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MAXIMAL AND MINIMAL CONDITIONS IN SEMIGROUPS

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In the following, a number of results about relations between maximal and minimal conditions in semigroups will be considered. A typical result of this sort for rings is the well-known theorem of Hopkins and Fuchs which says that the maximal condition on right ideals is implied by the minimal condition provided that the additive group contains no subgroups of type p^∞ ([1] p. 285). One or two of our results can be regarded as semigroup analogues of this theorem. In 0-simple semigroups there exists a strong parallelism between maximal and minimal conditions (cf. theorems 3 and 4) which seem to have no counterpart in rings.

As for the proofs, we refer to [2] and [3].

The minimal condition on the right ideals of a semigroup need not imply the maximal condition on right ideals. This can be seen from any semilattice which is an infinite well-ordered chain ; in such a semigroup the right ideals are in a one-one order-preserving correspondence with the elements. On the other hand, a semigroup S with the maximal and the minimal condition on right ideals has only finitely many right ideals. This is a consequence of the distributivity of the lattice of right ideals. The following result shows that only a feeble maximal condition needs to be added to the minimal condition in order to reach the finiteness of the set of right ideals. A right ideal Q will be called globally idempotent if $Q^2 = Q$.

THEOREM 1. - *For any semigroup* S , *the following are equivalent*

(a) S *satisfies the minimal condition on right ideals and the maximal condition on globally idempotent principal right ideals*,

(b) S *satisfies the maximal condition on right ideals and the minimal condition on principal right ideals*,

(c) S *has only finitely many right ideals*.

The equivalence (b) $\Longleftrightarrow$ (c) of the forgoing theorem is a consequence of the lattice properties of the set of right ideals. However, in the proof of (a) $\Longleftrightarrow$ (c) (cf. [3]) it seems necessary to take the semigroup structure into account.

The minimal condition on the right congruences of a semigroup need not imply the maximal condition on right congruences. Clearly, in any group the right congruences are in a one-one order-preserving correspondence with the subgroups, and there do exist groups with the minimal, but not the maximal condition on subgroups, e. g. the groups of type p^∞ . It can be shown that every right congruence of a subgroup of a semigroup S is induced by a right congruence of the whole semigroup S . This implies that the maximal (minimal) condition on the right congruences of a semigroup S carries over to each subgroup of S. Therefore, an analogue of the result of Hopkins and Fuchs can only be expected, if the bearing of the subgroups on the properties in question is somehow avoided. This will be attempted in different ways.

A semigroup S will be said to satisfy the weak minimal (maximal) condition if every non-empty set Σ of right congruences with the property that

$$\{\rho \cap (R \times R)\ ;\ \rho \in \Sigma\} \text{ is finite for every } \mathcal{R}\text{-class } R \text{ of } S$$

has a minimal (maximal) member.

It is not difficult to prove that the weak minimal condition is intermediate to the minimal condition on right ideals and the minimal condition on right congruences. In fact, the weak minimal condition lies properly between the latter conditions as can be seen from infinite semilattices that are well-ordered chains and from groups that do not satisfy the minimal condition on subgroups. Similarly, the weak maximal condition lies properly between the maximal condition on right ideals and the maximal condition on right congruences.

THEOREM 2. - <u>A semigroup</u> S <u>satisfies the weak minimal condition if, and only if,</u> S <u>has only finitely many right ideals. If this is the case, then</u> S <u>satisfies the weak maximal condition.</u>

As a consequence of theorem 2, we have that the minimal condition on right congruences implies the maximal condition on right ideals.

For any element a of a semigroup S, the set $a^\perp = \{(x, y) \in S \times S\ ;\ ax = ay\}$ is a right congruence of S. In a group or, more generally, in a left cancellative semigroup, the only right congruence of this type is the identity relation. In a completely 0-simple semigroup all right congruences of the form $a^\perp$ with $a \neq 0$ are incomparable. With respect to the following theorem, it should be observed that a 0-simple semigroup with a 0-minimal right ideal satisfies both the maximal and the minimal condition on principal right ideals.

THEOREM 3. - <u>For any</u> 0-<u>simple semigroup</u> S, <u>the following are equivalent</u>

(a) S <u>has a</u> 0-<u>minimal right ideal and</u> $\{a^\perp\ ;\ a \in S\}$ <u>has a minimal member,</u>

(b) S <u>has a</u> 0-<u>minimal right ideal and</u> $\{a^\perp\ ;\ a \in S,\ a \neq 0\}$ <u>has a maximal member,</u>

(c) S <u>has a maximal principal right ideal and</u> S <u>satisfies the maximal condition on right congruences of the form</u> $a^\perp$ $(a \in S)$

(d) S <u>has a maximal principal right ideal and a</u> 0-<u>minimal left ideal,</u>

(e) S <u>is completely</u> 0-<u>simple.</u>

The bicyclic semigroup satisfies the maximal condition on right ideals and the minimal condition on right congruences of the form $a^\perp$. One concludes from this that in (b) the words " 0-minimal" and "maximal" must not be replaced by "maximal" and "minimal", resp. The implication (a) $\Longrightarrow$ (c) could be replaced by the statement that, in a 0-simple semigroup, the minimal conditions on principal right ideals and on right congruences of the form $a^{\perp}$ together imply the corresponding maximal conditions. We do not know whether this holds in an arbitrary semigroup.

As a corollary to the foregoing theorem, it may be noted that a 0-simple semigroup with the maximal or minimal condition on right congruences is completely 0-simple. It is not difficult to show that the number of one-sided ideals of such a semigroup is finite. With respect to the following result, we remark that a semigroup S satisfies the maximal (minimal) condition on subgroups (i. e. on the partially ordered set of those subsemigroups that are groups) if, and only if, every maximal subgroup of S satisfies the maximal (minimal) condition on subgroups.

THEOREM 4. - Let S be 0-simple. S satisfies the minimal condition on right congruences together with the maximal condition on subgroups if, and only if, S satisfies the maximal condition on right congruences together with the minimal condition on subgroups.

The question of whether an arbitrary semigroup with the minimal condition on right congruences and with the maximal condition on subgroups satisfies the maximal condition on right congruences is still open. In view of this question and of the following theorem, it is notable that an infinite group satisfying both the maximal and the minimal condition on subgroups has not yet been found.

THEOREM 5. - If S is a semigroup that has no infinite subgroups and satisfies the minimal condition on right congruences, then S is finite.

REFERENCES

[1] FUCHS (L.). - Abelian groups, 3rd edition. - Oxford, Pergamon Press, 1960, reprinted 1967 (International Series of Monographs in pure and applied Mathematics, 12).

[2] HOTZEL (E.). - On semigroups with maximal conditions, Semigroup Forum, t. 11, 1975/76, p. 337-362.

[3] HOTZEL (E.). - On finiteness conditions in semigroups (submitted for publication).

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