CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 46, nº 2 (2005), p. 139-153
<http://www.numdam.org/item?id=CTGDC_2005__46_2_139_0>

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GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVI-2 (2005)

PRODUCTS IN THE CATEGORY OF APARTNESS SPACES

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Résumé. Les auteurs définissent une structure de séparation sur le produit de deux espaces de séparation ('apartness spaces') et analysent le rôle de la décomposabilité locale dans la théorie. Tout le travail est constructif – c'est-à-dire utilise une logique intuitionniste plutôt que classique.

1 Introduction

The axiomatic constructive theory of apartness spaces, originally introduced for point–set apartness in [3], was lifted to the set–set apartness context in [7] and [9]. Among the notable omissions from the latter paper was a notion of product apartness structure, corresponding in the set–set case to the point–set case covered in [3]. We rectify that omission in the present paper. We show that in the presence of a condition known as *local decomposability*, a condition that automatically holds under classical logic when the inequality relation is the denial of equality, both the point–set and the set–set notions of product space are categorical. Note that in this paper we discuss only products of two (and hence, by a simple extension, of finitely many) apartness spaces; we do not deal with products of infinite families.

Our work is entirely constructive, in that we use only intuitionistic logic. We expect that most of it can be formalised, with some effort, in a constructive (and predicative) set theory such as is found in [1]. For a classical–logic–based analogue of apartness spaces see [6].

2 Point–set apartness

We work throughout with a set X equipped with a binary relation $\neq$ of *inequality*, or *point–point apartness*, that satisfies the properties

$$\begin{aligned} x \neq y &\Rightarrow \neg(x = y), \\ x \neq y &\Rightarrow \quad y \neq x. \end{aligned}$$

A subset S of X has two natural complementary subsets:

- the **logical complement** $\neg S = \{x \in X : \forall y \in S \neg(x = y)\}$,
- the **complement** $\sim S = \{x \in X : \forall y \in S \ (x \neq y)\}$.

Initially, we are interested in the additional structure imposed on X by a relation $\mathbf{apart}(x, S)$ between points $x \in X$ and subsets S of X. For convenience we introduce the **apartness complement**

$$-S = \{x \in X : \mathbf{apart}(x, S)\}$$

of S; and, when A is also a subset of X, we write $A - S$ in place of $A \cap -S$.

We call X a **point–set apartness space** if the following axioms (introduced in [3]) are satisfied.

A1 $x \neq y \Rightarrow \mathbf{apart}(x, \{y\})$

A2 $\mathbf{apart}(x, A) \Rightarrow x \notin A$

A3 $\mathbf{apart}(x, A \cup B) \Leftrightarrow \mathbf{apart}(x, A) \wedge \mathbf{apart}(x, B)$

A4 $(\mathbf{apart}(x, A) \wedge -A \subset \sim B) \Rightarrow \mathbf{apart}(x, B)$

A5 $\mathbf{apart}(x, A) \Rightarrow \forall y \in X \ (x \neq y \vee \mathbf{apart}(y, A))$

We note, for future reference, that it can be deduced from axioms A5 and A2 that $-A \subset \sim A$; with this and the remaining axioms, if we also require the inequality on X to be **nontrivial**, in the sense that there exist x and y in X with $x \neq y$, then we can show that $X = -\emptyset$ (see [3], Section 2). Note also that if $\mathbf{apart}(x, \{y\})$, then $x \neq y$.

The morphisms in the category of point–set apartness spaces X and Y are those functions $f : X \to Y$ that are **continuous** in the sense that

$$\mathbf{apart}(f(x), f(S)) \Rightarrow \mathbf{apart}(x, S).$$

The canonical example of a point–set apartness space is a metric space (X, ρ), on which the inequality and apartness are respectively defined by

$$x \neq y \Leftrightarrow \rho(x, y) > 0$$

and

$$\mathbf{apart}\,(x, A) \Leftrightarrow \exists r > 0 \forall y \in A\,(\rho\,(x, y) \geqslant r)\,.$$

Functions between metric point–set apartness spaces are continuous, in the sense defined above, if and only if they are continuous in the usual ε–δ sense.

The metric–space example generalises to uniform spaces, but we defer further discussion of those until we deal with set–set apartness later. Another example of a point–set apartness is given by a T_1 topological space (X, τ) with a nontrivial inequality $\neq$; in this case, the apartness is defined by

$$\mathbf{apart}\,(x, A) \Leftrightarrow \exists U \in \tau\,(x \in U \subset \sim A)\,,$$

and we must postulate A5 since it need not hold in general (see [3]).

Let X_1 and X_2 be point–set apartness spaces, let X be their Cartesian product $X_1 \times X_2$, and, for example, let $\mathbf{x}$ denote the element (x_1, x_2) of X. The inequality relation on X is defined by

$$\mathbf{x} \neq \mathbf{y} \Leftrightarrow x_1 \neq y_1 \vee x_2 \neq y_2.$$

We define the **product point–set apartness structure** on X as follows:

$$\mathbf{apart}(\mathbf{x}, A) \Leftrightarrow \exists U_1 \subset X_1\, \exists U_2 \subset X_2\ (\mathbf{x} \in -U_1 \times -U_2 \subset \sim A)\,, \qquad (1)$$

where $-U_k$ is the apartness complement of U_k in the point–set apartness space X_k. We call X, equipped with this apartness structure, the **product of the point–set apartness spaces** X_1 and X_2.

Such product spaces are discussed in [3], where it is first verified that the product apartness structure really satisfies A1–A5. The proofs of a number of natural results on product point–set apartness spaces in [3] required the spaces X_1, X_2 (equivalently, the product space X) to be completely regular, according to a definition that we do not need here. We first state the main properties of product point–set apartness spaces, replacing complete regularity by the much weaker property of local decomposability; those results for which this replacement is made are proved in full.

A point–set apartness space X is said to be **locally decomposable** if

$$\forall x \in X \forall S \subset X \left(x \in -S \Rightarrow \exists T \left(x \in -T \wedge \forall y \in X \left(y \in -S \vee y \in T\right)\right)\right). \quad (2)$$

Local decomposability always holds classically: if $x \in -S$, then as $X = -S \cup \sim -S$ and $-S = -\sim -S$, we can take $T = \sim -S$. Every metric space (X, ρ) is locally decomposable: for if $x \in X - S$, then, choosing $r > 0$ such that $B(x, r) \subset -S$, we can take $T = \sim B(x, r/2)$ to obtain $x \in -T$ and $X = T \cup -S$. It also holds for a uniform space as defined in [7].

Lemma 1 *Let $X = X_1 \times X_2$ be the product of two point–set apartness spaces, and let $A_i \subset X_i$. Then $-A_1 \times X_2 = -(A_1 \times X_2)$ and $X_1 \times -A_2 = -(X_1 \times A_2)$.*

Proof. Since $-A_1 \times X_2 = -A_1 \times -\emptyset$ and

$$-A_1 \times X_2 \subset \sim A_1 \times X_2 \subset \sim (A_1 \times X_2),$$

the definition of the product apartness shows that $-A_1 \times X_2 \subset -(A_1 \times X_2)$. Conversely, given (x_1, x_2) in $-(A_1 \times X_2)$, use the definition of the product apartness to find subsets U_i of X_i such that

$$(x_1, x_2) \in -U_1 \times -U_2 \subset \sim (A_1 \times X_2).$$

If $\xi \in -U_1$, then

$$(\xi, x_2) \in -U_1 \times -U_2 \subset \sim (A_1 \times X_2),$$

so for all $\eta \in A_1$ we have $(\xi, x_2) \neq (\eta, x_2)$ and therefore $\xi \neq \eta$. Hence $x_1 \in -U_1 \subset \sim A_1$. It now follows from axiom A4 that **apart** (x_1, A_1). Thus $-(A_1 \times X_2) \subset -A_1 \times X_2$.

The other part of the lemma is proved similarly. ∎

Proposition 2 *Let $X = X_1 \times X_2$ be the product of two inhabited point–set apartness spaces. Then X is locally decomposable if and only if each X_k is locally decomposable.*

Proof. Suppose first that X is locally decomposable. Let $x_1 \in -U_1 \subset X_1$, and pick $x_2 \in X_2$. Then

$$\mathbf{x} = (x_1, x_2) \in -U_1 \times X_2 = -U_1 \times -\emptyset \subset \sim (U_1 \times X_2).$$

Hence, by definition of the apartness on X, $\mathbf{x} \in -(U_1 \times X_2)$. So, by the local decomposability of X, there exists $T \subset X$ such that

$$\mathbf{x} \in -T \wedge \forall \mathbf{y} \in X \left(\mathbf{y} \in -(U_1 \times X_2) \vee \mathbf{y} \in T\right).$$

Let

$$V_1 = \{\xi \in X_1 : (\xi, x_2) \in T\},$$

and consider any $\xi \in X_1$. Either $(\xi, x_2) \in -(U_1 \times X_2)$ and therefore (by Lemma 1) $\xi \in -U_1$, or else $(\xi, x_2) \in T$ and so $\xi \in V_1$. It remains to show that $x_1 \in -V_1$. To this end, since $\mathbf{x} \in -T$, we can find $W_i \subset X_i$ such that $\mathbf{x} \in -W_1 \times -W_2 \subset \sim T$. For any $\xi \in -W_1$ and $v \in V_1$ we have $(\xi, x_2) \in \sim T$ and $(v, x_2) \in T$; whence $\xi \neq v$. Thus $-W_1 \subset \sim V_1$. Since $x_1 \in -W_1$, it follows from axiom A4 that $x_1 \in -V_1$, as required. A similar proof shows that X_2 is locally decomposable.

Now suppose, conversely, that each X_k is locally decomposable. Consider $\mathbf{x} \in X$ and $S \subset X$ such that $\mathbf{x} \in -S$. Choose $U_1 \subset X_1$ and $U_2 \subset X_2$ such that $\mathbf{x} \in -U_1 \times -U_2 \subset \sim S$. Since $x_k \in -U_k$, there exists $V_k \subset X_k$ such that

$$x_k \in -V_k \wedge \forall x \in X_k \left(x \in -U_k \vee x \in V_k\right).$$

Let

$$T = (V_1 \times X_2) \cup (X_1 \times V_2).$$

Then $-V_1 \times -V_2 \in \sim T$, so, in particular, $\mathbf{x} \in -T$. On the other hand, for each $\xi \in X$, either $\xi_1 \in -U_1$ and $\xi_2 \in -U_2$, in which case $\xi \in -S$; or else we have either $\xi_1 \in V_1$ or $\xi_2 \in V_2$, when $\xi \in T$. Thus $X = -S \cup T$, and so X is locally decomposable. ∎

Corresponding to point–set apartness there is the opposite property of nearness, defined by

$$\mathbf{near}(\mathbf{x}, A) \Leftrightarrow \forall B \left(\mathbf{apart}(\mathbf{x}, B) \Rightarrow \exists \mathbf{y} \in A \ \mathbf{apart}(\mathbf{y}, B)\right).$$

Classically, if the inequality is the denial of equality, then $\mathbf{near}(\mathbf{x}, A)$ holds if and only if $\neg\mathbf{apart}(\mathbf{x}, A)$, which is equivalent to the condition

$$\forall U_1 \subset X_1 \forall U_2 \subset X_2 \left(\mathbf{x} \in -U_1 \times -U_2 \Rightarrow \exists \mathbf{y} \in (-U_1 \times -U_2) \cap A\right).$$

To discuss this constructively, we first observe that if X_i is a point–set apartness space and $U_i \subset X_i$ $(i = 1, 2)$, then

$$\begin{aligned} -U_1 \times -U_2 &= (-U_1 \times X_2) \cap (X_1 \times -U_2) \\ &= -(U_1 \times X_2) \cap -(X_1 \times U_2) \\ &= -\left((U_1 \times X_2) \cup (X_1 \times U_2)\right). \end{aligned}$$

Proposition 3 *Let $X = X_1 \times X_2$ be a product of two point-set apartness spaces, let $\mathbf{x}$ be a point of X, and let A be a subset of X. Then the following conditions are equivalent.*

(i) *for all $U_1 \subset X_1$ and $U_2 \subset X_2$ such that $\mathbf{x} \in -U_1 \times -U_2$, there exists $\mathbf{y} \in (-U_1 \times -U_2) \cap A$.*

(ii) $\mathbf{near}(\mathbf{x}, A)$.

Proof. For the proof that (i) implies (ii), see [3]. Conversely, if $U_1 \subset X_1$, $U_2 \subset X_2$, and $\mathbf{x} \in -U_1 \times -U_2$, then by the preceding observation, $\mathbf{x} \in -((U_1 \times X_2) \cup (X_1 \times U_2))$. If also $\mathbf{near}(\mathbf{x}, A)$, then there exists $\mathbf{y}$ in

$$A - ((U_1 \times X_2) \cup (X_1 \times U_2)),$$

which, again by our observation above, equals $(-U_1 \times -U_2) \cap A$. ■

Proposition 4 *Let $X = X_1 \times X_2$ be the product of two point-set apartness spaces, let $\mathbf{x} \in X$, and let $A \subset X$. Suppose that the following condition holds.*

(*) *There exist $V_1 \subset X_1$ and $V_2 \subset X_2$ such that $\mathbf{x} \in -V_1 \times -V_2$ and $A \subset (V_1 \times X_2) \cup (X_1 \times V_2)$.*

Then $\mathbf{apart}(\mathbf{x}, A)$. *Conversely, if the spaces X_1, X_2 are locally decomposable and* $\mathbf{apart}(\mathbf{x}, A)$, *then condition (*) holds.*

Proof. First assume (*), and construct V_1, V_2 with the stated properties. To prove that $\mathbf{apart}(\mathbf{x}, A)$, it suffices to note that

$$\begin{aligned} -V_1 \times -V_2 &= -(V_1 \times X_2) \cap -(X_1 \times V_2) \\ &\subset \sim (V_1 \times X_2) \cap \sim (X_1 \times V_2) \\ &\subset \sim A, \end{aligned}$$

where in the first line we have used the observation preceding Proposition 3.

Now assume, conversely, that the spaces X_1, X_2 are locally decomposable and that $\mathbf{apart}(\mathbf{x}, A)$. Choose $U_i \subset X_i$ such that

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim A.$$

For each i, we use the local decomposability of X_i to find $V_i \subset X_i$ such that $x_i \in -V_i$ and $X_i = -U_i \cup V_i$. Then

$$A \subset (V_1 \times X_2) \cup (X_1 \times V_2).$$

For if $(\xi_1, \xi_2) \in A$, then either $\xi_1 \in -U_1$ and $\xi_2 \in -U_2$, which is impossible, or else, as must be the case, $\xi_1 \in V_1$ or $\xi_2 \in V_2$. ■

Before going any further, we should prove

Lemma 5 *Let X_1, X_2 be point–set apartness spaces. Then the projection mappings $\mathrm{pr}_k : X_1 \times X_2 \longrightarrow X_k$ are continuous.*

Proof. Let **apart** $(\mathrm{pr}_1(\mathbf{x}), \mathrm{pr}_1(S))$, where $S \subset X = X_1 \times X_2$; then by Proposition 28 of [3], there exists $U_1 \subset X_1$ such that

$$x_1 = \mathrm{pr}_1(\mathbf{x}) \in -U_1 \subset {\sim}\mathrm{pr}_1(S).$$

Then $\mathbf{x} \in -U_1 \times X_2 = -U_1 \times -\emptyset$. Also, if $\mathbf{y} \in -U_1 \times -\emptyset$, then $y_1 \in -U_1$, so for all $\mathbf{z} \in S$, $y_1 \neq z_1$ and therefore $\mathbf{y} \neq \mathbf{z}$. Thus $-U_1 \times -\emptyset \subset {\sim} S$, and therefore **apart** $(\mathbf{x}, S)$. This proves the continuity of pr_1; that of pr_2 is established similarly. ■

We can now demonstrate the categoricity of the product of two point–set apartness spaces.

Proposition 6 *Let $X = X_1 \times X_2$ be the product of two locally decomposable point–set apartness spaces, and f a mapping of a point–set apartness space Y into X. Then f is continuous if and only if $\mathrm{pr}_i \circ f$ is continuous for each i.*

Proof. Assume that $\mathrm{pr}_i \circ f$ is continuous for each i. Let $y \in Y$ and $T \subset Y$ satisfy **apart** $(f(y), f(T))$. By Proposition 4, there exist $V_1 \subset X_1$ and $V_2 \subset X_2$ such that $f(y) \in -V_1 \times -V_2$ and

$$f(T) \subset (V_1 \times X_2) \cup (X_1 \times V_2).$$

Setting

$$T_1 = f^{-1}\left(f(T) \cap (V_1 \times X_2)\right),$$
$$T_2 = f^{-1}\left(f(T) \cap (X_1 \times V_2)\right),$$

we have $T \subset T_1 \cup T_2$. Moreover, for each i, $\mathrm{pr}_i \circ f(T_i) \subset V_i$ and therefore

$$\mathbf{apart}\,(\mathrm{pr}_i \circ f(y), \mathrm{pr}_i \circ f(T_i)).$$

Our continuity hypothesis now ensures that $y \in -T_1 \cap -T_2 \subset -T$.

The converse follows easily from Lemma 5. ∎

The local decomposability of the spaces X_1 and X_2 is used only once in the foregoing proof: namely, at the invocation of Proposition 4 in order to produce the sets V_1 and V_2. This suggests that instead of defining the product point–set apartness as we did, we might have defined it by taking **apart** $(\mathbf{x}, A)$ to mean that condition (*) of Proposition 4 holds. However, the status of axiom A4 for this notion of 'apart' remains undecided: we know of neither a proof that A4 holds nor a Brouwerian example indicating that A4 is not constructively derivable.

3 Set–set apartness

We now introduce a notion of apartness between subsets. We first assume that there is a **set–set pre–apartness** relation $\bowtie$ between pairs of subsets of X, such that the following axioms hold.

B1 $X \bowtie \emptyset$.

B2 $S \bowtie T \Rightarrow S \cap T = \emptyset$.

B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$

B4 $S \bowtie T \Rightarrow T \bowtie S$.

For the purposes of this paper, we then call X a **pre–apartness space**, or, if clarity demands, a **set–set pre–apartness space**. We then define

$$-S = \{x \in X : \{x\} \bowtie S\}. \qquad (3)$$

If, in addition to B1–B4, the axiom

B5 $x \in -S \Rightarrow \exists T(x \in -T \wedge \forall y(y \in -S \vee y \in T))$

holds, then we call $\bowtie$ a **(set–set) apartness** on X, and X a **(set–set) apartness space.**

The morphisms in the category of set–set (pre)apartness spaces are those functions $f : X \to Y$ that are **strongly continuous**, in the sense that

$$f(S) \bowtie f(T) \Rightarrow S \bowtie T.$$

Given a apartness space X, we define

$$x \neq y \Leftrightarrow \{x\} \bowtie \{y\}$$

and

$$x \bowtie S \Leftrightarrow \{x\} \bowtie S,$$

to obtain the inequality and the point–set apartness relation associated with the given set–set one. The point–set relation $\bowtie$ satisfies axioms **A1**–**A4**, where the apartness complement is defined as at (3). If the set–set relation also satisfies **B5**, then the point set apartness is locally decomposable and *a fortiori* satisfies **A5.**

An example of a set–set apartness space is afforded by a uniform space $(X, \mathcal{U})$, on which the inequality is defined by

$$x \neq y \Leftrightarrow \exists U \in \mathcal{U}\,((x,y) \notin U).$$

It is shown in [7] that the relation $\bowtie$ defined for subsets S, T of X by

$$S \bowtie T \Longleftrightarrow \exists U \in \mathcal{U}\ (S \times T \subset \sim U)$$

is a set–set apartness relation, Denoting by $\mathbf{apart}_{\mathcal{U}}$ the apartness relation associated with the topology $\tau_{\mathcal{U}}$ induced on X by $\mathcal{U}$, we can show that $x \bowtie S$ if and only if $\mathbf{apart}_{\mathcal{U}}(x, S)$.

The definition of the product of two set–set pre–apartnesses is much more complicated than that for point–set apartnesses. If $X = X_1 \times X_2$, where X_1 and X_2 are set–set pre–apartness spaces, then, taking the contrapositive of the definition used on page 23 of [6], we define the **product apartness** on X as follows. Two subsets A, B of X are **apart**, and we write $A \bowtie B$, if there exist finitely many subsets A_i $(1 \leqslant i \leqslant m)$ and B_j $(1 \leqslant j \leqslant n)$ of X such that

▷ $A \subset A_1 \cup \cdots \cup A_m,$

▷ $B \subset B_1 \cup \cdots \cup B_n$, and

▷ for all i, j either $\mathrm{pr}_1 A_i \bowtie \mathrm{pr}_1 B_j$ or $\mathrm{pr}_2 A_i \bowtie \mathrm{pr}_2 B_j$.

Proposition 7 *Let X_1, X_2 be pre-apartness spaces, and X their Cartesian product, define the relation* **apart** *as at* (1), *and let $\bowtie$ denote the product pre-apartness relation on X. Then $\{\mathbf{x}\} \bowtie S$ entails* **apart** $(\mathbf{x}, S)$. *Moreover, if both X_1 and X_2 are apartness spaces, then* **apart** $(\mathbf{x}, S)$ *entails* $\{\mathbf{x}\} \bowtie S$.

Proof. Assume first that $\{(x_1, x_2)\} \bowtie S$. Then there exist subsets B_j $(1 \leqslant j \leqslant n)$ of X such that $S \subset B_1 \cup \cdots \cup B_n$, and for each j either $x_1 \bowtie \mathrm{pr}_1 B_j$ or $x_2 \bowtie \mathrm{pr}_2 B_j$. Renumbering the sets B_j if necessary, we may assume that there exists $\nu \leqslant n$ such that $x_1 \bowtie \mathrm{pr}_1 B_j$ for $1 \leqslant j \leqslant \nu$, and $x_2 \bowtie \mathrm{pr}_2 B_j$ for $\nu + 1 \leqslant j \leqslant n$. Write

$$U = \mathrm{pr}_1 (B_1 \cup \cdots \cup B_\nu),$$
$$V = \mathrm{pr}_1 (B_{\nu+1} \cup \cdots \cup B_n).$$

Then $x_1 \in -U$ and $x_2 \in -V$. Given $x_1' \in -U$, $x_2' \in -V$, and $(\xi, \eta) \in S$, choose j such that $(\xi, \eta) \in B_j$. If $1 \leqslant j \leqslant \nu$, then $x_1' \neq \xi$; if $\nu + 1 \leqslant j \leqslant n$, then $x_2' \neq \eta$. Hence $(x_1', x_2') \neq (\xi, \eta)$. We now see that

$$(x_1, x_2) \in -U \times -V \subset \sim S$$

—in other words, **apart** $(\mathbf{x}, S)$.

Now assume that X_1, X_2 are actually set–set apartness spaces and that **apart** $(\mathbf{x}, S)$. So there exist $U_i \subset X_i$ with

$$\mathbf{x} \in -U_1 \times -U_2 \subset \sim S.$$

We can find sets $V_i \subset X_i$ such that

$$x_i \in -V_i \wedge \forall y \in X_i \, (y \in -U_i \vee y \in V_i). \tag{4}$$

Set

$$A_1 = \{\mathbf{x}\}, \quad B_1 = V_1 \times X_2, \quad B_2 = X_1 \times V_2.$$

For each ξ in S we have $\xi_i \in V_i$ for some i (for if $\xi_i \in -U_i$ for both i, then $\xi \in -U_1 \times -U_2 \in \sim S$, a contradiction); whence $S \subset B_1 \cup B_2$. Moreover, by our choice of V_i, we have $x_i \in -V_i$; whence $\{x_i\} = \mathrm{pr}_i(A_1) \bowtie V_i = \mathrm{pr}_i(B_i)$. Thus $\{\mathbf{x}\} \bowtie S$. ■

Having shown that our definition of product set–set apartness is compatible with the earlier definition of product point–set apartness, we still need to verify that, in the light of axioms B1–B5, we have actually defined an apartness on the Cartesian product $X = X_1 \times X_2$ of the sets underlying the apartness spaces X_1 and X_2. Once again, we keep track of the need, or otherwise, for axiom **B5**.

Proposition 8 *Let $X = X_1 \times X_2$ be the product of two set–set pre–apartness spaces. Then the relation $\bowtie$, defined as above for X, satisfies axioms **B1–B4**. If X_1, X_2 are apartness spaces, then $\bowtie$ also satisfies axiom **B5**.*

Proof. We verify each of the axioms in turn.

B1 Take $m = n = 1$, $A = X = X_1 \times X_2$, $B = \emptyset \subset X_1 \times X_2$, $A_1 = X_1 \times X_2$, and $B_1 = \emptyset \subset X_1 \times X_2$. Then $\mathrm{pr}_k(A_1) = X_k \bowtie \emptyset = \mathrm{pr}_k(\emptyset)$.

B2 Let $A, B \subset X = X_1 \times X_2$ and $A \bowtie B$. Choose the sets A_i, B_j as in the definition of $\bowtie$ on X. Supposing that $(x, y) \in A \cap B$, choose i, j such that $x \in A_i$ and $y \in B_j$. Then $(x, y) \in A_i \cap B_j$, so $x \in \mathrm{pr}_1(A_i) \cap \mathrm{pr}_1(B_j)$ and $y \in \mathrm{pr}_2(A_i) \cap \mathrm{pr}_2(B_j)$, which contradicts the properties of A_i and B_j.

B3 Let R, S, T be subsets of $X = X_1 \times X_2$. It is easy to prove that if $R \bowtie (S \cup T)$, then $R \bowtie S$ and $R \bowtie T$. Suppose then that $R \bowtie S$ and $R \bowtie T$. There exist subsets $R_1, \ldots, R_\alpha$ and $R'_1, \ldots, R'_\beta$ of R, subsets $S_1, \ldots, S_m$ of S, and subsets $T_1, \ldots, T_n$ of T such that

$$\begin{aligned} R &= R_1 \cup \cdots \cup R_\alpha = R'_1 \cup \cdots \cup R'_\beta, \\ S &= S_1 \cup \cdots \cup S_m, \\ T &= T_1 \cup \cdots \cup T_n, \end{aligned}$$

for $1 \leqslant i \leqslant \alpha$ and $1 \leqslant j \leqslant m$ there exists k such that $\mathrm{pr}_k(R_i) \bowtie \mathrm{pr}_k(S_j)$, and for $1 \leqslant i \leqslant \beta$ and $1 \leqslant j \leqslant n$ there exists k such that $\mathrm{pr}_k(R_i) \bowtie \mathrm{pr}_k(T_j)$. Let

$$P_{i,j} = R_i \cap R'_j \quad (1 \leqslant i \leqslant \alpha,\ 1 \leqslant j \leqslant \beta).$$

Then

$$R \subset \bigcup_{\substack{1 \leqslant i \leqslant \alpha \\ 1 \leqslant j \leqslant \beta}} P_{i,j}.$$

Also, if $l \leqslant m$, then, choosing k such that $\mathrm{pr}_k(R_i) \bowtie \mathrm{pr}_k(S_l)$, we see that

$$\left(\mathrm{pr}_k(R_i) \cap \mathrm{pr}_k\left(R'_j\right)\right) \bowtie \mathrm{pr}_k(S_l),$$

so $\mathrm{pr}_k(P_{i,j}) \bowtie \mathrm{pr}_k(S_l)$. Likewise, if $l \leqslant n$, then there exists k such that $\mathrm{pr}_k(P_{i,j}) \bowtie \mathrm{pr}_k(T_l)$. Since

$$S \cup T \subset \bigcup_{1 \leqslant i \leqslant m} S_i \cup \bigcup_{1 \leqslant j \leqslant n} T_j,$$

it follows that $R \bowtie (S \cup T)$.

B4 This is trivial.

This completes the proof that X is a pre–apartness space. Now suppose that X_1, X_2 are actually apartness spaces. Then, regarded as a point–set apartness space, each X_i is locally decomposable; whence, by Proposition 2, the product point–set apartness space X is locally decomposable. It follows from Proposition 7 that the product point–set apartness on X is precisely the point–set relation induced by the product set–set apartness on X. Hence X satisfies B5. ■

It is almost immediate from the definition of the product set–set pre–apartness structure on $X = X_1 \times X_2$ that the projection maps $\mathrm{pr}_i : X \to X_i$ are strongly continuous. We end the paper by showing that the product pre–apartness structure has the characteristic property of a categorical product.

Proposition 9 *Let $X = X_1 \times X_2$ be the product of two set–set pre–apartness spaces, and f a mapping of a set–set pre–apartness space Y into X. Then f is strongly continuous if and only if $\mathrm{pr}_i \circ f$ is strongly continuous for each i.*

Proof. Assume that $\mathrm{pr}_i \circ f$ is strongly continuous for each i. Let S, T be subsets of Y such that $f(S) \bowtie f(T)$, and choose finitely many subsets A_i $(1 \leqslant i \leqslant m)$ and B_j $(1 \leqslant j \leqslant n)$ of X such that

- $f(S) \subset A_1 \cup \cdots \cup A_m$,
- $f(T) \subset B_1 \cup \cdots \cup B_n$, and
- for all i, j either $\mathrm{pr}_1 A_i \bowtie \mathrm{pr}_1 B_j$ or $\mathrm{pr}_2 A_i \bowtie \mathrm{pr}_2 B_j$.

For all i, j write $S_i = f^{-1}(A_i)$ and $T_j = f^{-1}(B_j)$. Then $S \subset S_1 \cup \cdots \cup S_m$ and $T \subset T_1 \cup \cdots \cup T_n$. Moreover, for all i, j we have either $\mathrm{pr}_1 \circ f(S_i) \bowtie \mathrm{pr}_1 \circ f(T_j)$ or $\mathrm{pr}_2 \circ f(S_i) \bowtie \mathrm{pr}_2 \circ f(T_j)$; whence $S_i \bowtie T_j$, by our strong continuity hypothesis. It follows that $S \bowtie T$ and hence that f is strongly continuous.

The converse readily follows from the continuity of the projection maps. ■

Thus we have shown that under reasonable conditions, such as local decomposability in the point–set case, both the point–set product apartness and the set–set product (pre)apartness are categorical. This is another indication that the notion of apartness may have some constructive merit.

In earlier papers such as [3], we have required that point–set apartness spaces be nontrivial. This requirement would mean that in the context of product apartness structures, we would not have nullary products, which are the terminal objects in the category.

Much has been done in the four years since the theory of apartness spaces was first broached as a possible way of approaching topology constructively. In particular, connections between point–set apartness and topology, and between set–set apartness and uniformity, have been explored in some depth; see [3, 4, 7, 8, 9]. Among the interesting phenomena observable with intuitionistic logic is that if certain natural set–set apartness structures are induced (as they are under classical logic) by uniform structures, then the weak law of excluded middle,

$$\neg P \vee \neg\neg P,$$

holds; see [8] (Proposition 4.2 and Corollary 4.3). Thus, in a certain sense, the constructive theory of apartness spaces is larger than the theory of uniform spaces.

Of the major problems that remain in the foundations of apartness–space theory, the most significant and resistible to constructive attack is that of compactness. Since the Heine–Borel property fails to hold for the interval $[0, 1]$ in constructive mathematics plus Church's thesis ([2], Chapter 3), and since the sequential compactness of $[0, 1]$ is essentially nonconstructive, for metric and uniform spaces we define *compact* to mean *totally bounded and complete.* Lifting such ideas from the uniform– to the general apartness–space context appears to be a highly nontrivial exercise, for which we have as yet only partial solutions (see [5]).

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