

GROUPE DE TRAVAIL D'ANALYSE ULTRAMÉTRIQUE

MARIUS VAN DER PUT

Harmonic analysis on p -torsional groups

Groupe de travail d'analyse ultramétrique, tome 6 (1978-1979), exp. n° 14, p. 1-6

http://www.numdam.org/item?id=GAU_1978-1979__6__A8_0

© Groupe de travail d'analyse ultramétrique
(Secrétariat mathématique, Paris), 1978-1979, tous droits réservés.

L'accès aux archives de la collection « Groupe de travail d'analyse ultramétrique » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

HARMONIC ANALYSIS ON p -TORSIONAL GROUPS

(after A. M. M. Gommers)

by Marius van der PUT (*)
[Rijksuniversiteit, Utrecht]

The following is a presentation of results obtained by A. M. M. GOMMERS. Full details will appear in his forthcoming thesis prepared under the guidance of A. C. M. van ROOIJ.

1. The groups G that we consider are torsional, i. e. G satisfies the equivalent conditions :

(a) G is a commutative topological group, and G has a zero-dimensional open compact subgroup H such that G/H is a torsion group.

(b) G is a commutative, locally compact, zero-dimensional group such that every finite subset of G lies in a compact subgroup.

Let p be a prime number ; then G is called p -torsional (resp. p -free) when for any open compact subgroup H of G the group G/H is a p -torsion group (resp. has no p -torsion).

The field k is supposed to be a non-archimedean valued complete field with residue field $\bar{k}$ of characteristic p .

(1.1) LEMMA. - G has a unique decomposition as a topological product $G = G_1 \times G_2$, where G_1 is p -torsional and G_2 is p -free.

Proof. - For a compact zero-dimensional group G this decomposition is well known. In the general case, each open compact subgroup H of G has a unique decomposition $H_1 \times H_2$. Then $G_i = \bigcup \{H_i ; H \text{ open compact subgroup of } G\}$ ($i=1,2$) provides the unique decomposition of G .

(1.2) Remarks. - On the part G_2 of G there exists a (k -valued) Haar measure μ . Let $C_\infty(G_2)$ denote the Banach space of the continuous functions $G_2 \rightarrow k$ which are "zero at ∞ ", provided with the supremum norm. On $C_\infty(G_2)$ we have a convolution

$$(f * g)(a) = \int f(b) g(a - b) d\mu(b)$$

(*) Texte reçu le 12 mars 1979.
Marius Van der PUT, Mathematisch Instituut der Rijksuniversiteit, UTRECHT (Pays-Bas).

and $L(G_2)$ denotes $C_\infty(G_2)$ with the algebra structure given by the convolution.

Let us suppose, for convenience, that k is algebraically closed. Then the dual of G_2 is $\hat{G}_2$ = the continuous homomorphisms $G_2 \rightarrow k^*$, provided with the compact open topology. The Fourier theory ([2], [3]) states :

$$F : L(G_2) \rightarrow C_\infty(\hat{G}_2)$$

is an isometric isomorphism of Banach algebra's where the Fourier transform F is defined by :

$$F(f)(\chi) = \int f(b) \chi(-b) d\mu(b) \quad \text{with } f \in L(G_2) \quad \text{and } \chi \in \hat{G}_2.$$

On the part G_1 of G there is (in general) no Haar-measure. So $L(G_1)$ is meaningless. One studies instead $M(G_1)$. In general, $M(G)$ = the Banach space of tight measures on $G = \text{inj lim}\{C_\infty(H)'; H \text{ compact in } G\}$. In particular, if G is compact then $M(G) = C_\infty(G)'$ = the topological dual of the Banach space $C_\infty(G)$. On $M(G)$ the convolution is defined by

$$(\mu * \nu)(f) = \iint f(a+b) d\mu(a) d\nu(b).$$

If G is p -free then $M(G) \xrightarrow{\sim} BUC(\hat{G})$ = the bounded uniformly continuous functions on $\hat{G}$. This isomorphism is given by :

$$\mu \mapsto \hat{\mu} \quad \text{and} \quad \hat{\mu}(\chi) = \int \chi(a) d\mu(a), \quad \text{where } \mu \in M(G); \quad \chi \in \hat{G}.$$

In general, the algebra $M(G)$ is (morally speaking) determined by $M(G_1)$ and $M(G_2)$. Since the part $M(G_2)$ is well known as an algebra, the remaining part $M(G_1)$ will have most of our attention.

We can formulate the connection between $M(G)$, $M(G_1)$, $M(G_2)$ as follows :

(1.3) PROPOSITION. - If G_2 is compact then $M(G) \simeq M(G_1) \circ M(G_2)$ (as Banach algebra's) .

Proof. - The operation $\circ$ is a variant of the tensor product of Banach spaces. We define $\circ$ only for pairs (E, F') , where F' is the dual of some Banach space F .

Definition. - $E \circ F' = \text{proj lim}\{E \otimes F'_0; F_0 \text{ finite dimensional subspace of } F\}$. In our case, $M(G_2)$ is naturally given as the dual of $C_\infty(G_2)$. One easily verifies the formula when G_2 is finite (then $\circ$ and $\otimes$ agree). From this the general case follows.

(1.4) Remarks.

1° If G_2 is not compact then $M(G) \simeq \text{inj lim } M(G_1) \circ M(H_2)$, where H_2 runs in the set of all open compact subgroups of G_2 . The isomorphism is again an isomorphism of Banach algebras.

2° If G_2 is compact, and k is algebraically closed, then $M(G_2) = B(\hat{G}_2) =$

the bounded functions on $\hat{G}_2$. Proposition (1.3) yields

$$M(G) \simeq \prod_{\chi \in \hat{G}_2} M(G_1) * \chi \text{ and every } M(G_1) * \chi \cong M(G_1) .$$

3° In many cases, one can show that there is a $(1-1)$ -correspondance between the homomorphisms $\varphi : M(G) \rightarrow k$ and the pairs of homomorphisms

$$\varphi_i : M(G_i) \rightarrow k \quad (i = 1, 2) .$$

This holds for instance if k is not locally compact.

2. In this section, we assume that G is a p -torsional group.

Let T denote the discrete p -torsion group $\mathbb{Q}_p/\mathbb{Z}_p$. If the field k has characteristic 0 and is algebraically closed then we can identify T with the subgroup of k^* consisting of the elements of order p^n ($n \geq 0$).

For a p -torsional group G we define a dual $G^* =$ the continuous homomorphisms $G \rightarrow T$, provided with the compact open topology.

G^* is again p -torsional; $G \simeq G^{**}$; G is compact if, and only if, G^* is discrete.

There are two extreme cases for p -torsional groups :

Type (1) : G has no elements ($\neq 0$) of finite order.

Type (2) : The elements of finite order are dense in G .

For compact G one has : G is of type (1) if, and only if, G^* is a p -divisible group; G is of type (2) if, and only if, G^* has no p -divisible subgroups $\neq 0$. Further, if G is compact then $G = G_1 \times G_2$ where G_1 is of type (i). This follows from $G^* = H_1 \times H_2$ where H_1 is a maximal p -divisible subgroup of G^* and so $G = H_1^* \times H_2^*$.

The compact groups G of type (1) are easily determined : G^* is p -divisible and (as is well known) it follows that $G^* = T^{(I)}$ for some index set I . Then $G \cong \mathbb{Z}_p^I$ since $T^* = \mathbb{Z}_p$.

The compact groups G of type (2) (or their duals G^*) are very complicated in general. One can however prove the following :

(2.1) PROPOSITION. - Let G be compact, then there exists an exact sequence of topological groups

$$0 \rightarrow \mathbb{Z}_p^I \rightarrow G \rightarrow \prod_{j \in J} \mathbb{Z}/p^{n_j} \rightarrow 0 .$$

If $\sup(x_j) < \infty$ then the sequence splits topologically.

Next, we have the following :

(2.2) PROPOSITION. - The following properties of the p-torsional group G are equivalent :

- (a) G has no elements ($\neq 0$) of finite order (i. e. G of type (1)),
- (b) $\mathbb{Z}_p^I \subset G \subset \mathbb{Z}_p^I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ where $\mathbb{Z}_p^I$, with the product topology, is an open compact subgroup of G,
- (c) the norm on $M(G)$ is multiplicative,
- (d) for any $\mu \in M(G)$, $\mu \neq 0$, one has :

$$\mu \text{ is invertible in } M(G) \Leftrightarrow \|\mu\| = |\mu(G)|.$$

Proof. - Since $M(G) = \text{proj lim}\{M(H) ; H \text{ open compact subgroup of } G\}$, it suffices to consider compact groups G. In this case, (b) can be replaced by (b') : $G \cong \mathbb{Z}_p^I$.

Another argument shows that the general case will follow from the case where G is topologically finitely generated. Such a group has the form

$$G = \prod_{i=1}^n \mathbb{Z}_p / p^{m_i} \mathbb{Z}_p \text{ with } 0 < m_i \leq \infty.$$

In (2.3) and (2.4), $M(G)$ is explicitly given and one can verify (2.2).

(2.3) PROPOSITION. - Let $G = \mathbb{Z}_p^n$ then $M(G) \simeq k\langle X_1, \dots, X_n \rangle =$ the Banach algebra of all power series $\sum a_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with $\sup |a_\alpha| < \infty$.

Proof. - $C(\mathbb{Z}_p^n)$ has the orthonormal base

$$\left(\frac{X}{\alpha}\right) = \left(\frac{X_1}{\alpha_1}\right) \dots \left(\frac{X_n}{\alpha_n}\right),$$

considered as a function : $\mathbb{Z}_p^n \rightarrow k$. The isomorphism of (2.3) is given by the map

$$\mu \mapsto \sum_{\alpha} \mu\left(\frac{X}{\alpha}\right) X_1^{\alpha_1} \dots X_n^{\alpha_n}.$$

(2.4) COROLLARY. - Let $G = \prod_{i=1}^n \mathbb{Z}_p / p^{m_i} \mathbb{Z}_p$ then $M(G) = k\langle X_1, \dots, X_n \rangle / I$, where I is the ideal generated by $(X_i + 1)^{p^{m_i}} - 1$ (all i with $m_i \neq \infty$).

(2.5) Remark. - If G is compact then there exists a surjective map $\mathbb{Z}_p^I \rightarrow G$. Hence $M(G)$ is a quotient of $M(\mathbb{Z}_p^I) = k\langle X_i | i \in I \rangle$. If I is infinite then it is not clear what the kernel $M(\mathbb{Z}_p^I) \rightarrow M(G)$ should be.

3. We suppose in this section that G is a compact p-torsional group.

(3.1) PROPOSITION. - Suppose that k has characteristic p. Then

- (a) $M(G)$ has no idempotents $\neq 0, 1$,
- (b) any character $\chi : G \rightarrow k^*$ with open kernel is trivial,
- (c) if $\mu \in M(G)$ satisfies $\|\mu\| = |\mu(G)| \neq 0$, then μ is invertible and $\|\mu^{-1}\| = |\mu(G)|^{-1}$.

Proof. - Let $\mu = \mu^2 \in M(G)$, let H be an open compact subgroup of G , and let $\nu \in M(G/H)$ be the image of μ . Then $\nu^2 = \nu$ and $\nu = 0$ or 1 since $M(G/H)$ is a local ring. It follows easily that $\mu = 0$ or 1 . Statement (b) follows since k contains no p -th roots of unity. Statement (c) is easily seen for finite groups and follows from that special case.

Suppose now that k has characteristic zero (hence $k \supset \mathbb{Q}_p$). If k is algebraically closed then we can identify G^* with the characters $\chi : G \rightarrow k^*$ with open kernel. Further $M(G)$ can contain idempotents $\neq 0, 1$. Namely, let H be a finite subgroup of G , and let $\chi : H \rightarrow k^*$ be a character then

$$\mu_\chi = \frac{1}{p^n} \sum_{h \in H} \chi(-h) \delta_h \in M(H) \subset M(G),$$

where p^n is the order of H , is clearly an idempotent. For any finite set $E \subset H^*$ one can form

$$\mu_E = \sum_{\chi \in E} \mu_\chi.$$

In this way we have described all idempotents, with support in H . Now A. M. M. GOMMERS conjectures that there are no other idempotent elements in $M(G)$. We can state this as follows :

(3.2) CONJECTURE. - Every idempotent in $M(G)$ has finite support.

One has to work with G^* the characters of G to find a proof. The elements in G^* are linearly independent functions on G , but they are by no means orthogonal. This is the main difficulty in the verification of (3.2).

A. M. M. GOMMERS gives a proof of a special case :

(3.3) PROPOSITION. - For $G = (\mathbb{Z}/p)^I$ every element $\mu \in M(G)$ with $\mu = \mu^2$ has finite support.

We give some comment on the conjecture. Let G be a group of order p^n . Let $E \subset G^*$ be given, then

$$\mu_E = \sum_{\chi \in E} \mu_\chi = \sum_{g \in G} \left(\frac{1}{p^n} \sum_{\chi \in E} \chi(-g) \right) \delta_g$$

is an idempotent.

It has the property $\mu_E(\chi) = 1$ or 0 according to $\chi \in E$ or $\chi \notin E$. One sees that in general $\|\mu_E\| = p^n$. If μ_E has support in a subgroup H of G with order p^k , then $\|\mu_E\| \leq p^k$. This yields the following.

(3.4) CONJECTURE. - Let G be a group of order p^n , let $\mu \in M(G)$ be an idempotent with norm $\leq p^k$. If n is "large with respect to k " then μ has support in a proper subgroup of G .

We note that (3.4) implies (3.2). A first step towards (3.4) is estimating the

absolute value of sums of p^d -th roots of unity. This is done in :

(3.5) LEMMA. - Let $\omega \in k$ be a primitive p^d -th root of unity and let $\ell, n_i \in \mathbb{Z}$; $\ell \geq 1$. Then equivalent are :

$$(a) \quad \left| \sum_{i=0}^{p^d-1} n_i \omega^i \right| \leq \frac{1}{p^\ell},$$

$$(b) \quad \text{For all } 0 \leq i, j < p^d - 1 \text{ with } i \equiv j(p^{d-1}) \text{ one has } n_i \equiv n_j(p^\ell).$$

Proof. - (b) $\Rightarrow$ (a) follows easily from the minimal equation $(X^{p^d} - 1 / X^{p^{d-1}} - 1)$ satisfied by ω .

Further, we note that it suffices to show (a) $\Rightarrow$ (b) for $\ell = 1$; $\ell > 1$ follows easily by induction.

We consider $\mathbb{Z}[\omega] = \mathbb{Z}_p[\xi]$ where $\omega = 1 + \xi$. This is a subring of k . Since $|\xi|^{p^{d-1}(p-1)} = \frac{1}{p}$, it follows that the elements in $\mathbb{Z}_p[\xi]$ with absolute value $\leq \frac{1}{p}$ form the ideal $I = p\mathbb{Z}_p[\xi]$. Dividing by this ideal one finds :

$$\mathbb{Z}_p[\omega]/I = \mathbb{F}_p[T]/(1 + T^{p^{d-1}} + T^{2p^{d-1}} + \dots + T^{(p-1)p^{d-1}})$$

where T has image ω . Hence $\left| \sum_{i=0}^{p^d-1} n_i \omega^i \right| \leq \frac{1}{p}$ implies $\sum \bar{n}_i t^i = 0$ (where $\bar{n}_i$ is the image of n_i in $\mathbb{F}_p$).

This means

$$\sum \bar{n}_i T^i = (a_0 + a_1 T + \dots + a_{p-1} T^{p-1})(1 + T^{p^{d-1}} + T^{2p^{d-1}} + \dots + T^{(p-1)p^{d-1}})$$

for certain $a_0, a_1, \dots, a_{p-1} \in \mathbb{F}_p$. This is equivalent with statement (b) for $\ell = 1$.

REFERENCES

- [1] GOMMERS (A. M. M.). - Thesis, University of Nijmegen, 1979.
- [2] ROOIJ (A. C. M. van-). - Non-archimedean functional analysis. - New York and Basel, Marcel Dekker, 1978 (Pure and applied Mathematics. Dekker, 51).
- [3] SCHIKHOF (W. H.). - Non-archimedean harmonic analysis, Dissertation, Univ. Nijmegen, 1967.