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ON THE BEHAVIOR OF IGUSA'S LOCAL ZETA FUNCTION IN TOWERS OF FIELD EXTENSION

by Ben LICHTIN (*)

Introduction.

The purpose of this lecture is to discuss some of the work done in the last three years on the zeta function, ala Igusa, of hypersurfaces over local fields of characteristic zero (referred to below as "local field"). In particular, attention will be directed towards the recent work of D. Meuser [13] which describes the behavior of the zeta function in certain towers of local fields.

Section 1.

Let K be a local field, R its ring of integers and P the maximal ideal of R. Let $q = \#R/P$. Denote the norm on K by $|\ |_K$. (The subscript 'K' is dropped if it is clear which field is under discussion). Let $f \in K[x_1, \ldots, x_n]$. Using the Haar measure $d\mu_K$ on the additive group underlying K, Igusa defined a zeta function associated to f which extended to the mixed characteristic situation the classical Mellin transform introduced by Gel'fand-Shilov [7] over $\mathbb{R}, \mathbb{C}$.

Let S_K denote the Schwartz-Bruhat space of locally constant complex valued functions on K^n with compact support. For $s \in \mathbb{C}$ with $Re(s) > 0$ define the "local" zeta function to be

$$I(s, \varphi) = \int_{K^n - \{f=0\}} |f|^s \varphi \, d\mu_K$$

for $\varphi \in S_K$.

Over $\mathbb{R}, \mathbb{C}$, the two initial proofs [1,2] that this generalized Mellin transform possessed a meromorphic continuation with poles contained in finitely many arithmetic progressions of negative rationals used the embedded resolution of singularities theorem of Hironaka (i.e. Main Theorem II in the Annals '64 paper). Igusa observed [8] that these proofs could be adapted more or less straightforwardly to the situation over any local field K. In this way he showed that $I(s, \varphi)$ was extendable to a *rational* function in q^{-s} for each $\varphi \in S_K$.

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It is useful to observe that the difference in the size of the set of poles between $\mathbf{R}, \mathbb{C}$ and a local field, an extension of Q_p, is due solely to the rigid nature of the space of test functions in S_K. As shown in [10], if one enlarges the space of test functions to include functions of the form

$$|p| \cdot \varphi$$

with $p \in K[x_1, \ldots, x_n]$, $\varphi \in S_K$, then one can show that $I(s, -)$ will possess infinitely many poles, lying in finitely many arithmetic progressions of negative rationals.

Since Igusa's mid 70's papers, efforts to understand the zeta function have used either explicit resolutions of the function f to determine the set of poles of $I(s,-)$ (modulo the possibility of $s = -1$, usually) [6,9,11,12] or theorems from logic that give a general inductive (in the number of variables) procedure that allows one to write $I(s, \varphi)$ as a sum of resonably simply looking integrands over "cells" which are very complicated to describe [5]. Because Meuser's paper comes out of the first method we proceed to describe how one uses resolution data to understand $I(s,-)$.

From Hironaka one has that there exists an algebraic variety defined and smooth over K and projective and birational morphism $\pi : X \to K^n$ with the following properties.

1) There is a finite and disjoint open covering $\{\mathcal{U}_\alpha\}$ of X and a positive integer e such that each $\mathcal{U}_\alpha$ is K-analytically isomorphic to $\underbrace{P^e \times \cdots \times P^e}_{n}$.

2) By means of this local analytic isomorphism one can find coordinates $(y_1, \ldots, y_n)$ on $\mathcal{U}_\alpha$ so that

$$f \circ \pi(y_1, \ldots, y_n) = \left(\prod_{j=1}^{r} y_j^{N_j}\right) \cdot \epsilon_1(y)$$

$$\det d\pi(y_1, \ldots, y_n) = \left(\prod_{j=1}^{r} y_j^{n_j}\right) \cdot \epsilon_2(y)$$

3) Each $N_j \geq 1$ and $n_j \geq 0$.

4) ϵ_1, ϵ_2 are units on $\mathcal{U}_\alpha$ with the property that $\operatorname{ord} \epsilon_1 = c_1$, $\operatorname{ord} \epsilon_2 = c_2$.

Observe that the disjointness property is not obtainable over $\mathbf{R}, \mathbb{C}$.

Definition 1. Call the smallest integer e that one can use to satisfy (i) the "width" of the resolution π.

Definition 2. Let $\mathcal{A} = \{((c_1,c_2),(N_1,\dots,N_r),(n_1,\dots,n_r))$: $1 \le r \le n$ and there is an open set $\mathcal{U}_\alpha$ of X with coordinates $(y_1,\dots,y_n)$ so that properties (2) and (3) are satisfied with $\operatorname{ord}\epsilon_1 = c_1$, $\operatorname{ord}\epsilon_2 = c_2$ and the (N_j,n_j) multiplicities of $(f\circ\pi, \det d\pi)$ along the divisor $\{y_j = 0\}\cap\mathcal{U}_\alpha$. (In this case one says that $((c_1,c_2),(N_1,\dots,N_r),(n_1,\dots,n_r))$ corresponds to $\mathcal{U}_\alpha$)$\}$. One calls $\mathcal{A}$ the numerical data of the resolution $\pi : X \to K^n$.

One then has

$$I(s,\varphi) = \sum_\alpha q^{-(c_1 s+c_2)}\cdot\varphi\big|_{\mathcal{U}_\alpha}\cdot\int\limits_{P^e_\times\cdots\times P^e}\prod_1^r |y_j|^{N_j s+n_j}|dy_1\dots dy_n|$$

where $dy_1\dots dy_n$ is a maximal differential over $\mathcal{U}_\alpha$ and $|dy_1\dots dy_n|$ is the Borel measure associated to it.

One then checks that each summand equals

$$q^{-e(n-r)}\cdot(1-q^{-1})^r\cdot q^{-(c_1 s+c_2)}\cdot\varphi\big|_{\mathcal{U}_\alpha}\cdot\prod_{j=1}^r\left(\frac{q^{-e(N_j s+n_j+1)}}{1-q^{-(N_j s+n_j+1)}}\right).$$

A convenient way of expressing this when $\varphi = \chi_{R^n}$, the characteristic function for R^n, is this. Given an element $(\bar c,\overline{N},\bar n)\in\mathcal{A}$, let

$$\mathcal{N}(\bar c,\overline{N},\bar n) = \#\{\mathcal{U}_\alpha : \text{ the numerical data corresponding to } \mathcal{U}_\alpha \text{ is } (\bar c,\overline{N},\bar n)\}. \tag{1.2}$$

In addition let $r(\overline{N})$ = number of indices i for which $N_i \ne 0$, N_i a component of $\overline{N}$. Then

$$I(s,\chi_{R^n}) = q^{-e(n-r)}(1-q^{-1})^r\sum_{\substack{\{(\bar c,\overline{N},\bar n)\in\mathcal{A}:\\ r(\overline{N})=r\}}}\mathcal{N}(\bar c,\overline{N},\bar n)\cdot q^{-(c_1 s+c_2)}\prod_1^r\left(\frac{q^{-e(N_j s+n_j+1)}}{1-q^{-(N_j s+n_j+1)}}\right) \tag{1.3}$$

One obtains a similarly looking expression for the polar part of $I(S,\chi_{R^n})$ at a ratio λ. Let $\mathcal{I}_k(\lambda) = \{(\bar c,\overline{N},\bar n)$: for exactly k indices $i_1,\dots,i_k$ one has $\lambda = -(n_{i_j}+1)/N_{i_j}, j = 1,\dots,k\}$.

Then the coefficient of $(s-\lambda)^{-k}$ in the Laurent expansion of $I(s, \chi_{R^n})$ in a punctured neighborhood of λ is the expression

$$(1-q^{-1})^r \sum_{\{(\bar{c},\bar{N},\bar{n})\in I_k(\lambda):r(\bar{N})=r\}} \mathcal{N}(\bar{c},\bar{N},\bar{n})\cdot\frac{q^{-(c_1 s+c_2)}}{(\ell n q)^k}\cdot \prod_{\{j:\lambda\neq-\frac{(n_j+1)}{N_j}\}}\left(\frac{q^{-e(N_j\lambda+n_j+1)}}{1-q^{-(N_j\lambda+n_j+1)}}\right). \quad (1.4)$$

When one is lucky and willing to be very precise this local data can be assembled to lead to demonstrations that a particular ratio λ is a pole of $I(s, \chi_{R^n})$.

From the work done in this regard there has emerged an interesting conjecture that attempts to give a geometric significance to the poles of $I(s,-)$. A simplest version of this conjecture is this:

Let L be an algebraic number field $f \in L[x_1, \ldots, x_n]$. Via an embedding $L \hookrightarrow \mathbb{C}^n$, view f as a map $f : \mathbb{C}^n \to \mathbb{C}$. Let 0 be a critical value of f. Let T be the quasi-unipotent action of monodromy on the cohomologies of the vanishing cycle complex $R\Psi_f$ associated to f [3].

Now let P be a place of L and K the local field obtained by completing L at P. Let φ be an element of S_K with support in $P^m \times \cdots \times P^m$ for $m > 0$.

CONJECTURE. *The poles of the zeta function $I(s, \varphi)$ are related to the eigenvalues of T as follows.*

If ρ is a pole then $e^{-2\pi i\rho}$ is an eigenvalue of T action on some cohomology sheaf $\mathcal{H}^i(R\Psi_f)_p$, for some $1 \le i \le n-1$ and some point $p \in \{f = 0\}$ (in $\mathbb{C}^n$).

This conjecture has been verified in the following two cases.

1) *$f \in K[x_1, x_2]$ determines an irreducible element of $\overline{K}[[x_1, x_2]]$ [9,12]*

2) *$f \in K[x_1, \ldots, x_n]$ is "generic" with respect to its polyhedron [6,11].*

However, the verifications have been computational. What is completely missing is a conceptual bridge between the measure-thereoretic context in which the zeta function is defined and the p-adic cohomological context in which monodromy action on vanishing cycles is defined.

Section 2.

Let K_d be the unique unramified extension of K of degree d. A natural subject to investigate concerns the behavior of the zeta function

$$I_d(s,\varphi) = \int_{K_d^n - \{f=0\}} |f|_{K_d}^s \varphi \, d\mu_{K_d}\,, \text{ as } d \to \infty$$

for $f \in K[x_1,\ldots,x_n]$ and $\varphi = \chi_{R_d^n}$.

In [13], Meuser has determined this behavior. To describe the result it is necessary to introduce terminology.

Definition 3. Let $\{N_d\}_1^\infty$ be a sequence of numbers with N_d associated to field K_d. N_d is an invariant function of K_d if there are polynomials $F(y_1,\ldots,y_n)$, $G(y,\ldots,y_n) \in \mathbb{Z}[y_1,\ldots,y_n]$ and complex numbers $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$ such that

$$N_d = \frac{F(q^{d\alpha_1},\ldots,q^{d\alpha_n})}{G(q^{d\beta_1},\ldots,q^{d\beta_n})}$$

for all $d = 1,2,\ldots$

Basic examples.

1) $\{N_d\}$ satisfies a linear recursion over $\mathbb{C}$.

2) $N_d = \sum_{i=1}^M \tilde{N}_{d,i} \cdot \frac{q^{d\alpha_i}}{\prod_{j=1}^m (1-q^{d\beta_{ji}})}$ where $\{\tilde{N}_{d,i}\}$ satisfies a linear recursion over $\mathbb{C}$ and $\{\alpha_i\}$, $\{\beta_{ji}\}$ are in $\mathbb{C}$.

There are three steps in her argument.

A) Analyze the sets $\mathcal{A}_d$ of numerical data obtainable from an embedded resolution of f over each K_d.

Let $\pi_d : X_d \to K_d^n$ be such a resolution

B) Let $\mathcal{A} = \cup \mathcal{A}_d$. Analyze the sequence $N_d = \mathcal{N}_d(\overline{c},\overline{N},\overline{n})$ for each $(\overline{c},\overline{N},\overline{n}) \in \mathcal{A}$. Here $\mathcal{N}_d$ is counting the analogous number for π_d as counted by $\mathcal{N}$ in (1.2).

C) Let $e(d)$ be the width of resolution π_d (cf. Definition 1). Analyze the limiting behavior for $e(d)$.

As seen from (1.3), (1.4), one will be able to understand how $I_d(s, \chi_{R_d^n})$ behaves as $d \to \infty$ once there are satisfactory results for each of these steps. Note that one would replace $\mathcal{N}$ by $\mathcal{N}_d$, q by q^d and e by $e(d)$ in (1.3), (1.4) to obtain the expression for $I_d(s, \chi_{R_d^n})$.

It would be quite remarkable if in doing so one also knew

a) The set of distinct possible $(\overline{c}, \overline{N}, \overline{n})$ triples of tuples over all d was a finite set.

b) There was a uniform bound e on the $e(d)$.

c) $\mathcal{N}_d(\overline{c}, \overline{N}, \overline{n})$ was an invariant function of K_d for each $(\overline{c}, \overline{N}, \overline{n})$.

It turns out that (a)–(c) do indeed hold, (see [13, Theorem 2] for details). This implies a strong uniformity in the resolution as one goes up the tower of fields.

The conclusion of (A) is obtained straightforwardly because resolution behaves well with respect to field extension. The main result states

THEOREM 1. *$\mathcal{A} = \bigcup \mathcal{A}_d$ is a finite set of tuples of nonnegative integers.*

The main theorem for (B) states

THEOREM 2. *For each $(\overline{c}, \overline{N}, \overline{n}) \in \mathcal{A}$, $\mathcal{N}_d(\overline{c}, \overline{N}, \overline{n})$ is an invariant function of K_d.*

The main theorem for (C) states

THEOREM 3. *One can find a sequence of resolutions $\pi_d : X_d \to K_d^n$ for which there is an integer e independent of d so that $e(d) \leq e$.*

These results together with (1.4) imply the

MAIN THEOREM 4. *For any ratio λ, let $\rho_{d,k}$ denote the coefficient of $(s-\lambda)^{-k}$ in the Laurent expansion of $I_d(s, \chi_{R_d^n})$. Then $\rho_{d,k}$ is an invariant function of K_d.*

The proofs of (2) and (3) require a sequence of counting lemmas for which the reader is referred to [13] (Lemmas 3–6). Underlying them is a basic theorem whose elegant proof is due to Dwork. It is the one result which explicitly uses the unramified assumption for the extension K_d and upon which all the other counting lemmas rest. Thus, an extension of it to certain ramified extensions of K would automatically extend Theorems 2,3,4 to other types of towers of extensions of K.

A simplified verson of this theorem follows.

Let $g \in R[x_1, \ldots, x_n]$. Fix a positive integer e. Define $N_d(g,e) = \#\{\xi \in R_d^n \bmod (P_d^e)^n : g(\xi) \equiv 0(P_d^e)\}$. Let $Z_e(T) = \sum_{d=1}^{\infty} N_d(g,e)T^d/d$. One has

THEOREM 5. *$Z_e(T)$ is a rational function in T.*

From which one has immediately of course

COROLLARY. *$N_d(g,e)$ is an invariant function of K_d.*

To prove the theorem we introduce some notation.

Let M_d be the Teichmüller representatives of K_d. Let π be a uniformizing parameter for K. Set $h(x) = \frac{1}{\pi}[g(x_1^q, \ldots, x_n^q) - g^q(x_1, \ldots, x_n)]$.

LEMMA. *For $\alpha = (\alpha_1 \ldots, \alpha_n) \in M_d^n$, $g(\alpha) \equiv 0(P_d^e)$ iff $g(\alpha) \equiv 0(P_d^{e-1})$ and $h(\alpha) \equiv 0(P_d^{e-1})$.*

PROOF: One checks that $h \in R[x_1, \ldots, x_n]$. Now if $g(\alpha) \equiv 0(P_d^{e-1})$ write $g(\alpha) = \pi^{e-1} \cdot t$. Apply the K-automorphism σ of K_d which is the lifting of the Frobenius automorphism of $\mathbf{F}_{q^d}$ over $\mathbf{F}_q$.

One has

$$g(\sigma\alpha_1, \ldots, \sigma\alpha_n) = g(\alpha_1^q, \ldots, \alpha_n^q) \equiv \pi^{e-1}\sigma t(P_d^{e-1}) \equiv \pi^{e-1} \cdot t^q(P_d^e).$$

Of course, $g(\alpha)^q \equiv 0(P_d^e)$.

Thus, $h(\alpha) \equiv \pi^{e-2} \cdot t^q(P_d^e)$ and so, $h(\alpha) \equiv 0(P_d^{e-1})$ iff $t \equiv 0(P_d)$, that is, $g(\alpha) \equiv 0(P_d^e)$.

The proof of the theorem is based on this lemma and the observation that $N_d(g,e) = N_d^*(\tilde{g},e)$ where $N_d^*(\tilde{g},e) = \#\{\xi \in M_d^{ne} : \tilde{g}(\xi) \equiv 0(P_d^e)\}$ and $\tilde{g} \in R[x_{11},\ldots,x_{ne}]$ is defined by replacing each x_i by e variables x_{ij} taking values in M_d. That is, set

$$\tilde{g}(x_{11},x_{12},\ldots,x_{1e},\ldots,x_{n1},\ldots,x_{ne}) = g\left(\sum_{j=1}^{e} x_{1j}\pi^{j-1},\ldots,\sum_{j=1}^{e} x_{nj}\pi^{j-1}\right).$$

Using the lemma, it is now clear what to do. Set $\tilde{h}_1(x) = \frac{1}{\pi}[\tilde{g}(x_{11}^q,\ldots,x_{ne}^q) - \tilde{g}(x_{11},\ldots,x_{ne})^q]$. Then

$$\begin{aligned} N_d^*(g,e) &= N_d^*(\{\tilde{g},\tilde{h}_1\},e-1) \\ &= \#\{\xi \in M_d^{ne} : \tilde{g}(\xi) \equiv \tilde{h}_1(\xi) \equiv 0(P_d^{e-1})\} \end{aligned}$$

Repeating this $e-2$ more times one obtains $N_d^*(\tilde{g},e) = N_d^*(\{\tilde{g},\tilde{h}_1,\ldots,\tilde{h}_L\},1), L = 2^{e-1}$. Now, from Dwork's work on the Weil zeta function for arbitrary affine varieties [4], one knows that $N_d^*(\{\tilde{g},\ldots,\tilde{h}_L\},1)$ satisfies a linear recursion over $\mathbb{Z}$. This proves the theorem. ∎

Remark. There is a significant (rigid) analytic component to the proofs of Theorems 2,3, which will not be discussed in this survey. For it is a technical feature which would enlarge these notes considerably if treated in any detail. It suffices only to remark on the reason for its entry into the discussion.

One wants to construct an open cover $\{\mathcal{U}_\alpha\}$ for each X_d with the properties (1–4) described in section 1. Each of these are K_d-analytic properties. On the other hand, each X_d is an algebraic variety defined over K_d and obtained from the variety X_1 by field extension. It would be convenient to obtain a corresponding K_d-analytic variety $\mathcal{X}_d$, for each d, in a similar fashion. This would at least give one a systematic method of generating "analytically small" neighborhoods in each $\mathcal{X}_d$, in each of which one would hope to prove that these four properties are satisfied. This is almost the case.

What is possible is to obtain by a GAGA theorem a rigid analytic space $\tilde{X}_d$, corresponding to

X_d, an analytic morphism $\tilde{\pi}_d : \tilde{X}_d \to B^n(\overline{Q}_p)$, the unit (closed) ball in $\overline{Q}_p$, and a finite cover of $\tilde{X}_d$ by affinoid sets, the number of which is bounded independently of d. However, one then needs to determine that one can cover the K_d-rational points in an affinoid subset, defined by finitely many elements of a Tate algebra of the form $\frac{T_m(K_d)}{<G_1,\ldots,G_r>}$, by open sets which are K_d-analytically isomorphic to $P_d^{e(d)} \times \cdots \times P_d^{e(d)}$. Indeed, Theorem (3) implies that $e(d)$ can be chosen independently of d. To do this requires considerable care, the first major ingredient being Theorem (5). Once one has done this (cf. Lemma (6) of [13]), the proofs of Theorems (1–3) follow directly.

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