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AN APPLICATION OF NEWTON ITERATION PROCEDURE TO p -ADIC DIFFERENTIAL EQUATIONS

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This report is based on the author's lectures at Strasbourg, Padova, Grenoble, Groningen and Paris. The motivations of this research were explained in the papers to appear ([3],[5]) and the lecture-notes [4] (joint with S. SPERBER). Therefore, in this paper, we will report only on the technical part.

1. Preliminaries.

Let K be a field of characteristic zero complete with respect to an absolute value $|\cdot|$ which is non-trivial and ultrametric. The field of rational number, $\mathbb{Q}$, is a subfield of K , and we require that the restriction of $|\cdot|$ to $\mathbb{Q}$ is a p -adic absolute value for some prime number p . We normalize $|\cdot|$ so that $|p| = 1/p$.

For $\rho = \sum_{m=0}^{\infty} a_m x^m \in K[[x]]$, we set

$$|\rho|_0(r) = \sup_{m \geq 0} |a_m| r^m.$$

If $|\rho|_0(r_0) < +\infty$ for some positive constant r_0 , then ρ is convergent for $|x| < r_0$. The following lemma is fundamental throughout this report.

LEMMA 1. - Assume that $\rho_j = \sum_{m=0}^{\infty} a_{j,m} x^m \in K[[x]]$, $j = 1, 2, \dots$, with the properties :

- (i) $\lim_{j \rightarrow \infty} a_{j,m} = a_m$ exists for every m ;
- (ii) $|\rho_j|_0(r) \leq M(r)$ for $0 \leq r < r_0$, $j = 1, 2, \dots$, where r_0 is a positive number, and $M(r)$ is a non-negative number which depends only on r . Then,
 $\rho = \sum_{m=0}^{\infty} a_m x^m$ is convergent for $|x| < r_0$, and $\lim_{j \rightarrow \infty} |\rho_j - \rho|_0(r) = 0$ for
 $0 \leq r < r_0$. (Cf. B. DWORK [1].)

2. An example (a rough sketch).

Let us consider a non-linear differential equation

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$$(2.1) \quad x \, du/dx + \alpha u = f(x) + u^2 g(x, u),$$

where $\alpha \in K$, $f \in K[[x]]$, $g \in K[[x, u]]$, and f and g are convergent. We want to find a convergent power series $\rho \in K[[x]]$ which satisfies the equation (2.1). To do this, we try to construct ρ in the following form

$$(2.2) \quad u = \rho = \sum_{j=0}^{\infty} \rho_j, \quad \rho_j \in K[[x]].$$

Step 1. - First of all, ρ_0 is determined by the linear differential equation

$$(2.3) \quad x \, d\rho_0/dx + \alpha \rho_0 = f.$$

Step 2. - Change u by $u = \rho_0 + v$. Then (2.1) becomes

$$(2.1') \quad x \, dv/dx + \alpha v = \rho_0(x)^2 g(x, \rho_0(x)) + \rho_0(x) G(x) v + v^2 g_1(x, v),$$

where

$$G(x) = 2g(x, \rho_0(x)) + \rho_0(x) g_u(x, \rho_0(x)) \quad (g_u = \partial g / \partial u),$$

$$\begin{aligned} v^2 g_1(x, v) = & \rho_0(x)^2 \{g(x, \rho_0(x) + v) - g(x, \rho_0(x)) - g_u(x, \rho_0(x)) v\} \\ & + 2 \rho_0(x) v \{g(x, \rho_0(x) + v) - g(x, \rho_0(x))\} + v^2 g(x, \rho_0(x) + v). \end{aligned}$$

We determine ρ_1 by the linear part of (2.1')

$$(2.4) \quad x \, d\rho_1/dx + \alpha \rho_1 = \rho_0^2 g(x, \rho_0) + \rho_0 G(x) \rho_1.$$

The other ρ_j will be determined successively in a similar manner.

This is our Newton iteration procedure.

A closer look at equation (2.3). - If $f = \sum_{m=0}^{\infty} c_m x^m$ ($c_m \in K$), then ρ_0 is given by

$$(2.5) \quad \rho_0 = \sum_{m=0}^{\infty} \frac{c_m}{m + \alpha} x^m.$$

Assuming that $|f|_0(r) \leq M$ for $0 \leq r < r_0$, where r_0 and M are some positive numbers, we want to derive

$$(2.6) \quad |\rho_0|_0(r) \leq M \quad \text{for } 0 \leq r < r'_0,$$

for r'_0 a positive number, as large as possible, such that $0 < r'_0 \leq r_0$. To do this, we introduce two assumptions

$$(2.7) \quad c_m = 0 \quad \text{for } m < m_0,$$

$$(2.8) \quad |m + \alpha|^{-1} \leq C^{m^{1-\delta}} \quad (m \geq m_0),$$

where m_0 is a positive integer, C is a positive number greater than one, and δ is a positive number smaller than one, i. e. $C > 1$, $0 < \delta < 1$.

The assumption (2.8) may be called "non-Liouville property" of the exponent α . The condition (2.7) may be written

$$(2.7') \quad f \equiv 0 \pmod{x^{m_0}}.$$

Note that, if equation (2.1) admits a formal power series solution, then, we can change (2.1) so that condition (2.7) may be satisfied for any prescribed m_0 . Also note that any algebraic number α satisfies condition (2.8) for any δ if we choose C and m_0 suitably.

Under assumption (2.8), set

$$(2.9) \quad \rho_0 = (1/C)^{m_0^{-\delta}}$$

Then $0 < \rho_0 < 1$, and

$$\rho_0^m = (\rho_0^{m_0^\delta})^{m^{1-\delta}} = (C^{-(m/m_0)^\delta})^{m^{1-\delta}} \leq (1/C)^{m^{1-\delta}} \leq |m + \alpha| \quad \text{if } m \geq m_0.$$

Hence, under assumptions (2.7) and (2.8), we have

$$|\varphi_0|_0(r\rho_0) = \sup_{m \geq m_0} |m + \alpha|^{-1} |c_m| (r\rho_0)^m \leq \sup_{m \geq m_0} |c_m| r^m = |f|_0(r),$$

and

$$(2.6') \quad |\varphi_0|_0(r) \leq M \quad \text{for } 0 \leq r < r_0 \rho_0.$$

Equation (2.4) without φ_0 $G(t) \varphi_1$. - To simplify the explanation, we remove $\varphi_0 G(x) \varphi_1$ from the right-hand member of equation (2.4); i. e. we consider the equation

$$(2.10) \quad x \, d\varphi_1/dx + \alpha \varphi_1 = \varphi_0^2 g(x, \varphi_0).$$

We know already that

$$(2.11) \quad \varphi_0 \equiv 0 \pmod{x^{m_0}},$$

and that φ_0 satisfies (2.6'). First of all, (2.11) implies that

$$(2.12) \quad \varphi_0^2 g(\cdot, \varphi_0) \equiv 0 \pmod{x^{2m_0}}.$$

Hence, if we assume that g satisfies the condition

$$(2.13) \quad |\varphi_0^2 g(\cdot, \varphi_0)|_0(r) \leq M \quad \text{for } 0 \leq r < r_0 \rho_0,$$

we have

$$(2.14) \quad \begin{cases} \varphi_1 \equiv 0 \pmod{x^{2m_0}}, \\ |\varphi_1|_0(r) \leq M \quad \text{for } 0 \leq r < r_0 \rho_0 \rho_1, \end{cases}$$

$$\text{where } \rho_1 = (1/C)^{(2m_0)^{-\delta}} = \rho_0^{2^{-\delta}}.$$

Suppose that, proceeding inductively as above, we have defined for all $j \geq 0$,

$$\begin{cases} \varphi_j \equiv 0 \pmod{x^{2^j m_0}}, \\ |\varphi_j|_0(r) \leq M \quad \text{for } 0 \leq r < r_0 \rho_0 \rho_1, \dots, \rho_j. \end{cases}$$

where $\rho_j = \rho_{j-1}^{2^{-\delta}} = \rho_0^{2^{-j\delta}}$; set

$$\psi_j = \sum_{\ell=0}^j \rho_\ell, \quad \rho_\infty = \prod_{\ell=0}^{\infty} \rho_\ell = \rho_0^{(1-2^{-\delta})^{-1}} > 0.$$

Then $|\psi_j|_0(r) \leq M$ for $0 \leq r < r_0 \rho_\infty$, and ψ_j converges x -adically to

$$\rho = \sum_{\ell=0}^{\infty} \rho_\ell.$$

Therefore, by virtue of lemma 1, we conclude that ρ is convergent for $|x| < r_0 \rho_\infty$.

The argument of this section is not strictly speaking correct, since we removed $\rho_0 G(x) \rho_1$ from the right-hand member of equation (2.4). A correct treatment of equation (2.1) is given in SIBUYA-SPERBER ([2],[4]).

3. Typical results.

In this section, we shall give a rigorous treatment of a problem which is more general than the problem of section 2. We assume that K contains an element π such that

$$(3.1) \quad |\pi| = \left(\frac{1}{p}\right)^{(p-1)^{-1}}.$$

We consider the following situation.

(i) We are given $\alpha_1, \dots, \alpha_n \in K$ such that

$$(3.2) \quad |\alpha_j| \leq 1, \quad |m + \alpha_j|^{-1} \leq C^{m^{1-\delta}}, \quad |m + \alpha_1 - \alpha_j|^{-1} \leq C^{m^{1-\delta}}.$$

for $m \geq 2^k$ and $i, j = 1, \dots, n$, where k is a non-negative integer, and C and δ are positive numbers such that $C > 1$, $0 < \delta < 1$.

(ii) We are also given $a_1, \dots, a_n \in K[[x]]$ such that

$$(3.3) \quad a_j \equiv 0 \pmod{x}, \quad \left| \int_0^x t^{-1} a_j(t) dt \right|_0(r) < |\pi|,$$

for $0 \leq r < r_0$ and $j = 1, \dots, n$, where r_0 is a positive number, and where, for $a = \sum_{m=1}^{\infty} a_m x^m$, we have denoted $\sum_{m=1}^{\infty} (a_m/m) x^m$ by $\int_0^x t^{-1} a(t) dt$.

We define two sequences of numbers, $\{\sigma_h\}$ and $\{\tau_h\}$ by

$$(3.4) \quad \begin{cases} \sigma_1 = 1/C, & \tau_1 = (1/C)^{2(1-2^{-\delta})^{-1}} \\ \sigma_h = \sigma_{h-1}^2 \tau_{h-1}, & \tau_h = (\sigma_1^h \sigma_h)^{(1-2^{-\delta})^{-1}}. \end{cases}$$

Note that

$$(3.5) \quad 0 < \tau_h < \sigma_h < \tau_{h-1} < 1.$$

In this section, we shall prove the following two theorems.

THEOREM 1. - Assume that a differential operator $H = \sum_{j=0}^{n-1} b_j(x) \partial^j$ ($\partial = x d/dx$) satisfies the following conditions :

$$(3.6) \quad \begin{cases} b_j \in K[[x]] \text{ and } b_j \equiv 0 \pmod{x^{2^k}}, \\ |b_j|_0(r) < |\pi| \text{ for } 0 \leq r < r_0. \end{cases}$$

Then, there exists $\eta_1, \dots, \eta_n \in K[[x]]$ such that

$$(3.7) \quad \begin{cases} \eta_j \equiv 0 \pmod{x^{2^k}}, \\ \left| \int_0^{\cdot} t^{-1} \eta_j(t) dt \right|_0(r) < |\pi| \text{ for } 0 \leq r < r_0 \sigma_n^{2^{-k\delta}}, \quad j = 1, \dots, n, \end{cases}$$

and that

$$(3.8) \quad (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n) = H \\ = (\partial + \alpha_1 + a_1 - \eta_1) \dots (\partial + \alpha_n + a_n - \eta_n).$$

THEOREM 2. - Assume that

$$(3.9) \quad f \in K[[x]], \quad f \equiv 0 \pmod{x^{2^k}}, \quad |f|_0(r) < 1 \text{ for } 0 \leq r < r_0,$$

and that

$$(3.10) \quad \begin{cases} G = \sum_{\substack{\mu_0 + \dots + \mu_{n-1} \geq 2 \\ \mu_j \geq 0}} g_{\mu_0 \dots \mu_{n-1}}(x) v_0^{\mu_0} \dots v_{n-1}^{\mu_{n-1}} \in K[[x, v_0, \dots, v_{n-1}]], \\ \text{avec } g_{\mu_0 \dots \mu_{n-1}}(x) \in K[[x]], \\ |g_{\mu_0 \dots \mu_{n-1}}|_0(r) \leq |\pi| \text{ for } 0 \leq r < r_0. \end{cases}$$

Then, there exists a unique $\rho \in K[[x]]$ such that

$$(3.11) \quad \rho \equiv 0 \pmod{x^{2^k}},$$

and that

$$(3.12) \quad (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n)(\rho) = f + G(x, \rho, \partial \rho, \dots, \partial^{n-1} \rho).$$

Furthermore, this power series ρ also satisfies the condition

$$(3.13) \quad |\rho|_0(r) < 1 \text{ for } 0 \leq r < r_0 \tau_n^{2^{-k\delta}}.$$

Remark 1. - The power series ρ is a solution of a non-linear differential equation with purely Fuchsian linear part. This is a prototype of the most difficult situations in the study of p-adic non-linear problems. The most important part of theorem 2 is the estimate (3.13), i. e. the r-interval in which $|\rho|_0(r) < 1$ holds.

Remark 2. - Theorem 1 is a Hensel-type lemma. The problem of factorization of a linear differential operator is naturally reduced to a non-linear problem such as that of theorem 2. For example, if the order of the operator is two, the corresponding non-linear problem is a Riccati equation. In general, if the order of the operator is n , the order of the corresponding non-linear problem is $n - 1$. Taking advantage of this situation, we can prove theorem 1 and 2 simultaneously by an induction on n . Since the case $n = 1$ was treated in SIBUYA-SPERBER [2], we shall prove these theorems for $n \geq 2$. (Cf. also SIBUYA-SPERBER [4].)

4. Proof of theorem 1 for n .

In this section, assuming theorem 2 for $n - 1$, theorem 1 for $n = 1$, and theorem 1 for $n - 1$, we shall prove theorem 1 for n . Set

$$(4.1) \quad \begin{cases} L = (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_{n-1} + a_{n-1}), \\ \ell = \partial + \alpha_n + a_n. \end{cases}$$

We want to find $\eta \in K[[x]]$ and $\tilde{L} = \sum_{j=0}^{n-2} Y_j \partial^j$ ($Y_j \in K[[x]]$) such that

$$(4.2) \quad L\ell - H = (L - \tilde{L})(\ell - \eta).$$

The relation (4.2) is equivalent to the assertion that

$$(4.2') \quad L\ell(u) - H(u) = 0$$

for all u belonging to a sufficiently large extension of $K[[x]]$ such that

$$(\ell - \eta)(u) = 0.$$

Therefore, (4.2) is equivalent to the assertion that

$$(4.3) \quad L(u\eta) = H(u) \quad \text{for all such } u \text{ satisfying } \ell(u) = u\eta.$$

Observe that

$$(\partial + \alpha_j + a_j)(uv) = u(\partial + (\alpha_j - \alpha'_n) + (a_j - a_n) + \eta)(v),$$

if $\ell(u) = u\eta$. Hence

$$(4.4) \quad \begin{aligned} L(u\eta) = & u(\partial + (\alpha_1 - \alpha'_n) + (a_1 - a_n) + \eta) \dots \\ & \dots (\partial + (\alpha_{n-1} - \alpha'_n) + (a_{n-1} - a_n) + \eta)(\eta), \end{aligned}$$

if $\ell(u) = u\eta$. We can write

$$(4.4') \quad \begin{aligned} & (\partial + (\alpha_1 - \alpha'_n) + (a_1 - a_n) + \eta) \dots (\partial + (\alpha_{n-1} - \alpha'_n) + (a_{n-1} - a_n) + \eta)(\eta) \\ & = (\partial + (\alpha_1 - \alpha'_n) + (a_1 - a_n)) \dots (\partial + (\alpha_{n-1} - \alpha'_n) + (a_{n-1} - a_n))(\eta) \\ & \quad - \tilde{F}(x, \eta, \dots, \partial^{n-2} \eta), \end{aligned}$$

where

$$\tilde{F} = \sum_{\substack{\mu_0 + \dots + \mu_{n-2} \geq 2 \\ \mu_j \geq 0}} \tilde{F}_{\mu_0 \dots \mu_{n-2}}(x) v_0^{\mu_0} \dots v_{n-2}^{\mu_{n-2}} \in K[[x]][v_0, \dots, v_{n-2}],$$

$$\tilde{F}_{\mu_0 \dots \mu_{n-2}} \in K[[x]] \quad \text{and} \quad |\tilde{F}_{\mu_0 \dots \mu_{n-2}}|_0(r) \leq 1 \quad \text{for} \quad 0 \leq r < r_0.$$

On the other hand, if $\eta(u) = u\eta$, we have

$$\partial u = u(-\alpha_n - a_n + \eta), \quad \partial^2 u = u \{(-\alpha_n - a_n + \eta)^2 + \partial(-\alpha_n - a_n + \eta)\}, \quad \text{etc.}$$

Hence, $H(u)$ has the following form

$$(4.5) \quad H(u) = uF(x, \eta, \dots, \partial^{n-2} \eta),$$

where

$$F = \sum_{\substack{\mu_0 + \dots + \mu_{n-2} \geq 0 \\ \mu_j \geq 0}} F_{\mu_0 \dots \mu_{n-2}}(x) v_0^{\mu_0} \dots v_{n-2}^{\mu_{n-2}} \in K[[x]][v_0, \dots, v_{n-2}],$$

$$F_{\mu_0 \dots \mu_{n-2}} \in K[[x]], \quad F_{\mu_0 \dots \mu_{n-2}} \equiv 0 \pmod{x^{2^k}},$$

and

$$|F_{\mu_0 \dots \mu_{n-2}}|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0.$$

Thus, we derive from (4.3) the equation for η :

$$(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \dots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta) = F + \tilde{F}.$$

Set $\eta = \pi w$, and $\tilde{f}(x) = F_{0 \dots 0}(x)$, $\tilde{H} = \sum_{j=0}^{n-2} \tilde{b}_j(x) \partial^j$, where

$$\sum_{j=0}^{n-2} \tilde{b}_j(x) v_j = \sum_{\substack{\mu_0 + \dots + \mu_{n-2} = 1 \\ \mu_j \geq 0}} F_{\mu_0 \dots \mu_{n-2}}(x) v_0^{\mu_0} \dots v_{n-2}^{\mu_{n-2}},$$

and

$$\tilde{G}(x, v_0, \dots, v_{n-2}) = \sum_{\substack{\mu_0 + \dots + \mu_{n-2} \geq 2 \\ \mu_j \geq 0}} \{F_{\mu_0 \dots \mu_{n-2}}(x) + \tilde{F}_{\mu_0 \dots \mu_{n-2}}(x)\} v_0^{\mu_0} \dots v_{n-2}^{\mu_{n-2}}.$$

Then the equation for w is given by

$$(4.6) \quad (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \dots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(w) \\ = (1/\pi) \tilde{f} + \tilde{H}(w) + (1/\pi) \tilde{G}(x, \pi w, \pi \partial w, \dots, \pi \partial^{n-2} w)$$

Utilizing theorem 1 for $n-1$, we find $\tilde{\eta}_1, \dots, \tilde{\eta}_{n-1} \in K[[x]]$ such that

$$\tilde{\eta}_j \equiv 0 \pmod{x^{2^k}}, \quad \left| \int_0^\cdot t^{-1} \tilde{\eta}_j(t) dt \right|_0(r) < |\pi| \quad \text{for } 0 \leq r < r_0 \sigma_{n-1}^{2^{-k\delta}},$$

and that

$$\begin{aligned} & (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \dots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n)) - \tilde{H} \\ &= (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) - \tilde{\eta}_1) \dots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) - \tilde{\eta}_{n-1}). \end{aligned}$$

Then, applying to (4.6) theorem 2 for $n-1$, we find a unique solution $w(x)$ such that

$$\begin{cases} w \equiv 0 \pmod{x^{2^k}}, \\ |w|_0(r) < 1 \quad \text{for } 0 \leq r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k\delta}}. \end{cases}$$

Thus, we constructed η so that (4.3) is satisfied and

$$\begin{cases} \eta \equiv 0 \pmod{x^{2^k}}, \\ |\eta|_0(r) < |\pi| \quad \text{for } 0 \leq r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k\delta}}. \end{cases}$$

To compute $\tilde{L}$, we derive $\tilde{L}(\ell - \eta) = H - L\eta$. Putting

$$H - L\eta = \sum_{j=0}^{n-1} \hat{b}_j(x) \partial^j, \quad \hat{b}_j \in K[[x]],$$

we get

$$\begin{cases} \hat{b}_j \equiv 0 \pmod{x^{2^k}}, \\ |\hat{b}_j|_0(r) < |\pi| \quad \text{for } 0 \leq r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k\delta}}; \end{cases}$$

furthermore,

$$(4.3) \quad Y_{n-2} = \hat{b}_{n-1}, \quad Y_\mu = \hat{b}_{\mu+1} - \sum_{j=\mu+1}^{n-2} f_{j,\mu+1} Y_j, \quad \mu = 0, \dots, n-3,$$

where $f_{j,\mu} \in K[[x]]$, and $|f_{j,\mu}|_0(r) \leq 1$ for $0 \leq r < r_0(\sigma_{n-1} \tau_{n-1})^{2^{-k\delta}}$.

Finally, applying to $L - \tilde{L}$ theorem 1 for $n-1$, and to $\ell - \eta$ theorem 1 for $n=1$, and utilizing the inequality $\sigma_{n-1} < \sigma_1$, we complete the proof.

5. Proof of theorem 2 for n .

In this section, assuming theorem 1 for n , and theorem 2 for $n=1$, we shall prove theorem 2 for n . Setting

$$(5.1) \quad \psi_j = \sum_{\ell=0}^j \varphi_\ell = \psi_{j-1} + \varphi_j,$$

we determine $\psi_j \in K[[x]]$ by

$$\begin{aligned}
 (5.2) \quad & (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n)(\psi_j) \\
 & = f + G(x, \psi_{j-1}, \partial \psi_{j-1}, \dots, \partial^{n-1} \psi_{j-1}) \\
 & \quad + \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \dots, \partial^{n-1} \psi_{j-1}) \partial^i \rho_j,
 \end{aligned}$$

where $G_{v_i} = \partial G / \partial v_i$. This means that the ρ_j are determined by linear differential equations:

$$(5.3) \quad L_j(\rho_j) = f_j \quad (j = 0, 1, \dots),$$

where

$$(5.4) \quad \begin{cases} L_0 = (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n), \\ L_j = L_0 - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \dots, \partial^{n-1} \psi_{j-1}) \partial^i \quad (j \geq 1); \end{cases}$$

$$(5.5) \quad \begin{cases} f_0 = f \\ f_j = G(x, \psi_{j-1}, \dots, \partial^{n-1} \psi_{j-1}) - G(x, \psi_{j-2}, \dots, \partial^{n-1} \psi_{j-2}) \\ \quad - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-2}, \dots, \partial^{n-1} \psi_{j-2}) \partial^i \rho_{j-1}, \quad (j \geq 1) \end{cases}$$

where $\psi_\ell = 0$ if $\ell < 0$.

We want to construct the ρ_j so that

$$(5.6) \quad \begin{cases} \rho_j \equiv 0 \pmod{x^{2^{k+j}}}, \\ |\rho_j|_0(r) < 1 \quad \text{for } 0 \leq r < r_0 \sigma_1^{n 2^{-(k+j)\delta}} \prod_{\ell=0}^{j-1} (\sigma_1^n \sigma_n)^{2^{-(k+\ell)\delta}}. \end{cases}$$

To do this, set

$$(5.7) \quad L_j = L_{j-1} - H_j \quad (j \geq 1),$$

where by (5.4)

$$\begin{aligned}
 (5.8) \quad H_j = & \sum_{i=0}^{n-1} \{ G_{v_i}(x, \psi_{j-1}, \dots, \partial^{n-1} \psi_{j-1}) \\
 & - G_{v_i}(x, \psi_{j-2}, \dots, \partial^{n-1} \psi_{j-2}) \} \partial^i.
 \end{aligned}$$

Using an induction on j , we can achieve a factorization of L_j into linear factors, by virtue of theorem 1 for n , for

$$|x| < r_0 \prod_{\ell=0}^{j-1} (\sigma_1^n \sigma_n)^{2^{-(k+\ell)\delta}}.$$

Then, by using theorem 2 for $n = 1$ (n -times), we can achieve (5.6).

Thus, we get

$$|\psi_j|_0(r) < 1 \quad \text{for } 0 \leq r < r_0 \tau_n^{2^{-k\delta}}, \quad j = 0, 1, \dots,$$

and ψ_j converges x -adically to $\varphi = \sum_{j=0}^{\infty} \varphi_j$. Hence, by lemma 1 of section 1,

$$|\varphi|_0(r) < 1 \quad \text{for } 0 \leq r < r_0 \tau_n^{2^{-k\delta}}.$$

Finally, letting j tend to infinity on the both sides of (5.2), we complete the proof.

Results for more general cases, applications, and treatments of systems of differential equations were given in SIBUYA-SPERBER ([3],[4]).

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