:* We shall only prove this theorem in the case  $e=4$  and  $e'=2$ . The proof is similar in the general case.

Let  $f: \mathbb{N}_+ \rightarrow X^*$ ,  $g: \mathbb{N}_+ \rightarrow Y^*$ ,  $h: \mathbb{N}_+ \rightarrow Z^*$  and  $l: \mathbb{N}_+ \rightarrow T^*$  be four structure functions, such that  $X, Y, Z, T$  and  $\{b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_\infty, b_\infty, \#\}$  are five disjoint alphabets. We shall prove that

$$L_{f, g, h, l} > L_{f, h, l}$$

For that we choose a word  $w_1$  in  $a_3 S_h \uparrow_4 S_l$  and a positive integer  $n_g$  such that  $g(n_g)$  exists. Then we transform every word belonging to  $L_{f, g, h, l} \cap F_f \uparrow_2 (g(n_g) w_1^{n_g-1} a_3 F_h \uparrow_4 F_l)$  into a word of  $F_f \uparrow_2 F_h \uparrow_3 F_l$  by removing all the factors of the form  $g(n_g) w_1^{n_g-1} a_3$  and then by decreasing by one the subscripts of the letters  $b_3, a_4$  and  $b_4$ . The removed factors follow the occurrences of  $a_2$ .

Indeed this transformation is a bijection from

$$L_{f, g, h, l} \cap F_f \uparrow_2 (g(n_g) w_1^{n_g-1} a_3 F_h \uparrow_4 F_l)$$

onto  $L_{f, h, l}$ , and it can be performed by the reciprocal of a morphisme  $\phi$ .

Let us detail this. Let us define

$$\Omega = X \sqcup Y \sqcup Z \sqcup T \sqcup \{b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_\infty, b_\infty, \#\}.$$

Let  $n_g$  (resp.  $n_h$  and  $n_l$ ) be the least integer, for which  $g$  (resp.  $h$  and  $l$ ) is defined. Let

$$w_1 = a_3 h(n_h)(a_4 l(n_l) a_\infty^{n_l})^{n_h}$$

be the word in  $a_3 S_h \uparrow_4 S_l$  having a minimal number of occurrences of  $a_\infty$ .

Let

$$w_2 = g(n_g) w_1^{n_g - 1} a_3.$$

$w_2$  has been chosen such that

$$\begin{aligned} \forall u \in \Omega^*, \quad w_2 u \in (S_g \uparrow_3 S_h \uparrow_4 S_l) &\Leftrightarrow u \in (S_h \uparrow_4 S_l), \\ \forall u \in \Omega^*, \quad w_2 u \in (U_g \uparrow_3 U_h \uparrow_4 U_l) &\Leftrightarrow u \in (U_h \uparrow_4 U_l), \\ \forall u \in \Omega^*, \quad w_2 u \in (E_g \uparrow_3 E_h \uparrow_4 E_l) &\Leftrightarrow u \in (E_h \uparrow_4 E_l), \\ \forall u \in \Omega^*, \quad w_2 u \in (F_g \uparrow_3 F_h \uparrow_4 F_l) &\Leftrightarrow u \in (F_h \uparrow_4 F_l). \end{aligned}$$

We define the morphism

$$\varphi: (X \sqcup Z \sqcup T \sqcup \{b_1, a_2, b_2, a_3, b_3, a_\infty, b_\infty, \#\})^* \rightarrow \Omega^*$$

by

$$\begin{aligned} \varphi(x) &= x && \text{if } x \in (X \sqcup Z \sqcup T) \\ \varphi(b_1) &= b_1 \\ \varphi(a_2) &= a_2 w_2 \\ \varphi(b_2) &= b_3 \\ \varphi(a_3) &= a_4 \\ \varphi(b_3) &= b_4 \\ \varphi(a_\infty) &= a_\infty \\ \varphi(b_\infty) &= b_\infty. \end{aligned}$$

Then obviously

$$\begin{aligned} \varphi^{-1}(F_{f,g,h,l}) &= F_{f,h,l} \\ \varphi^{-1}(S_{f,g,h,l}) &= S_{f,h,l} \\ \varphi^{-1}(U_{f,g,h,l}) &= U_{f,h,l} \\ \varphi^{-1}(E_{f,g,h,l}) &= E_{f,h,l} \\ \varphi^{-1}(L_{f,g,h,l}) &= L_{f,h,l}. \end{aligned}$$

So that  $L_{f, g, h, l} \geq L_{f, h, l}$ . On the other hand we have

$$\bar{\rho}_{L_{f, g, h, l}}(n) \sim \bar{\rho}_{L_{f, h, l}}(n) \tilde{g}(n)$$

so that

$$\bar{\rho}_{L_{f, h, l}}(n) \in o(\bar{\rho}_{L_{f, g, h, l}}(n))$$

and we can conclude as in proof of theorem 12, that  $L_{f, h, l} \not\geq L_{f, g, h, l}$ .  $\square$

*E. g.* let  $k_1, \dots, k_e$  be a sequence of integers greater than 1. Let  $e' \in \{1, \dots, e\}$ . Then  $L_{k_1, \dots, k_e} > L_{k_1, \dots, \hat{k}_{e'}, \dots, k_e}$ .

**VII. OTHER EXAMPLES OF STRUCTURE FUNCTIONS**

**1. First example: a structure function leading to a context-free language whose rational index is  $\Theta(n \ln n)$**

DEFINITION 20: Let  $X_{\text{exp}} = \{a, b\}$  and

$$f_{\text{exp}}: \mathbb{N}_+ \rightarrow X_{\text{exp}}^*$$

$$i \mapsto bab^1 ab^3 ab^7 \dots ab^{2^{i-1}-1} = b \prod_{j=1}^{i-1} ab^{2^j-1}.$$

*I. e.*

$$f_{\text{exp}}(1) = b$$

$$f_{\text{exp}}(2) = bab$$

$$f_{\text{exp}}(3) = babab^3$$

$$f_{\text{exp}}(4) = babab^3 ab^7$$

$$f_{\text{exp}}(i+1) = f_{\text{exp}}(i) ab^{|f_{\text{exp}}(i)|}.$$

Let us show that  $f_{\text{exp}}$  is a structure function and  $x_{f_{\text{exp}}} = a$ :

•  $X_{\text{exp}}^* - f_{\text{exp}}(\mathbb{N}_+) = (X_{\text{exp}}^* - b(ab^*)^*) \cup \bigvee_{\neq} (X_{\text{exp}}^* | \cdot |, a, | \cdot |, b^*) (ab^*)^*$  so that according to lemma 9  $X_{\text{exp}}^* - f_{\text{exp}}(\mathbb{N}_+) \leq S_{\neq}$ .

- $\forall i \in \mathbb{N}_+, |f_{\text{exp}}(i)|_a = i - 1$ .
- $\forall i \in \mathbb{N}_+, |f_{\text{exp}}(i)| = 2^i - 1$ , so that

$$\lim_{i \rightarrow \infty} |f_{\text{exp}}(i)|/i = \infty \quad \text{and} \quad \tilde{f}_{\text{exp}}(n) = \lfloor \ln_2 n \rfloor.$$

Theorem 6 yields that  $L_{f_{\text{exp}}}$  is a context-free language and for large enough  $n$  we have

$$\rho_{L_{f_{\text{exp}}}}(n) = \bar{\rho}_{L_{f_{\text{exp}}}}(n) = n - 1 + n\tilde{f}_{\text{exp}}(n) = n - 1 + n \lfloor \ln_2 n \rfloor \sim n \ln_2 n.$$

**2. Second example: a structure function leading to a context-free language whose rational index is  $\Theta(n \ln \ln n)$**

Let us define a new notation in order to express the next examples.

DEFINITION 21: *If  $i \in \mathbb{N}_+$  and  $w$  is a word, such that  $|w| \leq 2^{i-1} - 2$ , then we define*

$$F_{\text{exp}}(i, w) = f_{\text{exp}}(i) b^{-|w|-1} cw,$$

i. e. a copy of  $f_{\text{exp}}(i)$  in which we have replaced the suffix  $b^{|w|+1}$  with  $cw$ . If  $|w| > 2^{i-1} - 2$  then  $f_{\text{exp}}(i)$  ends with too few  $b$ 's and  $F_{\text{exp}}(i, w)$  is not defined.

E. g.  $F_{\text{exp}}(4, d^2 f_{\text{exp}}(2)) = babab^3 abcd^2 bab$  and  $F_{\text{exp}}(3, d^2 f_{\text{exp}}(2))$  is not defined.

Hence, in particular

$$|F_{\text{exp}}(i, w)| = 2^i - 1$$

and

$$|F_{\text{exp}}(i, w)|_a = i - 1 + |w|_a.$$

LEMMA 18: *Let  $f : \mathbb{N}_+ \rightarrow X$  be a  $S_{\neq}$ -function. Let  $X'$  be a subset of  $X$ . Then the function  $g : i \mapsto F_{\text{exp}}(|f(i)|_{X'} + 1, f(i))$  is a  $S_{\neq}$ -function.*

Note that  $X$  and  $\{a, b, c\}$  are not necessarily disjoint.

*Proof:* Let us define  $Y = X \cup \{a, b, c\}$ . Let us define the rational transduction  $\tau : \{a, b\}^* \rightarrow Y^*$  whose graph is made of all the couples  $(w_1 b^{1+|w_2|}, w_1 cw_2)$  for  $w_1 \in \{a, b\}^*$  and  $w_2 \in Y^*$ . Then

$$\begin{aligned} Y^* - g(\mathbb{N}_+) &= (Y^* - X_{\text{exp}}^* c Y^*) \\ &\cup \tau(X_{\text{exp}}^* - f_{\text{exp}}(\mathbb{N}_+)) \\ &\cup X_{\text{exp}}^* c (Y^* - f(\mathbb{N}_+)) \\ &\cup \nabla_{\neq} (X_{\text{exp}}^*, | \cdot |_a, c, | \cdot |_{X'}, X^*). \end{aligned}$$

In this union the first term is regular. The two following terms are dominated by  $S_{\neq}$ , since  $f_{\text{exp}}$  and  $f$  are  $S_{\neq}$ -functions. And the last one is dominated by  $S_{\neq}$ . This proves that  $Y^* - g(\mathbb{N}_+) \leq S_{\neq}$ .  $\square$

LEMMA 19: Let  $f : \mathbb{N}_+ \rightarrow X$  be a  $S_\neq$ -function. Let  $X'$  be a subset of  $X$ . Let  $z$  be a letter, which does not belong to  $X$ . Then the function  $g : i \mapsto f(i)z^{|f(i)|_{X'}}$  is a  $S_\neq$ -function.

Proof: Let us define  $Y = X \cup \{z\}$ . Then

$$Y^* - g(\mathbb{N}_+) = (Y^* - X^*z^*) \cup (Y^* - f(\mathbb{N}_+))z^* \cup \nabla_\neq(X^*, | \cdot |_{X'}, \varepsilon, | \cdot |, z^*).$$

In this union the first term is regular. The second term is dominated by  $S_\neq$ , since  $f$  is a  $S_\neq$ -function. And the last one is dominated by  $S_\neq$ . This proves that  $Y^* - g(\mathbb{N}_+) \leq S_\neq$ .  $\square$

For  $f = f_{\text{exp}}$ ,  $X = X_{\text{exp}}$ ,  $X' = \{a\}$  and  $z = d$  this lemma yields, that

$$g_1: i \mapsto f_{\text{exp}}(i) d^{i-1}$$

is a  $S_\neq$ -function.

Lemma 18 yields for  $f = g_1$ ,  $X = \{a, b, d\}$  and  $X' = \{a, b\}$ , that

$$g_2: i \mapsto F_{\text{exp}}(2^i, f_{\text{exp}}(i) d^{i-1})$$

is a  $S_\neq$ -function.

Indeed  $g_2(i)$  is defined for every  $i \in \mathbb{N}_+$  and  $|g_2(i)|_d = i - 1$  and  $|g_2(i)| = 2^{2^i} - 1$ . So that  $\lim_{i \rightarrow \infty} |g_2(i)|/i = \infty$  and  $g_2$  is a structure function.

According to theorem 6,  $L_{g_2}$  is a context-free language, and for large enough  $n$  we have

$$\rho_{L_{g_2}}(n) = \bar{\rho}_{L_{g_2}}(n) = n - 1 + n\tilde{g}_2(n) = n - 1 + n \lfloor \ln_2 \ln_2 n \rfloor \sim n \ln_2 \ln n.$$

**3. Third example: a structure function leading to a context-free language whose rational index is  $\Theta(n \sqrt[k]{\ln n})$ .**

Let  $k$  be an integer greater than 1. For  $f = f_k$  and  $X = X' = X_k$  lemma 18 yields, that the function  $g_3 : i \mapsto F_{\text{exp}}(i^k, f_k(i))$  is a  $S_\neq$ -function. Indeed it is a structure function such that  $x_{g_2} = x_k$  and  $|g_3(i)| = 2^{i^k} - 1$ . According to theorem 6,  $L_{g_2}$  is a context-free language, and for large enough  $n$  we have

$$\rho_{L_{g_3}}(n) = \bar{\rho}_{L_{g_3}}(n) = n - 1 + n\tilde{g}_3(n) = n - 1 + n \lfloor \sqrt[k]{\ln_2 n} \rfloor \sim n \sqrt[k]{\ln_2 n}.$$

#### 4. Fourth example: a structure function leading to a context-free language whose rational index is $\Theta(n^{\sqrt{2}})$ .

We define  $g_4$  to be the partial function such that  $g_4(n)$  is defined only if  $n$  is a power of 2, and then

$$g_4(2^i) = F_{\text{exp}}(i+j, d^{2^i-1} f_{\text{exp}}(i) f_2(i) c f_2(j) a^{2i^2-j^2} b^{(j+1)^2-2i^2})$$

where  $j = \lfloor \sqrt{2} i \rfloor$ .

*Remark:*  $j$  is the only positive integer such that  $j^2 \leq 2i^2 < (j+1)^2$ .

LEMMA 20:  $g_4$  is a structure function verifying  $|g_4(2^i)| = 2^{\lfloor i(1+\sqrt{2}) \rfloor - 1}$  and  $x_{g_4} = d$ .

*Proof:* In order to prove that  $g_4$  is a structure function, we define

$$g'_4: i \mapsto d^{2^i-1} f_{\text{exp}}(i) f_2(i) c f_2(j) a^{2i^2-j^2} b^{(j+1)^2-2i^2}.$$

Let  $X = X_2 \sqcup \{a, b, c, d\}$ . We have  $g'_4(\mathbb{N}_+) \subset X^*$  and we are going to prove that  $X^* - g'_4(\mathbb{N}_+)$  is equal to the union  $B$  of the following eight languages:

$$\begin{aligned} B_1 &= X^* - d^* \{a, b\}^* X_2^* c X_2^* a^* b^+ \\ B_2 &= \nabla_{\neq} (d^*, | \cdot |, \varepsilon, | \cdot |, \{a, b\}^*) X_2^* c X_2^* a^* b^+ \\ B_3 &= d^* \nabla_{\neq} (\{a, b\}^*, | \cdot |_a, \varepsilon, | \cdot |_{x_2}, X_2^*) c X_2^* a^* b^+ \\ B_4 &= d^* (\{a, b\}^* - f_{\text{exp}}(\mathbb{N}_+)) X_2^* c X_2^* a^* b^+ \\ B_5 &= d^* \{a, b\}^* (X_2^* - f_2(\mathbb{N}_+)) c X_2^* a^* b^+ \\ B_6 &= d^* \{a, b\}^* X_2^* c (X_2^* - f_2(\mathbb{N}_+)) a^* b^+ \\ B_7 &= d^* \{a, b\}^* \nabla_{\neq} (X_2^* c, 2 | \cdot |_{x_2} + | \cdot |_c, \varepsilon, | \cdot |, X_2^* a^*) b^+ \\ B_8 &= d^* \{a, b\}^* X_2^* \nabla_{\neq} (c X_2^*, 3 | \cdot |_c + 2 | \cdot |_{x_2}, \varepsilon, | \cdot |, a^* b^+). \end{aligned}$$

- For any integer  $i$ ,  $g'_4(i)$  does not belong to this union because

$$\begin{aligned} g'_4(i) &\in d^* \{a, b\}^* X_2^* c X_2^* a^* b^+ \\ |d^{2^i-1}| &= 2^i - 1 = |f_{\text{exp}}(i)| \\ |f_{\text{exp}}(i)|_a &= i - 1 = |f_2(i)|_{x_2} \\ f_{\text{exp}}(i) &\in f_{\text{exp}}(\mathbb{N}_+) \\ f_2(i) &\in f_2(\mathbb{N}_+) \\ f_2(j) &\in f_2(\mathbb{N}_+) \end{aligned}$$

$$\begin{aligned}
 2 |f_2(i) c|_{x_2} + |f_2(i) c|_c &= 2 |f_2(i)| + 1 = 2(i^2 - 1) + 1 \\
 &= 2i^2 - 1 = (j^2 - 1) + (2i^2 - j^2) = |f_2(j) a^{2i^2 - j^2}| \\
 3 |cf_2(j)|_c + 2 |cf_2(j)|_{x_2} &= 3 + 2(j - 1) \\
 &= 2j + 1 = (2i^2 - j^2) + ((j + 1)^2 - 2i^2) = |a^{2i^2 - j^2} b^{(j + 1)^2 - 2i^2}|.
 \end{aligned}$$

This proves that  $g'_4(\mathbb{N}_+)$  and  $B$  are disjoint, *i. e.*

$$g'_4(\mathbb{N}_+) \subset X^* - B.$$

- Conversely let  $w$  be a word in  $X^* - B$ .  $w$  belongs to  $X^* - B_1$  *i. e.*

$$w \in d^* \{a, b\}^* X_2^* c X_2^* a^* b^+.$$

Since  $w$  belongs neither to  $B_4$  nor to  $B_5$  nor to  $B_6$ , we have

$$w \in d^* f_{\text{exp}}(\mathbb{N}_+) f_2(\mathbb{N}_+) c f_2(\mathbb{N}_+) a^* b^+,$$

*i. e.*

$$w = d^p f_{\text{exp}}(i') f_2(i) c f_2(j) a^q b^r,$$

for some  $i', i, j, r \in \mathbb{N}_+$  and  $p, q \in \mathbb{N}$ .

Since  $w$  does not belong to  $B_2$ , we have  $p = 2^{i'-1}$ .

Since  $w$  does not belong to  $B_3$ , we have  $i' - 1 = i - 1$  *i. e.*  $i' = i$ .

Since  $w$  does not belong to  $B_7$ , we have  $2i^2 - 1 = (j^2 - 1) + q$  *i. e.*  $q = 2i^2 - j^2$ .

Since  $w$  does not belong to  $B_8$ , we have  $2j + 1 = q + r$  *i. e.*  
 $r = (2j + 1) - (2i^2 - j^2) = (j + 1)^2 - 2i^2$ .

$q \geq 0$  and  $r > 0$  hence  $j^2 \leq 2i^2 < (j + 1)^2$ , *i. e.*  $j = \lfloor \sqrt{2} \rfloor$ . We have proved that  $w = g'_4(i)$ . Hence

$$g'_4(\mathbb{N}_+) \supset X^* - B.$$

We have proved that  $g'_4(\mathbb{N}_+) = X^* - B$  *i. e.*

$$X^* - g'_4(\mathbb{N}_+) = B.$$

$B_1$  is a regular language, and  $B_2 \dots B_8$  are languages dominated by  $S_\neq$ . This proves that  $g'_4$  is a  $S_\neq$ -function.

Since  $|g'_4(i)|_{\{x_2, c\}} = i + j - 1$ , lemma 18 yields that  $g_4$  is a  $S_\neq$ -function too.

$$\begin{aligned}
 |g'_4(i)| &= (2^i - 1) + (2^i - 1) + (i^2 - 1) + 1 + (j^2 - 1) \\
 &\quad + (2i^2 - j^2) + ((j + 1)^2 - 2i^2) \sim 2^{i+1} \in o(2^{i+j}).
 \end{aligned}$$



Hence  $g_4(2^i) = F_{\text{exp}}(i+j, g'_4(i))$  is defined when  $i$  is large enough.

We have

$$|g_4(2^i)|_d = 2^i - 1$$

and

$$|g_4(2^i)| = 2^{i+j} - 1 = 2^{i(1+\sqrt{2})} - 1$$

so that

$$\lim_{i \rightarrow \infty} |g_4(2^i)|/2^i = \infty.$$

Thus  $g_4$  is a structure function and  $x_{g_4} = d$ .  $\square$

Let  $n$  be an integer large enough for  $\tilde{g}_4(n)$  to exist. Then  $\tilde{g}_4(n)$  is the largest integer  $p$  such that

$$|g_4(p)| \leq n - 1.$$

Hence  $p$  is the largest power of 2, say  $2^i$ , such that

$$|g_4(2^i)| \leq n - 1.$$

This inequality is equivalent to the following ones:

$$\begin{aligned} 2^{i+\lfloor \sqrt{2}i \rfloor} - 1 &\leq n - 1, \\ \lfloor i + \sqrt{2}i \rfloor &\leq \log_2 n, \\ \lfloor i + \sqrt{2}i \rfloor &\leq \lfloor \log_2 n \rfloor, \\ i + \sqrt{2}i &< 1 + \lfloor \log_2 n \rfloor, \\ i &< (\sqrt{2} - 1) \lfloor 1 + \log_2 n \rfloor. \end{aligned}$$

This upper bound on  $i$  cannot be an integer, so that the largest  $i$  is

$$\lfloor (\sqrt{2} - 1) \lfloor 1 + \log_2 n \rfloor \rfloor \in (\sqrt{2} - 1) \log_2 n + O(1)$$

and the largest  $p$  is

$$\tilde{g}_4(n) = 2^{\lfloor (\sqrt{2} - 1) \lfloor 1 + \log_2 n \rfloor \rfloor} \in n^{\sqrt{2}-1} 2^{O(1)} = \Theta(n^{\sqrt{2}-1}).$$

Theorem 6 yields that  $L_{g_4}$  is a context-free language, such that for large enough  $n$  we have

$$\rho_{L_{g_4}}(n) = \bar{\rho}_{L_{g_4}}(n) = n - 1 + n \tilde{g}_4(n) = n - 1 + n 2^{\lfloor (1 + \log_2 n)(\sqrt{2} - 1) \rfloor} \in \Theta(n^{\sqrt{2}}).$$

This kind of construction may be generalized:

**5. Fifth example: structure functions leading to a context-free language whose rational index is  $\Theta(n^\lambda)$  for an algebraic number  $\lambda > 1$**

The main example of structure functions was the family of  $f_k$ 's. For any integer  $k$  greater than 1, we have  $\tilde{f}_k(n) \in \Theta(n^{1/k})$ . We extend this notation for other non integral numbers:

LEMMA 21: *Let  $\lambda$  be an irrational algebraic real number greater than 1. Then we can find a structure function  $f_\lambda$  such  $\tilde{f}_\lambda(n) \in \Theta(n^{1/\lambda})$ .*

Proof: Let  $P$  be a minimal polynomial of  $\lambda$ , i.e. a polynomial of minimal degree with integral coefficients such that  $P(\lambda) = 0$ . Let  $m$  be the degree of  $P$ . Let us assume

$$P(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_m t^m.$$

Since  $P$  is irreducible,  $\lambda$  is a simple root of  $P$ , i.e.

$$P(\lambda) = 0$$

and  $P'(\lambda) \neq 0$ , where  $P'$  is the derivative of  $P$ . If  $P'(\lambda) < 0$ , then we replace  $P$  by  $-P$  in order to ensure that

$$P'(\lambda) > 0.$$

$P'$  is a continuous function. Hence we can find two rational numbers  $p_1/q_1$  and  $p_2/q_2$  such that

$$1 \leq \frac{p_1}{q_1} < \lambda < \frac{p_2}{q_2},$$

$$\forall t \in \left[ \frac{p_1}{q_1}, \frac{p_2}{q_2} \right], \quad P'(t) > 0.$$

Hence

$$\forall t \in \left[ \frac{p_1}{q_1}, \lambda \right], \quad P'(t) < 0,$$

$$\forall t \in \left[ \lambda, \frac{p_2}{q_2} \right], \quad P'(t) > 0.$$

The integers  $p_1$ ,  $q_1$ ,  $p_2$  and  $q_2$  are now fixed, and we shall use them to define  $f_\lambda$ .

Let

$$n_1 = \left\lceil 1 / \min \left\{ \frac{p_2}{q_2} - \lambda, \lambda - \frac{p_1}{q_1} \right\} \right\rceil.$$

Let  $i$  be a positive integer. An integer  $j$  verifies the conditions

$$\begin{aligned} q_1 j - p_1 i &\geq 0 \\ p_2 i - q_2 (j+1) &\geq 0 \\ -i^m P(j/i) &> 0 \\ i^m P((j+1)/i) &> 0 \end{aligned} \tag{12}$$

if and only if it verifies

$$\begin{aligned} \frac{p_1}{q_1} \leq \frac{j}{i} < \frac{j+1}{i} \leq \frac{p_2}{q_2}, \\ P\left(\frac{j}{i}\right) < 0 < P\left(\frac{j+1}{i}\right), \end{aligned}$$

*i. e.*

$$\frac{p_1}{q_1} \leq \frac{j}{i} < \lambda < \frac{j+1}{i} \leq \frac{p_2}{q_2},$$

and then

$$j = \lfloor i\lambda \rfloor. \tag{13}$$

Furthermore, if  $i \geq n_1$  then (13) and (12) are equivalent, *i. e.*  $\lfloor i\lambda \rfloor$  is the only integer  $j$  verifying (12). If  $i < n_1$  then (12) may have no solution or it may have the unique solution  $\lfloor i\lambda \rfloor$ .

We define the two alphabets

$$\begin{aligned} D &= \{d_1, \dots, d_9\} \\ X &= \{x_{-1}, \dots, x_{m+1}, a, b, c, c'\}. \end{aligned}$$

The structure function  $f_\lambda$  will be defined on the alphabet

$$\Omega = D \sqcup X.$$

For every positive integer  $i$  for which (12) has a solution  $j$  we define

$$f'_\lambda(i) = d_1 c'^{2^i-1} d_2 f_{\text{exp}}(i) d_3 c^{j-1} d_4 a^{p_2 i - a_2(j+1)} d_5 a^{q_1 j - p_1 i} d_6 f_m(i)$$

$$d_7 f_m(j) d_8 b^{-i^m P(j/i)} d_9 \left( x_{-1} \left( \prod_{k=0}^m (a x_k^{j^k})^{i^{m-k}} \right) a x_{m+1} \right)^2 d_8$$

$$d_7 f_m(j+1) d_8 d_9 \left( x_{-1} \left( \prod_{k=0}^m (a x_k^{(j+1)^k})^{i^{m-k}} \right) a x_{m+1} \right)^2 d_8 b^{i^m P((j+1)/i)},$$

and

$$f_\lambda(2^i) = F_{\text{exp}}(j, f'_\lambda(i)).$$

The letters of  $D$  are used as separators.

The factor  $d_1 c'^{2^i-1}$  ensures that a letter occurs  $2^i - 1$  times in  $f'_\lambda(i)$  and thus in  $f_\lambda(2^i)$  too.

The factor  $d_2 f_{\text{exp}}(i)$  gives a relation between  $i$  and  $2^i$ .

The factor  $d_3 c^{j-1}$  ensures that a letters occurs  $j - 1$  times in  $f'_\lambda(i)$  so that we can define  $F_{\text{exp}}(j, f'_\lambda(i))$ .

The factors  $d_4 a^{p_2 i - a_2(j+1)}$ ,  $d_5 a^{q_1 j - p_1 i}$ ,  $d_8 b^{-i^m P(j/i)}$  and  $d_8 b^{i^m P((j+1)/i)}$  correspond to (12).

The factor  $d_6 f_m(i)$  gives a relation between  $i$  and  $i^k$  for every  $k \in [0, m]$ .

The factor  $d_7 f_m(j)$  gives a relation between  $j$  and  $j^k$  for every  $k \in [0, m]$ .

The factor  $x_{-1} \left( \prod_{k=0}^m (a x_k^{j^k})^{i^{m-k}} \right) a x_{m+1}$  is used to construct the number  $(j/i)^k i^m$ , which is the number of occurrences of  $x_k$ , from the numbers  $j^k$  and  $i^{m-k}$ , for every  $k$  in  $[0, m]$ . The factor  $(a x_k^{j^k})^{i^{m-k}}$  is preceded by  $x_{k-1}$  and followed by  $a x_{k+1}$  for every  $k$  in  $[0, m]$ . This explains what  $x_{-1}$  and  $a x_{m+1}$  are for.  $i^m P(j/i)$  is the linear combination of these numbers  $i^{m-k} j^k$ , whose coefficients are those of  $P$ . These coefficients may not have all the same sign, but in the equality

$$-i^m P\left(\frac{j}{i}\right) + \sum_{k=0}^m \max(0, \alpha_k) i^{m-k} j^k = \sum_{k=0}^m \max(0, -\alpha_k) i^{m-k} j^k + 0$$

both sides are sums of non-negative numbers. This is why this factor appears twice.

In the same way the number  $i^m P((j+1)/i)$  is built in the third line of the expression of  $f'_\lambda(i)$ .

Let  $K = (DX^*)^*$ . The language  $\Omega^* - f'_\lambda(\mathbb{N}_+)$  is the union of the following languages  $G_1, \dots, G_{12}$ .

$$\begin{aligned}
 G_1 &= \Omega^* - \left( d_1 c'^* d_2 \{a, b\}^* d_3 c^* d_4 a^+ d_5 a^+ d_6 X_m^* \right. \\
 &\quad \left. d_7 X_m^* d_8 b^+ d_9 \left( x_{-1} \left( \prod_{k=0}^m a(b x_k^+)^+ \right) a x_{m+1} \right)^2 d_8 \right. \\
 &\quad \left. d_7 X_m^* d_8 d_9 \left( x_{-1} \left( \prod_{k=0}^m a(b x_k^+)^+ \right) a x_{m+1} \right)^2 d_8 b^+ \right) \\
 G_2 &= K d_2 (\{a, b\}^* - f_{\text{exp}}(\mathbb{N}_+)) K \\
 G_3 &= K \{d_6, d_7\} (X_m^* - f_m(\mathbb{N}_+)) K \\
 G_4 &= \nabla_{\neq} (d_1 c'^*, | \cdot |, \varepsilon, | \cdot |, d_2 \{a, b\}^*) K \\
 G_5 &= K \nabla_{\neq} (d_3 c^* d_4 a^+, q_2 | \cdot |_{\{d_3, c, d_4\}^+} | \cdot |_a, K, p_2 | \cdot |_{\{d_6, x_m\}}, d_6 X_m) K \\
 G_6 &= K \nabla_{\neq} (d_3 c^*, q_1 | \cdot |, K, | \cdot |_a + p_1 | \cdot |_{\{d_6, x_m\}}, d_5 a^+ d_6 X_m) K \\
 G_7 &= K \nabla_{\neq} (d_2 \{a, b\}^*, | \cdot |_a, K, | \cdot |_{x_m}, d_6 X_m^*) K \\
 G_8 &= K \nabla_{\neq} (d_3 c^*, | \cdot |_c, K, | \cdot |_{x_m}, d_7 X_m^*) d_8 b^+ K \\
 G_9 &= K \nabla_{\neq} (d_3 c^*, | \cdot |, K, | \cdot |_{x_m}, d_7 X_m^*) d_8 K \\
 G_{10} &= K \bigcup_{k=0}^m \nabla_{\neq} (d_6 X_m^*, |\pi_{x_k}|, K d_9 X^* x_{k-1}, | \cdot |_a, (a x_k^+)^+ a x_{k+1} \Omega^* K \\
 G_{11} &= K \bigcup_{k=0}^m \nabla_{\neq} (d_7 X_m^*, |\pi_{x_{m-k}}|, d_8 b^* d_9 X^* a, | \cdot |, x_k^+) a \Omega^* K \\
 G_{12} &= K \nabla_{\neq} \left( d_8 b^* d_9 X^* x_{m+1}, | \cdot |_b + \sum_{k=0}^m \max(0, \alpha_k) | \cdot |_{x_k}, \varepsilon, \right. \\
 &\quad \left. | \cdot |_b + \sum_{k=0}^m \max(0, -\alpha_k) | \cdot |_{x_k}, x_{-1} X^* d_8 b^* \right) K.
 \end{aligned}$$

These twelve languages are dominated by  $S_{\neq}$ . Hence  $\Omega^* - f'_\lambda(\mathbb{N}_+) = \bigcup_{i=1}^{12} G_i$  is dominated by  $S_{\neq}$  too. So  $f'_\lambda$  is a  $S_{\neq}$ -function.

Since  $|f'_\lambda(i)|_c = j - 1$ , lemma 18 yields that  $f_\lambda$  is a  $S_{\neq}$ -function.

$$|f'_\lambda(i)| \sim 2^{i+1} \in o(2^j)$$

and  $f'_\lambda(i)$  is defined when  $i \geq n_1$ . Hence  $f_\lambda(2^i)$  is defined when  $i$  is large enough.

We have

$$|f_\lambda(2^i)|_{c'} = 2^i - 1$$

and

$$|f_\lambda(2^j)| = 2^j - 1 = 2^{i\lambda_j} - 1$$

so that

$$\lim_{i \rightarrow \infty} |f_\lambda(2^i)|/2^i = \infty.$$

Thus  $f_\lambda$  is a structure function and  $x_{f_\lambda} = c'$ .

Like in the fourth example we get

$$\tilde{f}_\lambda(n) = 2^{\lfloor 1 + \log_2 n/\lambda \rfloor} \in \Theta(n^{1/\lambda})$$

and

$$\rho_{L_{\tilde{f}_\lambda}}(n) \in \Theta(n^{1+1/\lambda}). \quad \square$$

**THEOREM 14:** *Let  $\lambda$  be an algebraic number greater than 1. Then there exists a context-free language  $L$  such that  $\rho_L(n) = \bar{\rho}_L(n) \in \Theta(n^\lambda)$ .*

*Proof:*  $\lambda$  may be expressed as  $\lambda = 1 + 1/\lambda_1 + \dots + 1/\lambda_e$ , where every  $\lambda_i$  is an irrational algebraic number greater than 1. Then lemma 21 and theorem 9 can be applied to copies of  $f_{\lambda_1}, \dots, f_{\lambda_e}$  on disjoint alphabets. This completes the proof.  $\square$

**THEOREM 15:** *Let  $\lambda$  and  $\mu$  be two algebraic numbers such that  $1 < \lambda < \mu$ . Then there exist two context-free languages  $L_\lambda$  and  $L_\mu$  such that:*

$$\rho_{L_\lambda}(n) = \bar{\rho}_{L_\lambda}(n) \in \Theta(n^\lambda),$$

$$\rho_{L_\mu}(n) = \bar{\rho}_{L_\mu}(n) \in \Theta(n^\mu),$$

$$L_\lambda < L_\mu.$$

*Proof:* We may have  $\mu = \lambda + 1/\lambda_{e+1} + \dots + 1/\lambda_e$  for some irrational algebraic numbers  $\lambda_{e+1} \dots \lambda_e$ , greater than 1. We define  $L_\lambda$  and  $L_\mu$  like in the previous proof. Theorem 13 yields, that  $L_\lambda < L_\mu$ .

We can also build structure functions  $f_\lambda$  such that  $\tilde{f}_\lambda(n) \in \Theta(n^{1/\lambda})$  for some transcendental numbers  $\lambda$ , e. g.  $\pi/\sqrt{6}$ :

**6. Sixth example: a structure function leading to a context-free language whose rational index is  $\Theta(n^{1+\sqrt{6}/\pi})$ .**

The construction of this structure function is based upon the equality

$$\frac{\pi^2}{6} = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

First we define the function

$$\alpha: \mathbb{N}_+ \rightarrow \mathbb{N}_+$$

$$i \mapsto \sum_{j=1}^i \left\lfloor \frac{i^2}{j^2} \right\rfloor.$$

We define then  $g_6$  to be the partial function such that  $g_6(n)$  is defined only if  $n$  is a power of 2, and then

$$g_6(2^i) = F_{\text{exp}} \left( \left\lfloor \sqrt{\alpha(i)} \right\rfloor, x_3^{\lfloor \sqrt{\alpha(i)} \rfloor - 1} \right. \\ \left. f_2 \left( \left\lfloor \sqrt{\alpha(i)} \right\rfloor \right) c a^{\alpha(i) - \lfloor \sqrt{\alpha(i)} \rfloor^2} b^{(\lfloor \sqrt{\alpha(i)} \rfloor + 1)^2 - 1 - \alpha(i)} \right. \\ \left. x_4^{2^i - 1} f_{\text{exp}}(i) f_2(i) \prod_{j=1}^i ((x_5 f_2(j))^{i^2/j^2} a^{i^2 \bmod j^2} b^{j^2 - 1 - (i^2 \bmod j^2)}) \right).$$

We can prove easily that  $g_6$ , like  $g_4$ , is a structure function, that  $x_{g_6} = x_4$ , and that  $|g_6(2^i)| = 2^{\lfloor \sqrt{\alpha(i)} \rfloor - 1}$ . We have

$$\alpha(i) \in i^2 \sum_{j=1}^{\infty} \frac{1}{j^2} + O(i) = \frac{\pi^2}{6} i^2 + O(i)$$

and thus

$$\left\lfloor \sqrt{\alpha(i)} \right\rfloor \in \frac{\pi}{\sqrt{6}} i + O(1)$$

so that

$$\tilde{g}_6(n) \in \Theta(n^{\sqrt{6}/\pi})$$

and

$$\bar{\rho}_{L_{g_6}}(n) \in \Theta(n^{1+\sqrt{6}/\pi}).$$

**7. Other examples and generalization**

• Let  $\mathcal{C}_\lambda$  be the set of context-free languages, whose extended rational index is in  $O(n^\lambda)$  for any real number greater than 1. It is a rational cone, i.e. it is closed for rational transductions. If  $1 < \lambda < \mu$  then you can find a rational number  $p/q$  between  $\lambda$  and  $\mu$ . There exists a context-free language whose rational index is in  $\Theta(n^{p/q})$ . This language belongs to  $\mathcal{C}_\mu - \mathcal{C}_\lambda$ . This proves that  $\mathcal{C}_\lambda$  is a proper sub-cone of  $\mathcal{C}_\mu$ . Hence the family  $(\mathcal{C}_\lambda)_{\lambda \in ]1, \infty[}$  is a strictly increasing family of cones with the same cardinality as  $\mathbb{R}$ .

• The structure functions  $g_2$  and  $g_4$  of second and fourth examples, and theorem 9 yield for instance that there exists a context-free language whose rational indexes for large enough  $n$  are:

$$\begin{aligned} n-1 + \tilde{g}_2(n)(n + \tilde{g}_4(n)(n + n\tilde{f}_5(n))) \\ = n-1 + \lfloor \ln_2 \ln_2 n \rfloor (n + 2^{\lfloor \ln_2 n \rfloor (\sqrt{2}-1)}) (n + n \lfloor \sqrt[5]{n} \rfloor) \\ \in \Theta(n^{\sqrt{2}+1/5} \ln_2 \ln_2 n). \end{aligned}$$

• We could, with this technique, build a context-free language, whose rational indexes are in  $\Theta(n^\pi)$ .

• The technique used in this paper can be sophisticated: We can replace the language  $S_\neq$ , omnipresent in this paper, by a generator of the rational cone of linear languages, like the only language solution of the equation  $L = aL\bar{a} \cup bL\bar{b} \cup \{\varepsilon\}$ , whose rational index is in  $\Theta(n^2)$ . Then the structure functions could involve decimal numbers and arithmetical computations on these numbers. In this way we can obtain a context-free language  $L$  such that  $\rho_L(n) = \bar{\rho}_L(n)$  and  $|\rho_L(n) - n^\pi| < 1$  for large enough  $n$ .

• Let  $\Lambda$  be the set of all the numbers  $\lambda \in ]1, \infty[$  such that there exists a context-free language whose rational index is  $\Theta(n^\lambda)$ . Since the non-isomorphic context-free languages form a denumerable set,  $\Lambda$  is denumerable too. However it holds all the algebraic numbers greater than 1, and seemingly any computable number greater than 1 like  $\pi, e, e+\pi, 2+\cos\sqrt[3]{e}+2+\ln 2$  or  $2+\ln \int_0^\pi \sqrt{8+\cos x} dx$ , for which there exists an efficient algorithm to



compute as many of its digits as you wish. So here is an open problem: can we find an explicit number in  $]1, \infty[-\Lambda$  ?

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