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p-ADIC TEICHMÜLLER SPACE AND SIEGEL HALFSPACE

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In order to study the space $\mathcal{M}_n$ of Mumford curves of genus n and the space $\mathcal{A}_n$ of principally polarized abelian varieties which can be represented as analytic tori we introduce the p-adic Teichmüller space $\mathcal{T}_n$ and the Teichmüller modular group Ψ_n as well the p-adic Siegel halfspace $\mathcal{H}_n$ and the Siegel modular group $\Gamma_n \cong \text{PGL}_n(\mathbb{Z})$. One will arrive at the result that the orbit space $\mathcal{T}_n \text{ mod } \Psi_n$ is the space $\mathcal{M}_n$ of Mumford curves and that the orbit space $\mathcal{H}_n \text{ mod } \Gamma_n$ is the space $\mathcal{A}_n$ of polarized abelian varieties.

In this paper, we will only describe the main points of the construction of the analytic space $\mathcal{T}_n$ and the transformation group Ψ_n as well as the construction of the analytic space $\mathcal{H}_n$ and the transformation group Γ_n . A great deal of questions remain open.

1. Conjugacy classes of homomorphisms.

(1.1) Homomorphism classes. - Let X, Y be groups, let A be a subgroup of the group $\text{Aut } X$ of automorphisms of X and B a subgroup of $\text{Aut } Y$.

Denote by (X, Y) the set of all group homomorphisms $\zeta : X \rightarrow Y$. A acts on (X, Y) by composition of mappings from right :

$$X \xrightarrow{\alpha} X \xrightarrow{\zeta} Y.$$

If $\alpha \in A$, $\zeta \in (X, Y)$, then $\zeta \circ \alpha \in (X, Y)$. The set of equivalence classes ζA will be denoted by ${}_A[Y, X]$.

The group B acts on (X, Y) by composition of mappings from left :

$$X \xrightarrow{\zeta} Y \xrightarrow{\beta} Y.$$

If $\beta \in B$, $\zeta \in (X, Y)$, then $\beta \circ \zeta \in (X, Y)$. The set of equivalence classes $\beta \zeta$ will be denoted by $(X, Y]_B$.

For $\alpha \in A$, $\beta \in B$, $\zeta \in (X, Y)$, we have

$$(\beta \circ \zeta) \circ \alpha = \beta \circ (\zeta \circ \alpha)$$

because composition of mappings is associative. Therefore A (resp. B) acts canonically on $(X, Y]_B$ (resp. ${}_A[X, Y]$).

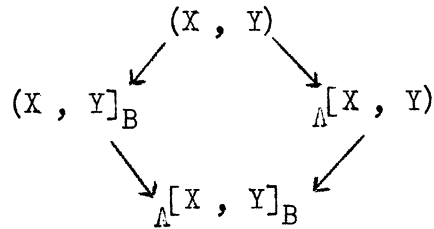
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Obviously, $(X, Y]_B \bmod A = {}_A[X, Y) \bmod B$.

We denote this set by ${}_A[X, Y]_B$. Its elements are the double cosets $B\zeta A$.

One gets a canonical commutative diagram



where each arrow denotes the canonical equivalence class mapping.

(1.2) Isotropy groups. - We consider the following isotropy groups :

$$\mathfrak{I}_A(\zeta) := \{\alpha \in A ; \zeta \circ \alpha = \zeta\}$$

$$\mathfrak{I}_B(\zeta) := \{\beta \in B ; \beta \circ \zeta = \zeta\}$$

$$\mathfrak{I}_A(B\zeta) := \{\alpha \in A ; B\zeta \circ \alpha = B\zeta\}$$

$$\mathfrak{I}_B(\zeta A) := \{\beta \in B ; \beta \circ \zeta A = \zeta A\}.$$

PROPOSITION 1. - $\mathfrak{I}_A(\zeta)$ is a normal subgroup of $\mathfrak{I}_A(B\zeta)$. $\mathfrak{I}_B(\zeta)$ is a normal
subgroup of $\mathfrak{I}_B(\zeta A)$ and

$$\mathfrak{I}_A(B\zeta)/\mathfrak{I}_A(\zeta) \cong \mathfrak{I}_B(\zeta A)/\mathfrak{I}_B(\zeta).$$

Proof.

1° Let $\alpha_0 \in \mathfrak{I}_A(\zeta)$, $\alpha \in \mathfrak{I}_A(B\zeta)$. Then there is a $\beta \in B$ such that $\zeta \circ \alpha = \beta \circ \zeta$.
Now $\zeta \circ \alpha_0 = \zeta$ and

$$\zeta \circ \alpha \circ \alpha_0 \circ \alpha^{-1} = \beta \circ \zeta \circ \alpha_0 \circ \alpha^{-1} = \beta \circ \zeta \circ \alpha^{-1} = \zeta \circ \alpha \circ \alpha^{-1} = \zeta$$

which shows that $\alpha \alpha_0 \alpha^{-1} \in \mathfrak{I}_A(\zeta)$. Thus $\mathfrak{I}_A(\zeta)$ is a normal subgroup of $\mathfrak{I}_A(B\zeta)$.

2° Let $\beta_0 \in \mathfrak{I}_B(\zeta)$, $\beta \in \mathfrak{I}_B(\zeta A)$. Then there is a $\alpha \in A$ such that $\beta \circ \zeta = \zeta \circ \alpha$.
Now $\beta_0 \circ \zeta = \zeta$ and

$$\beta^{-1} \beta_0 \beta \zeta = \beta^{-1} \beta_0 \zeta \alpha = \beta^{-1} \zeta \alpha = \beta^{-1} \beta \zeta = \zeta$$

which shows that $\beta^{-1} \beta_0 \beta \in \mathfrak{I}_B(\zeta)$. Thus $\mathfrak{I}_B(\zeta)$ is a normal subgroup of $\mathfrak{I}_B(\zeta A)$.

3° It is an easy exercise to show that the mapping which associates to $\alpha \in \mathfrak{I}_A(B\zeta)$ the residue class $\bar{\beta}$ in $\mathfrak{I}_B(\zeta A)/\mathfrak{I}_B(\zeta)$ of a $\beta \in B$ which satisfies $\zeta \alpha = \beta \circ \zeta$ induces an isomorphism $\mathfrak{I}_A(B\zeta)/\mathfrak{I}_A(\zeta)$ on to $\mathfrak{I}_B(\zeta A)/\mathfrak{I}_B(\zeta)$.

(1.3) Schottky groups. - Let now K be an algebraically closed field together with a non-trivial complete ultrametric valuation. Let E_n be a non-abelian free group of rank $n \geq 2$ together with a fixed basis $e_1, \dots, e_n$.

We consider $PSL_2(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in K ; ad - bc = 1 \right\}$ as a K -algebraic group. Denote by tr^2 the regular function on $PSL_2(K)$ which has the value $(a+d)^2$ at the point $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $PSL_2(K)$ is called hyperbolic if $|tr^2(\begin{pmatrix} a & b \\ c & d \end{pmatrix})| = |a+d|^2 > 1$. The set $(E_n, PSL_2(K))$ of all group homomorphisms $\zeta : E_n \rightarrow PSL_2(K)$ will be identified with the n -fold product $PSL_2^n(K) = PSL_2(K) \times \dots \times PSL_2(K)$ of $PSL_2(K)$: any $w = (w_1, \dots, w_n) \in PSL_2^n(K)$ determines a unique homomorphism $\zeta_w : E_n \rightarrow PSL_2(K)$ which satisfies $\zeta_w(e_i) = w_i$ for all i .

The action of $\text{Aut } E_n$ on $(E_n, PSL_2(K))$ when it is identified with $PSL_2^n(K)$ can be described as follows: let $\alpha \in \text{Aut } E_n$, $\alpha(e_i)$ is a reduced word in the letter $e_1, \dots, e_n$: we substitute w_j for e_j and obtain an element w'_i for each i . Then

$$(w_1, \dots, w_n) \times \alpha = (w'_1, \dots, w'_n).$$

This explicit description shows that α is a biregular transformation of the K -algebraic space $PSL_2^n(K)$.

Definition. - A homomorphism $\zeta : E_n \rightarrow PSL_2(K)$ is called Schottky homomorphism, if $\zeta(e)$ is hyperbolic for any $e \in E_n$, $e \neq 1$.

Denote by $\mathcal{S}_n$ the set of Schottky homomorphisms. As a subset of $PSL_2^n(K)$ it is given by infinitely many inequalities. More precisely: we fix $e \in E_n$, and consider the mapping $\zeta \rightarrow \zeta(e)$. It is a regular mapping $\phi_e : PSL_2^n(K) \rightarrow PSL_2(K)$ and $f_e = tr^2 \phi_e$ is a regular function on $PSL_2^n(K)$. Then

$$\mathcal{S}_n = \{w \in PSL_2^n(K) ; |f_e(w)| > 1 \text{ for all } e \in E_n, e \neq 1\}.$$

One can give explicit expressions for the mapping ϕ_e and the function f_e . e is determined by the finite sequence $\epsilon(1), \dots, \epsilon(r)$ with $\epsilon(i) \in \{-1, \dots, +n\}$ and $\epsilon(i+1) \neq -\epsilon(i)$ such that

$$e = \prod_{i=1}^r e_{\epsilon(i)} \text{ with } e_{-i} = e_i^{-1}.$$

Let

$$\phi_e \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

we will give expression for x_{kl} as polynomials in $a_1, \dots, d_n$.

$N_{kl} :=$ set of all sequences $s = ((i_1, j_1), (i_2, j_2), \dots, (i_r, j_r))$ such that $i_{v+1} = j_v$ for all v and $i_v, j_v \in \{1, 2\}$. For any such s , we consider the product

$$x(s) := x_{i_1 j_1}^{(1)} \cdots x_{i_r j_r}^{(r)}$$

with

$$x_{i_v j_v}^{(v)} := \begin{cases} a_\varepsilon(i) ; & (i_v, j_v) = (1, 1) \\ b_\varepsilon(i) ; & (i_v, j_v) = (1, 2) \\ c_\varepsilon(i) ; & (i_v, j_v) = (2, 1) \\ d_\varepsilon(i) ; & (i_v, j_v) = (2, 2) \end{cases}$$

with $a_{-i} = + d_i$, $b_{-i} = - b_i$, $c_{-i} = - c_i$, $d_{-i} = a_i$. Then

$$\begin{aligned} x_{11} &= \sum_{s \in N_{11}} x(s) \\ x_{12} &= \sum_{s \in N_{12}} x(s) \\ x_{21} &= \sum_{s \in N_{21}} x(s) \\ x_{22} &= \sum_{s \in N_{12}} x(s) . \end{aligned}$$

The proof readily follows by induction on r .

We consider homomorphism classes as in §(1.1) for $A = \text{Aut } E_n$, $B =$ group of inner automorphisms of $\text{PSL}_2(K) \cong \text{PSL}_2(K)$. The set $\bar{S}_n = \text{Aut } E_n [S_n]$ of classes $\zeta \circ \text{Aut } E_n$ with $\zeta \in S_n$ is just simply the set of Schottky subgroups of $\text{PSL}_2(K)$ of rank n (see [1], chapter I, (1.6)). Because if $\zeta \in S_n$, $\alpha \in \text{Aut } E_n$, then the image $\text{Im}(\zeta \circ \alpha)$ does not depend on α .

If, on the other hand, Γ is a Schottky subgroup of $\text{PSL}_2(K)$ of rank n , then by definition there is a $\zeta \in S_n$ such that $\text{Im } \zeta = \Gamma$. If $\zeta' \in S_n$ also satisfies $\text{Im } \zeta' = \Gamma$, then we note that $(\zeta|\Gamma)^{-1} : \Gamma \rightarrow E_n$ is a group homomorphism and $\alpha = (\zeta|\Gamma)^{-1} \circ \zeta' \in \text{Aut } E_n$ and $\zeta \circ \alpha = \zeta'$.

2. Hyperbolic fractional linear transformations.

(2.1). Let $\underline{P} = K \cup \{\infty\}$ be the projective line over K and $\underline{P} \times \underline{P} - \underline{P} = \{(x, y) \in \underline{P} \times \underline{P} : x \neq y\}$ be the complement of the diagonal in the product $\underline{P} \times \underline{P}$.

In order to determine the regular functions on $\underline{P} \times \underline{P} - \underline{P}$, we introduce the following affine charts :

$$U_{11} = \{(x, y) \in \underline{P} \times \underline{P} - \underline{P}; x \neq \infty, y \neq \infty\}$$

$$U_{12} = \{(x, y) \in \underline{P} \times \underline{P} - \underline{P}; x \neq \infty, y \neq 0\}$$

$$U_{21} = \{(x, y) \in \underline{P} \times \underline{P} - \underline{P}; x \neq 0, y \neq \infty\}$$

$$U_{22} = \{(x, y) \in \underline{P} \times \underline{P} - \underline{P}; x \neq 0, y \neq 0\}.$$

The algebras $\mathcal{O}(U_{ij})$ of regular functions on U_{ij} are the following

$$\mathcal{O}(U_{11}) = K[x, y, \frac{1}{x-y}].$$

$$\mathcal{O}(U_{12}) = K[x, \frac{1}{y}, \frac{1}{1-\frac{x}{y}}]$$

$$\mathcal{O}(U_{21}) = K[\frac{1}{x}, y, \frac{1}{1-\frac{y}{x}}]$$

$$\mathcal{O}(U_{22}) = K[\frac{1}{x}, \frac{1}{y}, \frac{1}{\frac{1}{x}-\frac{1}{y}}].$$

PROPOSITION 2. - $\mathcal{O}(\underline{P} \times \underline{P} - \underline{P}) = K[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}]$.

Proof. - The functions

$$\frac{1}{x-y} = \frac{\frac{1}{y}}{\frac{x}{y}-1} = \frac{\frac{1}{x}}{1-\frac{y}{x}} = \frac{\frac{1}{x} \times \frac{1}{y}}{\frac{1}{y}-\frac{1}{x}}$$

$$\frac{x}{x-y} = \frac{\frac{x}{y}}{\frac{x}{y}-1} = \frac{1}{1-\frac{y}{x}} = \frac{\frac{1}{y}}{\frac{1}{y}-\frac{1}{x}}$$

$$\frac{xy}{x-y} = \frac{x}{\frac{x}{y}-1} = \frac{y}{1-\frac{y}{x}} = \frac{1}{\frac{1}{y}-\frac{1}{x}}$$

are clearly regular on each U_{ij} and are thus regular on $\underline{P} \times \underline{P} - \underline{P}$. Therefore the K -algebra $K[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}]$ generated by $\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}$ is a subalgebra of $\mathcal{O}(\underline{P} \times \underline{P} - \underline{P})$. Let now f be a regular function on $\underline{P} \times \underline{P} - \underline{P}$.

Let f have a representation $f = g(x, y)/(x-y)^n$ with a polynomial $g(x, y) \in K[x, y]$ in the variables x, y . Let

$$g(x, y) = \sum_{\nu, \mu} g_{\nu\mu} x^\nu y^\mu.$$

Then

$$f = \sum g_{\nu\mu} \frac{x^\nu y^\mu}{(x-y)^n} = \sum g_{\nu\mu} \frac{x^\nu y^{\mu-n}}{(\frac{x}{y}-1)^n}$$

which shows that $f \in \mathcal{O}(U_{12}) = K[x, 1/y, 1/(1 - x/y)]$ if, and only if, $g_{\nu\mu} = 0$ whenever $\mu > n$.

In the same way, one proves that $f \in \mathcal{O}(U_{21})$ if, and only if, $g_{\nu\mu} = 0$ whenever $\nu > n$. But if $n \geq \nu \geq \mu$, then

$$\frac{x^\nu y^\mu}{(x-y)^n} = \left(\frac{xy}{x-y}\right)^\mu \times \left(\frac{x}{x-y}\right)^{\nu-\mu} \times \left(\frac{1}{x-y}\right)^{n-\nu},$$

which shows that $\frac{x^\nu y^\mu}{(x-y)^n} \in K\left[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}\right]$.

If $n \geq \mu \geq \nu$, then

$$\frac{x^\nu y^\mu}{(x-y)^n} = \left(\frac{xy}{x-y}\right)^\nu \times \left(\frac{y}{x-y}\right)^{\mu-\nu} \times \left(\frac{1}{x-y}\right)^{n-\mu}.$$

As $\frac{y}{x-y} = \frac{x}{x-y} - 1$, we obtain also

$$\frac{x^\nu y^\mu}{(x-y)^n} \in K\left[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}\right].$$

As g is a linear combination of functions in $K\left[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}\right]$, it is also in this algebra, which proves

$$\mathcal{O}(\underline{P} \times \underline{P} - \underline{P}) \subseteq K\left[\frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}\right].$$

(2.2). Let $SL_2(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in K ; ad - bc = 1 \right\}$ and $K_* = K - \{0\}$. We consider the mapping

$$\pi : K_* \times (\underline{P} \times \underline{P} - \underline{P}) \longrightarrow SL_2(K)$$

given by

$$\pi(\tau, x, y) = \begin{pmatrix} a(\tau, x, y) & b(\tau, x, y) \\ c(\tau, x, y) & d(\tau, x, y) \end{pmatrix}$$

$$a(\tau, x, y) = \frac{\tau^{-1}x - \tau y}{x-y}$$

$$b(\tau, x, y) = \frac{(\tau - \tau^{-1})xy}{x-y}$$

$$c(\tau, x, y) = \frac{\tau^{-1} - \tau}{x-y}$$

$$d(\tau, x, y) = \frac{\tau x - \tau^{-1}y}{x-y}$$

$$\begin{aligned}
a(\tau, x, y) d(\tau, x, y) &= \frac{x^2 + y^2 - (\tau^{-2} + \tau^2)xy}{(x - y)^2} \\
&= \frac{(x - y)^2 - (\tau^{-2} - 2 + \tau^2)xy}{(x - y)^2} \\
&= 1 - \frac{(\tau^{-1} - \tau)^2 xy}{(x - y)^2} \\
&= 1 + b(\tau, x, y) c(\tau, x, y)
\end{aligned}$$

which shows that indeed $\pi(\tau, x, y) \in \text{SL}_2(K)$.

Properties of π :

$$\pi(-\tau, x, y) = -\pi(\tau, x, y)$$

$$\pi(\tau^{-1}, y, x) = +\pi(\tau, x, y)$$

$$\pi(\tau, x, y) \times \pi(\tau^{-1}, x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi(\tau_1, x, y) \times \pi(\tau_2, x, y) = \pi(\tau_1 \tau_2, x, y)$$

$$\pi(1, x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi(-1, x, y) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\tau, \tau' \neq \pm 1$, then $\pi(\tau, x, y) = \pi(\tau', x', y')$ if, and only if, either $\tau = \tau', x' = x, y' = y$ or if $\tau' = +\tau^{-1}, y' = x, x' = y$.

Let $\sigma = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \text{SL}_2(K)$. σ acts on $\underline{P}$ by $\sigma(x) = \frac{a_0 x + b_0}{c_0 x + d_0}$.

Then

$$\pi(\tau, \sigma(x), \sigma(y)) = \sigma \pi(\tau, x, y) \sigma^{-1}$$

$$\pi(\tau, x, y)(x) = x, \quad \pi(\tau, x, y)(y) = y$$

$$\pi(K_* \times (\underline{P} \times \underline{P} - \underline{P})) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(K) ; a + d \neq \pm 2 \right\} \cup \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

The trace $\text{tr } \pi(\tau, x, y) = \tau + \tau^{-1}$.

π is a morphism of K -algebraic spaces,

π induce a 2-sheeted unramified covering,

from $(K - \{0, 1, -1\}) \times (\underline{P} \times \underline{P} - \underline{P})$ onto the affine subdomain $\text{SL}'_2(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(K) ; (a + d)^2 \neq 4 \right\}$ of non-parabolic matrices.

Let

$$\mathcal{O}_1 = \mathcal{O}(K - \{0, 1, -1\} \times (\underline{P} \times \underline{P} - \underline{P})) = K[\tau, \tau^{-1}, \frac{1}{\tau-1}, \frac{1}{\tau+1}, \frac{1}{x-y}, \frac{x}{x-y}, \frac{xy}{x-y}]$$

and

$$\mathcal{O}_2 = \mathcal{O}(\mathrm{SL}_2^1(K)) = K[a, b, c, d, \frac{1}{a+d-2}, \frac{1}{a+d+2}]/(ad - bc - 1).$$

The mapping π induces a K -algebra homomorphism

$$\pi_* : \mathcal{O}_2 \longrightarrow \mathcal{O}_1$$

which is injective.

PROPOSITION 3. - $\mathcal{O}_1$ is a free $\mathcal{O}_2$ -module, generated by 1 and τ .

Proof. - Let M be the $\mathcal{O}_2$ -module, generated by 1 and τ . One has $\tau \notin \mathcal{O}_2$, as for any polynomial $f(\tau, x, y) \in \mathcal{O}_2$, we have the condition $f(\tau, x, y) = f(\tau^{-1}, y, x)$.

Now $\tau + \tau^{-1} = a + d$ and $\tau^2 - (a + d)\tau + 1 = 0$ which shows that τ is quadratic over $\mathcal{O}_2$. Thus $\tau^2 \in M$ and more generally all powers τ^i of τ , $i \in \mathbb{Z}$, are in M .

Thus M is a $\mathcal{O}_2$ -algebra.

$$\frac{(\tau - \tau^{-1})}{(a + d - 2)(a + d + 2)} = \frac{\tau - \tau^{-1}}{(\tau + \tau^{-1}) - 4} = \frac{1}{\tau - \tau^{-1}} \in M$$

$$\frac{\tau(1 - \tau^{-1})}{\tau - \tau^{-1}} = \frac{1}{1 + \tau^{-1}} \in M$$

$$\frac{\tau(1 + \tau^{-1})}{\tau - \tau^{-1}} = \frac{1}{1 - \tau^{-1}} \in M$$

$$\frac{1}{x - y} = \frac{c}{\tau^{-1} - \tau} \in M$$

$$\frac{xy}{x - y} = \frac{b}{\tau^{-1} - \tau} \in M$$

$$\frac{x}{x - y} = \frac{a - \tau}{\tau^{-1} - \tau} \in M.$$

This shows that $M = \mathcal{O}_1$.

(2.3). Let $\mathrm{SL}_2^{\mathrm{hb}}(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) ; |a + d| > 1 \right\}$ be the subdomain of $\mathrm{SL}_2(K)$ of hyperbolic matrices and

$$\mathrm{Hb}(K) := \{(\tau, x, y) \in K_* \times (\underline{P} \times \underline{P}) - \underline{P} \mid 0 < |\tau| < 1\}.$$

Then π induces an analytic mapping

$$\pi' : \mathrm{Hb}(K) \longrightarrow \mathrm{SL}_2^{\mathrm{hb}}(K).$$

PROPOSITION 4. - π' is bianalytic.

LEMMA. - There exists a power series $\tau(s) \in \mathbb{Z}[[s]]$ such that $\tau(s) + \frac{1}{\tau(s)} = \frac{1}{s}$.

Proof. - Define $\tau_0 = 0$, $\tau_1 = 1$ and, for $k \geq 1$,

$$\tau_{k+1} := - \sum_{i=1}^{k-1} (-1)^i \tau_i \tau_{k-i} \in \mathbb{Z}.$$

Then $\tau(s) := \sum_{i=1}^{\infty} \tau_i s^i$ satisfies the equation $\tau(s) + \frac{1}{\tau(s)} = s^{-1}$.

One gets $\tau_i = 0$ if i is even and $\tau_i > 0$ if i is odd and

$$\tau(s) = s + s^3 + 2s^5 + 5s^7 + 14s^9 + 42s^{11} + 132s^{13} + 429s^{15} + \dots$$

Another way to prove this lemma : you remark that $\tau(s)$ satisfies a quadratic equation :

$$\tau(s)^2 - \frac{1}{s} \tau(s) + 1 = 0.$$

Thus if $\text{char } K \neq 2$:

$$\left(\tau(s) - \frac{1}{2s}\right)^2 = \frac{1}{4s^2} - 1$$

$$\tau(s) = \frac{1}{2s} \pm \frac{1}{2s} \sqrt{1 - 4s^2}$$

$$\tau(s) = \frac{1}{2s} \pm \frac{1}{2s} \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-1)^i (4s^2)^i.$$

If you choose the right sign for the square root, you get

$$\tau(s) = \frac{1}{2} \sum_{i=1}^{\infty} \binom{\frac{1}{2}}{i} (-1)^{i+1} \times 4^i s^{2i-1}$$

now

$$\binom{\frac{1}{2}}{i} = \frac{1}{2^i} \binom{\frac{1}{2}}{i-1} \quad \text{and} \quad \binom{-\frac{1}{2}}{j} = \frac{\frac{1}{2^j} (-1)^j (2j-1)!}{j! \times 2^{j-1} (j-1)!}.$$

Thus

$$\tau_{2i-1} = \frac{1}{2} (-1)^{i+1} 4^i \frac{\frac{1}{2^{i-1}} (-1)^{i-1} (2i-3)!}{2^i (i-1)! 2^{i-2} (i-2)!}$$

$$\tau_{2i-1} = 2 \frac{(2i-3)!}{i! (i-2)!} = 2 \binom{2i-2}{i-2} \times \frac{1}{2i-2} = \frac{1}{i-1} \binom{2i-2}{i-2} = \frac{1}{i} \binom{2i-2}{i-1}.$$

The proof of proposition 4 is immediate with the help of the lemma and of proposition 3. The inverse of π' can be given explicitly, namely :

$$\begin{aligned}\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \tau \left(\frac{1}{a+d} \right) = \tau \\ \frac{1}{x-y} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{c}{\tau^{-1} - \tau} \\ \frac{xy}{x-y} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{b}{\tau^{-1} - \tau} \\ \frac{x}{x-y} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{a - \tau}{\tau^{-1} - \tau} \cdot\end{aligned}$$

(2.4). Let $\mathrm{PSL}_2(K) = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \}$ be the projective special linear group and

$$\mathrm{PSL}_2^{\mathrm{hb}}(K) = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^{\mathrm{hb}}(K) \} .$$

Now

$$\tau \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \tau \left(-\frac{1}{a+d} \right) = -\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\pi(-\tau, x, y) = -\pi(\tau, x, y) .$$

COROLLARY. - The mapping

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \left(\tau^2 \left(\frac{1}{a+d} \right), \frac{a-\tau}{c}, \frac{b}{a-\tau} \right)$$

gives a bianalytic mapping

$$\mathrm{PSL}_2^{\mathrm{hb}}(K) \longrightarrow \mathrm{Hb}(K) .$$

3. Teichmüller space.

(3.1) The set $\mathbb{S}_n$ of Schottky homomorphisms $\zeta : E_n \longrightarrow \mathrm{PSL}_2(K)$ will be identified with a subset of $\mathrm{Hb}^n(K) = \{ w = (w_1, \dots, w_n) ; w_i \in \mathrm{Hb}(K) \}$. We identify $\mathrm{Hb}(K)$ through the inverse of mapping $\pm \pi$ with $\mathrm{PSL}_2^{\mathrm{hb}}(K)$. As $\zeta(e_i)$ is hyperbolic, we get $(\zeta(e_1), \dots, \zeta(e_n)) \in \mathrm{Hb}^n(K)$.

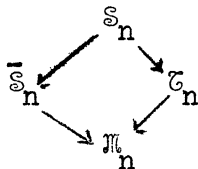
We study the actions of $\mathrm{Aut} E_n$ and of $\mathrm{Aut}^i \mathrm{PSL}_2(K)$ on $\mathbb{S}_n$. Let

$$\bar{\mathbb{S}}_n = \mathrm{Aut} E_n [\mathbb{S}_n]$$

$$\mathcal{C}_n = (\mathbb{S}_n]_{\mathrm{PSL}_2(K)}$$

$$\mathcal{M}_n = \mathrm{Aut} E_n [\mathbb{S}_n]_{\mathrm{PSL}_2(K)} .$$

Then we have a commutative diagram



and group actions of $PSL_2(K)$ on $\bar{S}_n$ and of $Aut E_n$ on T_n .

While $\bar{S}_n$ corresponds biuniquely with the set of Schottky subgroups of $PSL_2(K)$ of rank n (see (1.3)), the set T_n consists of normed Schottky homomorphisms.

PROPOSITION 5. - T_n can be identified with

$$\{w = (w_1, \dots, w_n) \in Hb^n(K) ; w \in S_n ;$$

$$w_i = (t_i, x_i, y_i) \in Hb(K) ; x_1 = 0 ; y_1 = \infty ; y_2 = 1\}.$$

Proof. - Let $\sigma(z) = (y_2 - y_1)/(y_2 - x_1) \times (z - x_1)/(z - y_1)$ be the fractional-linear transformation which maps x_1 to 0, y_1 to ∞ and y_2 to 1.

Now

$$\sigma\pi(t_1, x_1, y_1) \sigma^{-1} = \pi(t_1, 0, \infty)$$

$$\sigma\pi(t_2, x_2, y_2) \sigma^{-1} = \pi(t_2, \sigma(x_2), 1)$$

(see properties of π in (2.2)).

If $w \in S_n$ with $x_1 = 0$, $y_1 = \infty$, $y_2 = 1$ and $\sigma \in PSL_2(K)$ such that $\sigma \circ w \circ \sigma^{-1} = (w'_1, \dots, w'_n)$, $w'_i = (t'_i, x'_i, y'_i)$, $x'_1 = 0$, $y'_1 = \infty$, $y'_2 = 1$, then $\sigma(0) = 0$, $\sigma(\infty) = \infty$, $\sigma(1) = 1$ for which one concludes $\sigma = id$.

We consider T_n now as a subset of K^{3n-3} : a point $w \in T_n$ is given by the coordinates $(t_1, \dots, t_n, x_2, x_3, y_3, \dots, x_n, y_n) \in K^{3n-3}$. The kernel of ineffectivity of the action of $Aut E_n$ on T_n contains the inner automorphisms.

Let $\Psi_n := Aut E_n / Aut^1 E_n$ be the group of outer automorphisms.

Then the action of $Aut E_n$ induces an action of Ψ_n on T_n .

(3.2). Let $w = (w_1, \dots, w_n)$, $w_i = (t_i, x_i, y_i)$, be a variable point of $Hb^n(K)$. In order to get shorter formulas we also write x_{-i} for y_i . Let

$$u_{ijk} = \frac{\frac{x_j - x_i}{x_j - y_i}}{\frac{x_k - x_i}{x_k - y_i}} = \frac{(x_j - x_i)(x_k - y_i)}{(x_j - y_i)(x_k - x_i)}$$

for $i \in \{1, \dots, n\}$, $j, k \in \{^+1, \dots, ^+n\}$ and $^+j \neq i$, $^+k \neq i$.

We can consider u_{ijk} to be a meromorphic function on $Hb^n(K)$. It is an analytic function without zeroes on the subdomain $Hb_0^n(K) := \{w \in Hb^n(K) ; x_i \neq x_j \text{ for all } i \neq j ; i, j \in \{\pm 1, \dots, \pm n\}\}$ as

$$\frac{x_j}{x_j - x_k}, \frac{y_j}{x_j - x_k}, \frac{1}{x_i - x_k}$$

are analytic on $Hb_0^n(K)$. This can be seen as in the proof of proposition 2.

Now

$$u_{ijk} = \frac{x_j x_k}{(x_j - y_i)(x_k - x_i)} + \frac{x_i y_i}{(x_j - y_i)(x_k - x_i)} - \frac{x_j y_i}{(x_j - y_i)(x_k - x_i)} - \frac{x_i x_k}{(x_j - y_i)(x_k - x_i)}$$

and each term is clearly analytic on $Hb_0^n(K)$. As $\frac{1}{u_{ijk}} \neq u_{ijk}$ it has no zeroes on $Hb_0^n(K)$.

Let $\mathcal{B}_n := \{w \in Hb_0^n(K) ; |t_i| < |u_{ijk}(w)| < |t_i|^{-1} \text{ for all } i ; k \in \{\pm 1, \dots, \pm n\} ; i \in \{1, \dots, n\} ; i \neq \pm j ; i \neq \pm k\}$.

PROPOSITION 6. - $\mathcal{B}_n \subseteq \mathcal{S}_n$.

Proof. - Let $\gamma_i = \pm \pi(\sqrt{t_i}, x_i, y_i)$ and $v_i(z) = (z - x_i)/(z - y_i)$. Then $v_i(\gamma_i z) = t_i v_i(z)$. If now $\rho_i = \sup_{j \neq i} |v_i(x_j)|$ and $\rho'_i = \inf_{j \neq i} |v_i(x_j)|$, then $|t_i| < \rho'_i / \rho_i \leq 1$.

Let $\rho''_i > \rho_i$ such that $|t_i| \rho''_i < \rho'_i$. Fix x_j with $\rho_i = |v_i(x_j)|$, and let

$$F_i = \{z \in \mathbb{P} ; \rho''_i |t_i| \leq |v_i(z)| \leq \rho'_i\}$$

and

$$F = \bigcap_{i=1}^n F_i.$$

It is easy to see now that $\gamma_1, \dots, \gamma_n$ generates a Schottky group of rank n , and that F is a fundamental domain for this group (see [1], chapter I, (4.1.3)).

PROPOSITION 7. - The action of $\text{Aut } E_n$ on $\mathcal{B}_n$ satisfies :

(i) $\mathcal{B}_n \circ \alpha = \mathcal{B}_n$ if α is an inner automorphism of E_n .

(ii) There is only a finite number of classes $\alpha \in \text{Aut } E_n / \text{Aut}^1 E_n$ of automorphisms $\alpha \in \text{Aut } E_n$ such that

$$(\mathcal{B}_n \circ \alpha) \cap \mathcal{B}_n \neq \emptyset.$$

(iii) $\bigcup_{\alpha \in \text{Aut } E_n} \mathcal{B}_n \circ \alpha = \mathcal{S}_n$.

Proof. - The proof of (i) relies on the fact that the cross ratios u_{ijk} are invariant with respect to fractional linear transformations. The proof of (iii) is a corollary to [1] (chapter I, (4.3)).

In order to prove (ii) one has to introduce the canonical tree T for a Schottky group Γ . One has to use the fact that the geometric base systems and the fundamental domains $\subset T$ (see [2], p. 263, bottom), for this action correspond to the systems in $\mathcal{B}_n$.

If F is a fundamental domain (= maximal subtree) $\subset T$ for the action of T , there are only a finite number of other fundamental domains F' such that $F \cap F' \neq \emptyset$. From this one can conclude (ii).

(3.3). Let now $\bar{\mathcal{B}}_n = \mathcal{B}_n \cap \mathcal{C}_n$. Because $\bar{\mathcal{B}}_n$ is an analytic polyhedron $\subset K^{3n-3}$, it has a canonical analytic structure as subdomain of K^{3n-3} . For any $\psi \in \Psi_n$, also $\psi(\bar{\mathcal{B}}_n)$ is an analytic polyhedron $\subset K^{3n-3}$.

We consider the covering $\{\psi(\bar{\mathcal{B}}_n) ; \psi \in \Psi_n\}$ and put on $\mathcal{C}_n$ the analytic structure which is isomorphic on $\psi(\bar{\mathcal{B}}_n)$ to the canonical one given there.

We call $\mathcal{C}_n$ Teichmüller space and Ψ_n Teichmüller modular group.

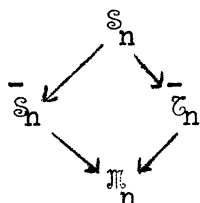
THEOREM 1. - Ψ_n acts discontinuously on $\mathcal{C}_n$.

One has to prove that the covering $\{\psi(\bar{\mathcal{B}}_n) ; \psi \in \Psi_n\}$ is admissible (see [4], p. 194, bottom), which means the following holds: if $\phi ; X \rightarrow K^{3n-3}$ is an analytic mapping of an affinoid space X into K^{3n-3} is given with $\phi(X) \subseteq \mathcal{C}_n$, then there is a finite set $\{\psi_1, \dots, \psi_r\}$ of elements of Ψ_n such that $\phi(X) \subset \bigcup_{i=1}^r \psi_i(\bar{\mathcal{B}}_n)$. It follows from the method given in [2], [3] and in the proof of the proposition in [1] (chapter I, (4.1.3)).

That the action of Ψ_n is discontinuous follows from proposition 2, (ii).

I will not work out the details as it seems to make more sense to prove the stronger statement that $\mathcal{C}_n$ is a Stein manifold.

Remark. - One should construct analytic structures on $\bar{\mathcal{S}}_n$ and on $\mathcal{M}_n$ such that the mappings of the diagram



are analytic quotient maps. It seems likely that the spaces $\mathcal{S}_n$, $\mathcal{C}_n$, $\bar{\mathcal{S}}_n$ may be even $\mathcal{M}_n$ are Stein spaces.

4. Siegel halfspaces

(4.1). Let $\mathcal{H}_n$ be the set of all symmetric $n \times n$ matrices $x = (x_{ij})$ with $x_{ij} = x_{ji} \in K_* = K - \{0\}$ for which the real matrix $(-\log |x_{ij}|)$ is positive definite. $\mathcal{H}_n$ is a subset of the space $\mathcal{S}_n(K_*)$ of all symmetric $n \times n$ matrices

$x = (x_{ij})$ with entries $x_{ij} \in K_*$. We identify $S_n(K_*)$ with the algebraic torus $K_*^{n(n+1)/2}$ by identifying the matrix $x = (x_{ij})$ with the $(n(n+1)/2)$ -tuple $(x_{11}, x_{12}, \dots, x_{21}, x_{22}, \dots, x_{nn})$.

Let $a = (a_{ij})$ be a $n \times r$ matrix with entries $a_{ij} \in \mathbb{Z}$. For $x \in S_n(K_*)$, we define $x^a = (y_{ij})$, $y_{ij} := \prod_{k=1}^r x_{ik}^{a_{kj}}$.

Then x^a is $n \times r$ matrix with entries in K_* . Similarly, one defines ${}^a x = (z_{ij})$ if a is a $r \times n$ matrix with entries $\in \mathbb{Z}$ by $z_{ij} := \prod_{k=1}^n x_{kj}^{a_{ik}}$.

This matrix operations satisfy the usual rules of matrix calculations.

If a is a $n \times n$ matrix and if a^t denotes the transpose of a , then ${}^a x^a$ is a symmetric $n \times n$ matrix $\in S_n(K_*)$ whenever $x \in S_n(K_*)$.

Moreover if $x \in \mathcal{H}_n$, then ${}^a x^a \in \mathcal{H}_n$ if $\det a \neq 0$. The mapping Φ_a which sends

$$x \mapsto {}^a x^a \text{ of } S_n(K_*) \rightarrow S_n(K_*)$$

is a morphism of algebraic spaces as the entries of ${}^a x^a$ are monomials in the variables x_{ij} . Because of $\Phi_a \circ \Phi_b = \Phi_{ba}$ one gets that $\Gamma_n := \{\Phi_a; a \in GL_n(\mathbb{Z})\}$ is a group of automorphisms of the K -algebraic space $S_n(K_*)$. It is easy to see that $\Gamma_n \cong PGL_n(\mathbb{Z})$.

(4.2). If $k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$ is a column vector with $k_i \in \mathbb{Z}$ and $x \in S_n(K_*)$, then $k^t x^k$ is an element of K_* . It is the value of the multiplicative quadratic form associated to x at the point k . We write $x[k] = k^t x^k$. Denote by M_n the set of all matrices $x \in S_n(K_*)$ which satisfy the following conditions:

For each i and all $k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \mathbb{Z}^n$ for which the greatest common divisor of the numbers $k_i, k_{i+1}, \dots, k_n$ is 1, we have

$$1 > |x_{ii}| \geq |x[k]|.$$

We call M_n Minkowski domain. It consists of those matrices x for which the associated real matrices $(-\log |x_{ij}|)$ are half-reduced in the sense of Minkowski.

For any $x \in M_n$, we have $|x[k]| \leq |x_{11}| < 1$. This allows to conclude that $x \in \mathcal{H}_n$. Thus $M_n \subseteq \mathcal{H}_n$. It is a simple consequence of the definition that $\bigcup_{\Phi \in \Gamma_n} \Phi(M_n) = \mathcal{H}_n$ (see for example [6], chapter II, § 3).

An important theorem of classical reduction theory says, that M_n is actually defined by a finite number of inequalities (see [6], chapter II, § 5, theorem 10). This means, there are finite sets $F_1, \dots, F_n \subset \mathbb{Z}^n$ such that

$$M_n = \{x \in S_n(K_*) ; 1 > |x_{11}| ; |x_{ii}| \geq |x[k]| \text{ for all } k \in F_i, \text{ all } i\}.$$

Example. — $M_2 = \{x \in S_2(K_*) ; 1 > |x_{11}| \geq |x_{22}| ; |x_{22}| \leq |x_{12}|^2 \leq |x_{22}|^{-1}\}.$

(4.3). The Minkowski domains are analytic polyhedra in $S_n(K_*)$.

They are therefore quasi-Stein subdomains of $K_*^{n(n+1)/2}$ in the sense of [5], § 2. Thus there is a canonical analytic structure on M_n .

Each $\phi \in \Gamma_n$ is an automorphism of the K -algebraic space $S_n(K_*)$ and thus also an analytic automorphism of $S_n(K_*)$. Thus $\phi(M_n)$ is also a quasi-Stein subdomain of $S_n(K_*)$ and we have a canonical analytic structure on $\phi(M_n)$. Thus we have defined an analytic atlas $\{\phi(M_n); \phi \in \Gamma_n\}$ on $\mathcal{H}_n$. We put on $\mathcal{H}_n$ the analytic structure given by this atlas. We call $\mathcal{H}_n$ together with this analytic structure the Siegel halfspace and Γ_n the Siegel modular group.

THEOREM 2. - $\mathcal{H}_n$ is an analytic manifold on which Γ_n acts discontinuously.

The proof of the fact that Γ_n acts discontinuously is left to the reader. It can be deduced from results in [6] (chapter II, § 5, especially statement 4 on page 67).

It means that for any affinoid polyhedron P of $S_n(K_*)$ which lies in $\mathcal{H}_n$ the set $\{\phi \in \Gamma_n; \phi(P) \cap P \neq \emptyset\}$ is finite.

Remark. - It is very likely that the set $\mathcal{H}_n/\Gamma_n$ of Γ_n -orbits in $\mathcal{H}_n$ can be given a canonical analytic structure such that the quotient mapping is locally bianalytic outside the ramification set. One can prove that $\mathcal{H}_n$ is a Stein manifold. It seems possible that even $\mathcal{H}_n/\Gamma_n$ is a Stein space.

(4.4) One of the more interesting points in the study of these topics is the period mapping q which is an analytic mapping $\mathcal{C}_n \rightarrow \mathcal{H}_n$ compatible with the actions of the Teichmüller and of the Siegel modular group (see [3], 8). Thus q induces a mapping $\bar{q}: \mathcal{C}_n/\psi_n \rightarrow \mathcal{H}_n/\Gamma_n$. Local properties of $\bar{q}$ have been studied in [3] (see for example Satz 7).

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