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EQUIVARIANT GOTTLIEB GROUPS

by Marek GOLASIŃSKI and Daciberg LIMA GONÇALVES*

Résumé

Nous développons le diagramme de groupes de Gottlieb $G_n(\boldsymbol{X})$ et $G_n(X)$, pour $n \geq 1$, et $\boldsymbol{X}$ un espace de diagrammes et X un espace équivariant respectivement. Nous donnons quelques-unes de leurs propriétés, étendant celles du cas nonéquivariant. Ensuite, au moyen de la G-fibration universelle $p_\infty : E_\infty \to B_\infty$, nous établissons un lien entre $F_n(F)$ et les homomorphismes de connexion déterminés par une G-fibration $E \to B$ de fibre F.

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Introduction

The Gottlieb groups $G_n(X)$ of a connected pointed space X were defined in [7, 8]; first $G_1(X)$ and then $G_n(X)$ for all $n \geq 1$. As a result of [7], the group $G_1(X)$ plays a central role in the study of Jiang subgroups applied in fixed point theory. In [16], P. Wong considered a concept of equivariant Jiang subgroups which play a relevant role in the equivariant Nielsen theory for equivariant maps. The higher Gottlieb groups $G_n(X)$ are related in [8, 9] to the problem of sectioning fibrations

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with fibre X. For instance, if the Gottlieb group $G_n(X)$ is trivial then there is a homotopy section for every fibration over the sphere $\mathbb{S}^{n+1}$, with fibre X.

The objectives of this paper include the development of an equivariant concept of Gottlieb groups which will extend the relation with the Jiang subgroups to the equivariant case as well as allowing analyses of sectioning for equivariant fibrations. Section 1 serves as an introduction for the rest of the paper. We recall the definition of Gottlieb groups and examine the relationship between evaluation subgroups $\widetilde{G}_n(X)$ and $G_n(X)$ for any space X and $n \geq 1$. In particular, we show that for any k-space X the Gottlieb group $G_n(X)$ is isomorphic to the evaluation subgroup $\widetilde{G}_n(X)$ for all $n \geq 1$.

By [4] the homotopy category of G-spaces is equivalent to the homotopy category of diagram spaces indexed by the canonical orbits $\mathcal{O}(G)$ and in [3, 11] diagram of spaces indexed by different categories are studied. Therefore, we have first developed, in Section 2, the diagram of Gottlieb groups $\boldsymbol{G}_n(\boldsymbol{X})$ corresponding to an $\mathbb{I}$-diagram of spaces $\boldsymbol{X}$ and $n \geq 1$, and present some of their properties extending those in [7, 8]. We note that if $\mathbb{I}$ is an EI-category then the diagram $\boldsymbol{G}_n(\boldsymbol{X})$ gives rise to a complex of groups (Remark 2.1). Thus the all results on complexes of groups presented for instance in [2] can be used to study these diagrams $\boldsymbol{G}_n(\boldsymbol{X})$ of groups. Then, we generalize (Proposition 2.4) the result presented in [8] by relating $\boldsymbol{G}_1(\boldsymbol{X})$ with the diagram $\boldsymbol{\pi}_1(\boldsymbol{X})$ of fundamental groups for a diagram $\boldsymbol{X}$ of aspherical CW-spaces.

For an equivariant space X, diagrams of Gottlieb groups $\boldsymbol{G}_n(X)$ determining a complex of groups are discussed in Section 3. We present two equivalent definitions of those diagrams, relate them with the Gottlieb groups of some orbit spaces and the equivariant Jiang groups developed in [16]. The category $\mathcal{O}(G)$ of canonical orbits is an EI-category for any group G. Thus we deduce that the diagram $\boldsymbol{G}_n(X)$ gives rise to a complex of groups for any G-space X. At the end, some examples of those diagrams are stated.

Section 4 indicates how to extend Gottlieb's results presented in [9, 12] to the equivariant case. We make use of the universal G-fibration $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ constructed in [15] to enunciate (Theorem 4.1) a relation between $G_n(F)$ and connecting homomorphisms determined by a G-

fibration $E \to B$ with fibre F. We also investigate the diagram $G_1(\mathcal{B}_\infty)$ and deduce (Theorem 4.3) that $G_1(\mathcal{B}_\infty)$ is trivial provided that the fibre of $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ is an equivariant Eilenberg-MacLane space $K(\pi, n)$.

We have included certain results interpreted for complexes of groups. Although these are not central to our principal applications we feel they are suggestive of a link that deserves further study.

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1 Nonequivariant backgrounds

Throughout this section spaces will be connected and pointed. The n-th *Gottlieb group* $G_n(X)$ of a space X, defined in [7, 8] for $n \geq 1$, is the subgroup of the n-th homotopy group $\pi_n(X)$ containing all elements which can be represented by a map $\alpha : \mathbb{S}^n \to X$ from the n-sphere $\mathbb{S}^n$ and such that $\alpha \vee \mathrm{id}_X : \mathbb{S}^n \vee X \to X$ extends (up to homotopy) to a map $F : \mathbb{S}^n \times X \to X$. The map $F : \mathbb{S}^n \times X \to X$ is called an *associated map* for $\alpha : \mathbb{S}^n \to X$. If CX is the space of all self-maps of the space X (with the compact-open topology) then the evaluation map $\mathrm{ev} : CX \to X$ at the base point is continuous, and one can form the group $\widetilde{G}_n(X) = \mathrm{Im}\,(\mathrm{ev}_n : \pi_n(CX, \mathrm{id}_X) \to \pi_n(X))$. Certainly, both groups are isomorphic if X is locally compact.

Now compare the groups $G_n(X)$ and $\tilde{G}_n(X)$ for any space X. First, recall some facts on k-spaces (see e.g. [6, Appendix]). A subset $A \subseteq X$ is said to be *compactly closed*, if for every compact space C and map $f : C \to X$ the preimage $f^{-1}(A)$ is closed in C. A space is said to be a *k-space* whenever all its compactly closed subsets are closed. The property of being a k-space is preserved by closed subspaces and identifications. All metric spaces and CW-spaces are k-spaces. For an arbitrary space X there is a *k-ification* kX, the space having the same underlying set as X, but with the topology given by taking as closed sets the compactly closed sets with respect to the topology of X. Note that the identity map $kX \to X$ is continuous and a weak homotopy equivalence. Moreover, if Y is a k-space then a map $f : Y \to X$ is continuous if and only if the map $f : Y \to kX$ is continuous. If X and

Y are k-spaces then their product $X \times Y$ (in the category of k-spaces) is given by $k(X \times_c Y)$, where $X \times_c Y$ is the Cartesian product (endowed with the product topology). If at least one of the spaces X or Y is locally-compact Hausdorff then $X \times Y = X \times_c Y$. The category of k-spaces is both complete and cocomplete, and also has mapping spaces satisfying the exponential law. More precisely, for any k-spaces X and Y, the mapping space $L(Y, X) = k(C(Y, X))$, where $C(Y, X)$ is the space of all continuous maps $Y \to X$ with the compact-open topology.

Proposition 1.1 (1) *If X is a k-space then there is an isomorphism*

$$G_n(X) \xrightarrow{\cong} \tilde{G}_n(X);$$

(2) *If X is any space then there exist monomorphisms*

$$G_n(X) \longrightarrow G_n(kX) \xrightarrow{\cong} \tilde{G}_n(kX) \longleftarrow \tilde{G}_n(X);$$

(3) *If X is a Hausdorff space then there is a commutative diagram*

$$\begin{array}{ccc} G_n(X) & \longrightarrow & \tilde{G}_n(X) \\ \downarrow & & \downarrow \\ G_n(kX) & \xrightarrow{\cong} & \tilde{G}_n(kX) \end{array}$$

of monomorphisms for any $n \geq 1$.

Proof: (1) is a direct consequence of the exponential law in the category of k-spaces.

(2) For a map $\alpha : \mathbb{S}^n \to X$ there is its lifting $\tilde{\alpha} : \mathbb{S}^n \to kX$, since the sphere $\mathbb{S}^n$ is compact. If $F : \mathbb{S}^n \times X \to X$ is an associated map for α then a lifting $\tilde{F} : \mathbb{S}^n \times kX \to kX$ of the composite map $\mathbb{S}^n \times kX \to \mathbb{S}^n \times X \xrightarrow{F} X$ is an associated map for $\tilde{\alpha}$. Thus, a map $G_n(X) \to G_n(kX)$ has been defined.

Note that the canonical map $kX \to X$ yields an isomorphism $\pi_n(kX) \xrightarrow{\cong} \pi_n(X)$ and the diagram

$$\begin{array}{ccc} G_n(X) & \longrightarrow & G_n(kX) \\ \downarrow & & \downarrow \\ \pi_n(X) & \xleftarrow{\cong} & \pi_n(kX) \end{array}$$

commutes, where the vertical maps are given by inclusions. Hence, the top map $G_n(X) \to G_n(kX)$ is a monomorphism. Analogously, one can define a map $\tilde{G}_n(X) \to \tilde{G}_n(kX)$ and show that this is also a monomorphism.

(3) If the space X is Hausdorff then an associated map $F : \mathbb{S}^n \times X \to X$ for $\alpha : \mathbb{S}^n \to X$ determines an adjoint map $\bar{F} : \mathbb{S}^n \to CX$ and the rest follows from (2).

□

For k-spaces X and Y and a map $f : X \to Y$, the evaluation map $\mathrm{ev} : L(Y,X) \longrightarrow X$ at the base point y_0 in Y induces a homomorphism

$$\mathrm{ev}_n : \pi_n(L(Y,X), f) \longrightarrow \pi_n(X, f(y_0))$$

for any $n \geq 1$. The subgroups $J(Y,X;f) = \mathrm{ev}_1(\pi_1(L(Y,X),f))$, called the *Jiang subgroups* of f in $L(Y,X)$, play an important role in fixed point theory.

If X is a CW-space then by [1] there is a universal fibration $p_\infty : E_\infty \to B_\infty$ with fibre F_∞ homotopy equivalent to X. Let $\partial_\infty : \pi_{n+1}(B_\infty) \to \pi_n(F_\infty)$ be the connecting homomorphism from the homotopy exact sequence of the fibration p_∞. The following result is presented in [8].

Theorem 1.2 (1) *If X and Y are k-spaces then there is an inclusion*

$$G_n(X, f(y_0)) \subseteq \mathrm{ev}_n(\pi_n(L(Y,X), f))$$

for any $n \geq 1$, a map f in $L(Y,X)$ and the base point y_0 in Y;

(2) *If X is a CW-space then $G_n(F_\infty) = \partial_\infty(\pi_{n+1}(B_\infty))$. Thus, for any fibration*

$$X \to E \to B,$$

$\partial(\pi_{n+1}(B)) \subseteq G_n(X)$, where $\partial : \pi_{n+1}(B) \to \pi_n(X)$ is the connecting homomorphism from the homotopy exact sequence of this fibration.

Consequently, if $G_n(X) = 0$, then every fibration over the sphere $\mathbb{S}^{n+1}$, with fibre X, has a homotopy section. The group $G_n(X)$ is the intersection of all subgroups of $\pi_n(X)$ which are the image of a homomorphism induced by an evaluation map. The group $G_n(X)$ is also the union of all subgroups of $\pi_n(X)$ each of them is the image of the connecting homomorphism arising in the homotopy exact sequence of a fibration with fibre X.

2 Gottlieb groups of diagrams

If $\sigma : I \to X$ is a path in X with $\sigma(0) = x_0$ and $\sigma(1) = x_1$ then by [8] the induced isomorphism of homotopy groups $\sigma_* : \pi_n(X, x_1) \to \pi_n(X, x_0)$ restricts to an isomorphism $\sigma_* : G_n(X, x_1) \to G_n(X, x_0)$. Unfortunately, as was shown in [8], it is not true that any map $f : X \to Y$ of spaces induces a map from $G_n(X)$ to $G_n(Y)$.

Let now $\Pi(X)$ be the fundamental groupoid of the space X and $\mathbb{G}pd$ the category of groups. Thus, in the light of the result above, one gets a local coefficient system

$$\boldsymbol{G}_n(X) : \Pi(X) \longrightarrow \mathbb{G}pd$$

such that $\boldsymbol{G}_n(X)(x) = G_n(X, x)$ for all points x in the space X and $n \geq 1$.

Let $\mathbb{I}$ be a small category with the set $\mathrm{Ob}(\mathbb{I})$ as its objects and $\boldsymbol{X}$ a contravariant functor from $\mathbb{I}$ into k-spaces, called an $\mathbb{I}$-*diagram* of spaces. The notions of well-pointing, path-connectivity, extension property and others can be easily extended from spaces into $\mathbb{I}$-diagrams of spaces in an obvious way and the homotopy theory of $\mathbb{I}$-diagrams has been developed in [14]. The object-wise product $\boldsymbol{X} \times \boldsymbol{Y}$ of two $\mathbb{I}$-diagrams $\boldsymbol{X}$ and $\boldsymbol{Y}$ forms the categorical product. Observe that, for these two $\mathbb{I}$-diagrams, the "hom-set" $\mathrm{Nat}(\boldsymbol{Y}, \boldsymbol{X})$ in the category of $\mathbb{I}$-diagrams is contained in the product $\prod_{i \in \mathrm{Ob}(\mathbb{I})} L(\boldsymbol{Y}(i), \boldsymbol{X}(i))$ of k-spaces. Endow $\mathrm{Nat}(\boldsymbol{Y}, \boldsymbol{X})$ with the k-ification of the induced topology from the product $\prod_{i \in \mathrm{Ob}(\mathbb{I})} L(\boldsymbol{Y}(i), \boldsymbol{X}(i))$. Moreover, any set yields a k-space with respect to the discrete topology. Define the *mapping* $\mathbb{I}$-*diagram* $\boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X})$ by taking

$$\boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X})(i) = \mathrm{Nat}(\boldsymbol{Y} \times \mathbb{I}(-, i), \boldsymbol{X}),$$

for any object i in $\mathrm{Ob}(\mathbb{I})$ and in the obvious way on morphisms in $\mathbb{I}$, where $\mathbb{I}(-, i)$ is the Yoneda functor determined by i in $\mathrm{Ob}(\mathbb{I})$. Then, for $\mathbb{I}$-diagrams $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, there is a bijection

$$\mathrm{Nat}(\boldsymbol{Y} \times \boldsymbol{Z}, \boldsymbol{X})) \cong \mathrm{Nat}(\boldsymbol{Z}, \boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X}))$$

natural in $\boldsymbol{X}$ and $\boldsymbol{Y}$. Consequently, the category of $\mathbb{I}$-diagrams is cartesian closed and the evaluation map $\mathbf{exp} : \boldsymbol{Y} \times \boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X}) \to \boldsymbol{X}$ can be derived, for any $\mathbb{I}$-diagrams $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Given an $\mathbb{I}$-diagram $\boldsymbol{X}$ of spaces, one can form the associated *fundamental category* $\Pi(\mathbb{I},\boldsymbol{X})$ studied for example in [11, p. 144] for the orbit category $\mathbb{I} = \mathcal{O}(G)$ of a group G and then generalized in [14] for any small category $\mathbb{I}$. For a map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, let $\Pi(\boldsymbol{f}) : \Pi(\mathbb{I},\boldsymbol{X}) \to \Pi(\mathbb{I},\boldsymbol{Y})$ be the induced functor. Our aim is to define a $\Pi(\mathbb{I},\boldsymbol{X})$-diagram of groups

$$\boldsymbol{G}_n(\boldsymbol{X}) : \Pi(\mathbb{I},\boldsymbol{X}) \to \mathbb{G}pd.$$

If $\mathbb{S}^n$ is the n-sphere with a base-point s_0 and $\boldsymbol{\Lambda} : \mathbb{S}^n \times \boldsymbol{X} \to \boldsymbol{X}$ a map of $\mathbb{I}$-diagrams such that $\boldsymbol{\Lambda}|_{s_0\times \boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ is homotopic to the identity $\mathrm{id}_{\mathbf{X}}$, then for any object i in $\mathrm{Ob}\,(\mathbb{I})$ and x_i in $\boldsymbol{X}(i)$, one gets an associated map $\boldsymbol{\Lambda}(i) : \mathbb{S}^n \times \boldsymbol{X}(i) \to \boldsymbol{X}(i)$ for the map $\alpha_i : \mathbb{S}^n \to \boldsymbol{X}(i)$ given by $\alpha_i(s) = \boldsymbol{\Lambda}(i)(s,x_i)$ for all s in $\mathbb{S}^n$. Let $\boldsymbol{G}_n(\boldsymbol{X}(i),x_i)$ be the subset of the homotopy group $\pi_n(\boldsymbol{X}(i),x_i)$ containing all elements which can be represented by those maps $\alpha_i : \mathbb{S}^n \to \boldsymbol{X}(i)$. Say that $\boldsymbol{\Lambda}$ is an $\mathbb{I}$-*associated map*. It is easy to see that $\boldsymbol{G}_n(\boldsymbol{X}(i),x_i)$ is a subgroup not only of $\pi_n(\boldsymbol{X}(i),x_i)$ but also of the Gottlieb group $G_n(\boldsymbol{X}(i),x_i)$. If $(\varphi,\sigma) : (i,x_i) \to (j,x_j)$ is a map in $\Pi(\mathbb{I},\boldsymbol{X})$ then there is an induced map $(\varphi,\sigma)^* : \boldsymbol{G}_n(\boldsymbol{X}(j),x_j) \to \boldsymbol{G}_n(\boldsymbol{X}(i),x_i)$. Thus one obtains a $\Pi(\mathbb{I},\boldsymbol{X})$-diagram $\boldsymbol{G}_n(\boldsymbol{X})$ of groups given by $\boldsymbol{G}_n(\boldsymbol{X})(i,x_i) = \boldsymbol{G}_n(\boldsymbol{X}(i),x_i)$ for any object (i,x_i) in $\Pi(\mathbb{I},\boldsymbol{X})$.

Recall that a small category $\mathbb{I}$ is called an *EI-category* ([11]) if any endomorphism in $\mathbb{I}$ is an isomorphism. Note that for any *EI*-category $\mathbb{I}$ and an $\mathbb{I}$-diagram $\boldsymbol{X}$ of spaces a skeleton $\Pi'(\mathbb{I},\boldsymbol{X})$ of the associated fundamental category $\Pi(\mathbb{I},\boldsymbol{X})$ is a category without loops. Then, according to [2], the geometric realization $B\Pi'(\mathbb{I},\boldsymbol{X})$ of the nerve of $\Pi'(\mathbb{I},\boldsymbol{X})$ (called the *classifying space* of $\Pi'(\mathbb{I},\boldsymbol{X})$) is an ordered simplicial cell complex. Thus one may easily check

Remark 2.1 If $\mathbb{I}$ is an *EI*-category and $\boldsymbol{X}$ an $\mathbb{I}$-diagram of spaces then the restriction of the $\Pi(\mathbb{I},\boldsymbol{X})$-diagram

$$\boldsymbol{G}_n(\boldsymbol{X}) : \Pi(\mathbb{I},\boldsymbol{X}) \to \mathbb{G}pd$$

to the skeleton $\Pi'(\mathbb{I},\boldsymbol{X})$ gives rise to a complex of groups $G'_n(B\Pi'(\mathbb{I},\boldsymbol{X}))$ (see [2] for the definition) on the ordered simplicial cell complex $B\Pi'(\mathbb{I},\boldsymbol{X})$.

For an $\mathbb{I}$-diagram $\boldsymbol{Y}$ of spaces, fix y_i in $\boldsymbol{Y}(i)$ for any i in $\mathrm{Ob}(\mathbb{I})$. Then the exponential map $\mathbf{exp} : \boldsymbol{Y} \times \boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X}) \to \boldsymbol{X}$ gives rise to an evaluation map

$$\mathrm{ev} : \boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X}) \to \boldsymbol{X}.$$

Any map $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ of $\mathbb{I}$-diagrams composed with the projection $\boldsymbol{Y} \times \mathbb{I}(-, i) \to \boldsymbol{Y}$ yields an element in $\boldsymbol{L}(\boldsymbol{X}, \boldsymbol{Y})(i)$ for any i in $\mathrm{Ob}(\mathbb{I})$ and consequently, one gets a base point in the mapping diagram $\boldsymbol{L}(\boldsymbol{X}, \boldsymbol{Y})$ denoted also by $\boldsymbol{f}$. Thus, the induced homomorphism

$$\mathbf{ev}(i)_n : \pi_n(\boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X})(i), \boldsymbol{f}) \longrightarrow \pi_n(\boldsymbol{X}(i), \boldsymbol{f}(y_i))$$

occurs for any $n \geq 1$ and i in $\mathrm{Ob}(\mathbb{I})$. The arguments given for Proposition 1.1 and the results presented in [8, Proposition 1-4 and Theorem 1-7] generalize without much difficulty to the case of $\mathbb{I}$-diagrams, for instance:

Theorem 2.2 (1) *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathbb{I}$-diagram of spaces then there is an inclusion*

$$\boldsymbol{G}_n(\boldsymbol{X}(i), \boldsymbol{f}(y_i)) \subseteq \mathbf{ev}(i)_n(\pi_n(\boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{X})(i), \boldsymbol{f}))$$

for any $n \geq 1$, a map $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ of $\mathbb{I}$-diagrams and a point y_i in $\boldsymbol{Y}(i)$ with i in $\mathrm{Ob}(\mathbb{I})$*;*

(2) *If $\boldsymbol{r} : \boldsymbol{Y} \to \boldsymbol{X}$ is a homotopy retract of well-pointed and path connected $\mathbb{I}$-diagrams then there is a natural transformation*

$$\boldsymbol{G}_n(\boldsymbol{Y}) \to \boldsymbol{G}_n(\boldsymbol{X}) \circ \Pi(\boldsymbol{r}).$$

In particular, if $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ is a homotopy equivalence of well-pointed and path-connected $\mathbb{I}$-diagrams with $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ a homotopy inverse then there are equivalences of diagrams

$$\boldsymbol{G}_n(\boldsymbol{Y}) \cong \boldsymbol{G}_n(\boldsymbol{X}) \circ \Pi(\boldsymbol{f})$$

and

$$\boldsymbol{G}_n(\boldsymbol{X}) \cong \boldsymbol{G}_n(\boldsymbol{Y}) \circ \Pi(\boldsymbol{g}).$$

Proof: (1) It is a straightforward extension of Proposition 1.1 to $\mathbb{I}$-diagrams.

(2) Let α_i be in $G_n(\boldsymbol{X}(i), x_i)$ with i in $\mathbb{I}$. By the well-pointness property of $\boldsymbol{Y}$, one may assume that the image $\boldsymbol{r}_*(\alpha_i)$ of α_i is in $\pi_n(\boldsymbol{Y}(i), \boldsymbol{r}(x_i))$.

Set $\boldsymbol{\Lambda} : \mathbb{S}^n \times \boldsymbol{X} \to \boldsymbol{X}$ for an $\mathbb{I}$-associated map to α_i. Then define a map

$$\boldsymbol{\Lambda}' : \mathbb{S}^n \times \boldsymbol{Y} \to \boldsymbol{Y}$$

by letting $\boldsymbol{\Lambda}'(s, y) = \boldsymbol{r}\boldsymbol{\Lambda}(s, \boldsymbol{t}(y))$, where $\boldsymbol{t} : \boldsymbol{Y} \to \boldsymbol{X}$ is a right homotopy inverse to $\boldsymbol{r} : \boldsymbol{X} \to \boldsymbol{Y}$. Then one can mimic the proof of [8, Proposition 1-4] and use the extension homotopy property of the inclusion $* \times \boldsymbol{Y} \hookrightarrow \mathbb{S}^n \times \boldsymbol{Y}$ to find an associated map for $\boldsymbol{r}_*(\alpha_i)$ and show that $\boldsymbol{r}_*(\alpha_i)$ is in $G_n(\boldsymbol{Y}(i), \boldsymbol{r}(x_i))$.

The rest of the proof goes over from the nonequivariant case [8, Theorem 1-7] verbatim.

□

Given $\mathbb{I}$-diagrams $\boldsymbol{X}$ and $\boldsymbol{Y}$ of spaces the projection maps $\boldsymbol{p} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{X}$ and $\boldsymbol{q} : \boldsymbol{X} \times \boldsymbol{Y} \to \boldsymbol{Y}$ determine a functor $(\Pi(\boldsymbol{p}), \Pi(\boldsymbol{q})) : \Pi(\mathbb{I}, \boldsymbol{X} \times \boldsymbol{Y}) \to \Pi(\mathbb{I}, \boldsymbol{X}) \times \Pi(\mathbb{I}, \boldsymbol{Y})$. Therefore, one can conclude

Corollary 2.3 *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathbb{I}$-diagrams of spaces then there is an equivalence of diagrams*

$$\boldsymbol{G}_n(\boldsymbol{X} \times \boldsymbol{Y}) \cong (\boldsymbol{G}_n(\boldsymbol{X}) \times \boldsymbol{G}_n(\boldsymbol{Y})) \circ (\Pi(\boldsymbol{p}), \Pi(\boldsymbol{q})).$$

Gottlieb has shown in [8] that $G_1(X) \subseteq \mathfrak{Z}(\pi_1(X))$, the center of the group $\pi_1(X)$, for any space X and $G_1(X) = \mathfrak{Z}(\pi_1(X))$, if X has the homotopy type of an aspherical CW-space X. Note that an element α in $\pi_1(X)$ is in $\mathfrak{Z}(\pi_1(X))$ if and only if the map $\varphi : \mathbb{Z} \times \pi_1(X) \to \pi_1(X)$ given by $\varphi(n, g) = \alpha^n g$, for (n, g) in $\mathbb{Z} \times \pi_1(X)$ is a homomorphism of groups, where $\mathbb{Z}$ denotes the integers. Let now $\boldsymbol{\pi}_1(\boldsymbol{X})$ be the $\Pi(\mathbb{I}, \boldsymbol{X})$-diagram of fundamental groups, for an $\mathbb{I}$-diagram $\boldsymbol{X}$ of spaces. Thus one has the following extension of Gottlieb's result.

Proposition 2.4 (1) *If $\boldsymbol{X}$ is an $\mathbb{I}$-diagram of spaces then any $\mathbb{I}$-associated map $\boldsymbol{\Lambda} : \mathbb{S}^1 \times \boldsymbol{X} \to \boldsymbol{X}$ gives a rise to a map*

$$\boldsymbol{\lambda} : \mathbb{Z} \times \boldsymbol{\pi}_1(\boldsymbol{X}) \to \boldsymbol{\pi}_1(\boldsymbol{X})$$

of $\Pi(\mathbb{I}, \boldsymbol{X})$*-diagrams restricting to the identity on* $\pi_1(\boldsymbol{X})$;

(2) *If* $\boldsymbol{X}$ *is an* $\mathbb{I}$*-diagram of aspherical CW-spaces then any map* $\lambda : \mathbb{Z} \times \pi_1(\boldsymbol{X}) \to \pi_1(\boldsymbol{X})$ *of* $\Pi(\mathbb{I}, \boldsymbol{X})$*-diagrams restricting to the identity on* $\pi_1(\boldsymbol{X})$ *gives a rise to an* $\mathbb{I}$*-associated map*

$$\Lambda : \mathbb{S}^1 \times \boldsymbol{X} \to \boldsymbol{X}.$$

3 Gottlieb groups of equivariant spaces

Let now G be a finite group and X a G-space. For a subgroup $H \subseteq G$, let X^H be the fixed-point subspace. Then one can form the following index category $\mathcal{O}(G, X)$ (called in [3, p. 72] the *orbit category over* X; the associated discrete category $\Pi_0(G, X)$, called the *component category over* X, has been studied in [3, p. 73] and [11, p. 99]):

(1) the set $\mathrm{Ob}(\mathcal{O}(G, X))$ of objects consists of pairs $(G/H, x)$ with H a subgroup of G and x in X^H,

(2) morphisms $(G/H, x) \to (G/K, y)$ are given by G-maps $\phi : G/H \to G/K$ such that $\overline{\phi}(y) = x$, with $\overline{\phi} : X^K \to X^H$ as the induced map of the fixed-point subspaces.

Consider the $\mathcal{O}(G, X)$-diagram $\boldsymbol{X}$ of spaces such that $\boldsymbol{X}(G/H, x) = X^H_x$ with X^H_x as the path-connected component of X^H consisting of the point x. If $\Pi(G, X)$ denotes the fundamental category $\Pi(\mathcal{O}(G, X), \boldsymbol{X})$ studied already in the previous section then, in light of that section, one gets $\Pi(G, X)$-systems of Gottlieb groups $\boldsymbol{G}_n(\boldsymbol{X})$ for all $n \geq 1$.

On the other hand, if $\mathbb{S}^n$ is the n-sphere with a base-point s_0 and the trivial action of G, and $\Lambda : \mathbb{S}^n \times X \to X$ a G-map such that $\Lambda|_{s_0 \times X} : X \to X$ is G-homotopic to the identity map id_X then for any subgroup $H \subseteq G$ and x in X^H one gets an associated map $\Lambda^H : \mathbb{S}^n \times X^H \to X^H$ for the map $\alpha^H : \mathbb{S}^n \to X^H$ given by $\alpha^H(s) = \Lambda^H(s, x)$ for all s in $\mathbb{S}^n$. Let $\boldsymbol{G}_n(X^H, x)$ be the subset of the homotopy group $\pi_n(X^H, x_i)$ containing all elements which can be represented by those maps $\alpha^H : \mathbb{S}^n \to X^H$. It is easy to see that $\boldsymbol{G}_n(X^H, x)$ is a subgroup not only of $\pi_n(X^H, x)$ but also of the Gottlieb group $G_n(X^H, x)$, and there is a map $\boldsymbol{G}_n(X^H, x) \to G_n(X, x)$ determined by the inclusion $X^H \subseteq X$.

Let $C_G X$ be the space (with the compact-open topology) of all self G-maps of the G-space X. Given a subgroup $H \subseteq G$ and a point x

in X^H consider the evaluation map $\mathrm{ev}_H : (C_G X, \mathrm{id}_X) \to (X^H, x)$ given by $\mathrm{ev}_H(f) = f^H(x)$ for any f in $C_G X$, with f^H as the induced self map of the fixed-point subspace X^H. Then $\boldsymbol{G}_n(X^H, x) \cong \mathrm{Im}((\mathrm{ev}_H)_* : \pi_n(C_G X, \mathrm{id}_X) \to \pi_n(X^H, x))$ provided that X is a k-space. Observe that there is an isomorphism $\pi_n(C_G X, \mathrm{id}_X) \cong \pi_n(\mathrm{aut}_G X, \mathrm{id}_X)$ for any $n \geq 0$ and the monoid $\mathrm{aut}_G X$ of self G-homotopy equivalences of the G-space X. Finally, one obtains the $\Pi(G, X)$-diagram $\boldsymbol{G}_n(X)$ given by $\boldsymbol{G}_n(X)(G/H, x) = \boldsymbol{G}_n(X^H, x)$ for any object $(G/H, x)$ in $\Pi(G, X)$.

Observe that the category $\mathcal{O}(G)$ of canonical orbits is an EI-category and consequently $\mathcal{O}(G, X)$ is also such a category, for any G-space X. Hence, Remark 2.1 yields

Remark 3.1 For any G-space X the $\Pi(G, X)$-diagram $\boldsymbol{G}_n(X)$ gives rise to a complex of groups $G'_n(B\Pi'(G, X))$ on the classifying space $B\Pi'(G, X)$ of a skeleton $\Pi'(G, X)$ of the category $\Pi(\mathbb{I}, X)$.

As may readily be seen, Theorem 2.2 and Corollary 2.3 can be reformulated for G-spaces as well. Moreover, for G-spaces there is an appropriate version of Proposition 2.4 and one may also state

Proposition 3.2 *If X is a G-CW space then there is an isomorphism of $\Pi(G, X)$ diagrams $\boldsymbol{G}_n(\boldsymbol{X}) \cong \boldsymbol{G}_n(X)$ for all $n \geq 1$.*

Proof: Let X be a G-CW-space and $\boldsymbol{X}$ the associated $\mathcal{O}(G, X)$-diagram of spaces. Then any G-map $\Lambda : \mathbb{S}^n \times X \to X$ determines, in an obvious way, an $\mathcal{O}(G, X)$-map $\boldsymbol{\Lambda} : \mathbb{S}^n \times \boldsymbol{X} \to \boldsymbol{X}$. Then $\boldsymbol{\Lambda}(G/H, x)(s, x) = \Lambda^H(s, x)$ for any s in $\mathbb{S}^n$ and x in X^H. Consequently, one gets a natural transformation

$$\Phi : \boldsymbol{G}_n(X) \to \boldsymbol{G}_n(\boldsymbol{X}).$$

By virtue of the Elmendorf classifying functor [4], an $\mathcal{O}(G, X)$-map $\boldsymbol{\Lambda} : \mathbb{S}^n \times \boldsymbol{X} \to \boldsymbol{X}$ yields a G-map $\Lambda : \mathbb{S}^n \times X \to X$, where $\boldsymbol{\Lambda}$ and Λ are related as above. Thus, one gets a natural transformation

$$\Psi : \boldsymbol{G}_n(\boldsymbol{X}) \to \boldsymbol{G}_n(X)$$

which is an inverse to Φ.

□

For a G-space X, a point x in X and the trivial subgroup $E \subseteq G$, let $\boldsymbol{G}_n(X,x) = \boldsymbol{G}_n(X^E,x)$ for all $n \geq 1$. One can easily mimic the nonequivariant case to shown that the group $\boldsymbol{G}_1(X,x)$ coincides with the G-Jiang subgroup $J_G(X,x)$ (of the identity map on X) studied by P. Wong in [16] for X being a connected G-ENR. If $H \subseteq G$ is a subgroup of G then there is an action of the Weyl group $WH = NH/H$ on the fixed-point subspace X^H of a G-space X. If X/G is its orbit space then the quotient map $p : X \to X/G$ restricts to $p_H : X^H \to X^H/WH$. Then one can prove the result.

Proposition 3.3 (1) *If X is a G-space then for any subgroup $H \subseteq G$ there is an inclusion*

$$(p_H)_*\boldsymbol{G}_n(X^H,x) \subseteq (p_H)_*\pi_n(X^H,x) \cap G_n(X^H/WH, p_H(x))$$

for any $n \geq 1$ and x in X;

(2) *If X is a Hausdorff space with a free G-action then*

$$p_*\boldsymbol{G}_n(X,x) = p_*\pi_n(X,x) \cap G_n(X/G, p(x))$$

for any $n \geq 1$ and x in X. Thus there is an isomorphism

$$p_*\boldsymbol{G}_n(X,x) \cong G_n(X/G, p(x))$$

for all $n > 1$.

Proof: (1) is straightforward to verify.

To prove (2) one has to check the opposite inclusion. For $n > 1$ it follows from the fact that the quotient map $p : X \to X/G$ is a covering.

For $n = 1$, take an element α in the group $p_*\pi_1(X,x) \cap G_1(X/G, p(x))$ and an associated map $F : \mathbb{S}^1 \times X/G \to X/G$. If $F' : I \times X/G \to X/G$ is the corresponding cyclic homotopy then a lifting $F'' : I \times X \to X$ of the composite map $F' \circ (\mathrm{id}_I \times p) : I \times X \to X/G$ is also cyclic, since the element α is in $p_*\pi_1(X,x)$.

To complete the proof one has to show that $F'' : I \times X \to X$ is a G-map. Take a point (t,x) in $X \times I$, an element g in G and a path $\sigma : I \to X$ such that $\sigma(0) = x$ and $\sigma(1) = gx$. From a lifting of the composite map $F' \circ (\mathrm{id}|_{[0,t]} \times p) \circ (\mathrm{id}|_{[0,t]} \times \sigma) : [0,t] \times I \to X/G$ the result follows. □

Some calculations of $G_1(X)$ for a free G-action on an acyclic space X and its relation with equivariant fixed point theory were presented in [5]. At the end of this section, Proposition 3.2 is used to present

Example 3.4 (1) Consider the 2-dimensional torus $\mathbb{T}^2$ with a free action of the cyclic group $\mathbb{Z}/2$. Its orbit space is the Klein Bottle, K. Then, as it was shown in [7], $G_1(K) = \mathfrak{Z}(\pi_1(K)) = \mathbb{Z}$ the group of integers and $\pi_1(\mathbb{T}^2) \cap G_1(K) = \mathbb{Z}$. By virtue of the last proposition one gets that $\boldsymbol{G}_1(\mathbb{T}^2) = \mathbb{Z}$. On the other hand $G_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$.

(2) J. Pak and M.H. Woo considered in [13] the generalized lens spaces $L_{2n+1}(p)$ to prove that

$$G_{2n+1}(L_{2n+1}(p)) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, 3, \\ 2\mathbb{Z}, & \text{for any other } n. \end{cases}$$

Hence $\boldsymbol{G}_{2n+1}(\mathbb{S}^{2n+1}) = G_{2n+1}(L_{2n+1}(p))$.

(3) Consider the product $\mathbb{T}^2 \times I$ with I the unit interval and the map $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ given by $\phi(x, y) = (\bar{x}, \bar{y})$ for (x, y) in $\mathbb{T}^2$, where $\bar{z}$ is the conjugation of z as an element in the field of complex numbers. Take the quotient space $X = \mathbb{T}^2 \times I/\sim$, where the equivalent relation is given by $(x, y, 0) \sim (\bar{x}, \bar{y}, 1)$ for all (x, y) in $\mathbb{T}^2$. The involution $s : \mathbb{T}^2 \times I \to \mathbb{T}^2 \times I$ given by $s(x, y, t) = (x, y, 1 - t)$ for (x, y) in $\mathbb{T}^2$ and t in I determines a semi-free action of the cyclic group $\mathbb{Z}_2$ on the space X. Then $X^{\mathbb{Z}_2} = \mathbb{T}^2 \times \{1/2\} \cup \{[(1,1,0)]_\sim, [(-1,1,0)]_\sim, [(1,-1,0)], [(-1,-1,0)]_\sim\}$ and one gets a fibre sequence

$$\mathbb{T}^2 \times \{1/2\} \to X \to \mathbb{S}^1.$$

Consequently, the space X is aspherical, the canonical map $\boldsymbol{G}_1(X^{\mathbb{Z}_2}, x) \to G_1(X, x)$ is a monomorphism and the induced short exact sequence $0 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(X, x) \to \mathbb{Z} \to 0$ splits with x as a base point in $\mathbb{T}^2 \times \{1/2\}$. Thus, the group $\pi_1(X, x)$ is a semi-direct product of $\mathbb{Z} \oplus \mathbb{Z}$ with the cyclic group $\mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $g(m, n) = (-m, -n)$ for any (m, n) in $\mathbb{Z} \oplus \mathbb{Z}$ and the generator g of $\mathbb{Z}$. The presentation

$$< g, g_1, g_2 \,|\, [g_1, g_2] = 1, g g_1 g^{-1} = g_1^{-1}, g g_2 g^{-1} = g_2^{-1} >$$

of the group $\pi_1(X,x)$ yields that the cyclic group $<(0,0,g^2)>$ is contained in the center $\mathfrak{Z}(\pi_1(X,x))$. On the other hand, if (m,n,g^k) is in $\mathfrak{Z}(\pi_1(X,x))$ then, in particular, $(0,0,g^{-1})(m,n,g^k)(0,0,g)=(m,n,g^k)$. Therefore, $m=n=0$ and k is even since $(0,0,g)$ is not in $\mathfrak{Z}(\pi_1(X,x))$ and, in view of [7], one can say that $G_1(X,x) = 2\mathbb{Z}$. Finally, $G_1(X^{\mathbb{Z}_2},x) = 0$ as a subgroup of the trivial group $G_1(X^{\mathbb{Z}_2},x)\cap G_1(X,x)=0$, since $G_1(X^{\mathbb{Z}_2},x)=\mathbb{Z}\oplus\mathbb{Z}$.

4 Gottlieb groups and equivariant fibrations

For a finite group G, one refers to [15] to recall some notions. An *equivariant fibration over* a G-space B is a G-map $p: E \to B$ which satisfies the G-covering homotopy property. A G-map $p: E\to B$ is a *G-fibration with fibre F* if for each b in B there is an action on the space F of the isotropy subgroup G_b of b such that $p^{-1}(b)$ is G_b-homotopy equivalent to F with respect to the given action. For a G-fibration $p: E\to B$ and a subgroup $H\subseteq G$, in virtue of [9, Corollary 1] (cf. also [12, Theorem 2.2]) there is an inclusion $\partial\pi_{n+1}(B^{G_b},b)\subseteq G_n(F^{G_b},x)$ for all $n\geq 1$ and x in $p^{-1}(b)$, where ∂ is the connecting homomorphism. Moreover, if $\boldsymbol{\pi}_{n+1}(B^{G_b},b)\subseteq \pi_{n+1}(B^{G_b},b)$ is the preimage of the subgroup $\boldsymbol{G}_n(F^{G_b},x)\subseteq G_n(F^{G_b},x)$ by the map ∂, then there is also an inclusion $\partial\boldsymbol{\pi}_{n+1}(B^{G_b},b)\subseteq \boldsymbol{G}_n(F^{G_b},x)$.

If F is a compact CW-space with some collection of actions on F of subgroups of G then, from [15, Section 2], there is a G-fibration $\wp_\infty:\mathcal{E}_\infty\to\mathcal{B}_\infty$ with fibre F being universal for all G-fibrations with fibre F and over a "suitable" base G-space. Moreover, pointed G-fibrations give rise to a homotopy functor. Then one may apply the same techniques as in [1] to derive (by means of Brown's representability result) the existence of a universal G-fibration $\wp_\infty:\mathcal{E}_\infty\to\mathcal{B}_\infty$ for G-fibrations with fibre F being a pointed CW-space. In particular, for pointed G-fibrations with the trivial G-action on base spaces, the space $\mathcal{B}_\infty$ has the homotopy type of the classifying space $\mathrm{B}\,(\mathrm{aut}_G F)$ of the monoid $\mathrm{aut}_G F$ of self G-homotopy equivalences of F.

From now, let F be a G-CW-space with the collection of restricted actions of all subgroups of G and $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ the corresponding universal G-fibration. Then, one has an equivariant version of [9, Theorem 2].

Theorem 4.1 *If F is a compact G-CW-space then $G_n(F^H, x) = \bigcup \mathrm{Im}(\partial : \pi_{n+1}(B^H, p(x)) \to \pi_n(F^H, x))$ for any subgroup $H \subseteq G$ and a point x in F^H, where the union runs over all G-fibrations $F \to E \xrightarrow{p} B$ with B satisfying the properties stated in* [15].

Proof: From Section 3, one knows that there is an isomorphism $G_n(F^H, x) \cong \mathrm{Im}((\mathrm{ev}_H)_* : \pi_n(C_G F, \mathrm{id}_F) \to \pi_n(F^H, x)$ for any $n \geq 1$, a subgroup $H \subseteq G$ and a point x in F^H, where $C_G F$ is the space of self G-maps of the G-space F and $\mathrm{ev}_H : (C_G X, \mathrm{id}_X) \to (X^H, x)$ the evaluation map at the point x in X^H. But, from the construction of the universal G-fibration $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ it follows that $\pi_{n+1}(\mathcal{B}_\infty, \wp_\infty(x)) \cong \pi_n(C_G F, \mathrm{id}_F)$.

Every G-fibration $F \to E \xrightarrow{p} B$ is obtained by pulling back the universal G-fibration $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ via a classifying G-map $B \to \mathcal{B}_\infty$. The pullback provides, in particular, a commutative square

$$\begin{array}{ccc} \pi_{n+1}(B^H, p^H(x)) & \longrightarrow & \pi_{n+1}(\mathcal{B}^H_\infty, \wp^H_\infty(x)) \cong \pi_n(C_G F, \mathrm{id}_F) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_H)_*} \\ \pi_n(F^H, x) & = & \pi_n(F^H, x). \end{array}$$

Therefore, any connecting map ∂ factors through the universal one and the proof is complete.

□

For a subgroup $H \subseteq G$, let $\wp^H_\infty : \mathcal{E}^H_\infty \to \mathcal{B}^H_\infty$ be the fibration given by fixed-point subspaces and $q : E \to B$ the fibration induced from $\wp^H_\infty : \mathcal{E}^H_\infty \to \mathcal{B}^H_\infty$ via a map $\alpha : B \to \mathcal{B}^H_\infty$. Follow [9, 10] to generalize some notations. For k-spaces A and B let $L(A, B)$ be the associated mapping space studied in Section 1 and $L(A, B; \phi)$ its path component containing a map $\phi : A \to B$. Let $L^*(E, \mathcal{E}^H_\infty, \alpha)$ be the space of fibre preserving maps $\tilde{f} : E \to \mathcal{E}^H_\infty$ which carry each fibre of q into a fibre of $\wp^H_\infty$ by a homotopy equivalence and cover maps $f : B \to \mathcal{B}^H_\infty$ are homotopic to $\alpha : B \to \mathcal{B}^H_\infty$. By [9], there is a continuous map

$$\Phi : L^*(E, \mathcal{E}^H_\infty; \alpha) \to L(B, \mathcal{B}^H_\infty; \alpha).$$

Lemma 4.2 *If $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ is a G-fibration universal for G-fibrations with fibre F and $\alpha : B \to \mathcal{B}_\infty^H$ a map with a subgroup $H \subseteq G$ then the homotopy groups $\pi_n(L^*(E, \mathcal{E}_\infty^H; \alpha))$ are trivial for all $n \geq 1$.*

Proof: As in the proof of [9, Lemma 2], let X be a compact CW-space and $\bar{g} : X \to L^*(E, \mathcal{E}_\infty^H; \alpha)$ a quasi-continuous map. Then the commutative triangle

$$\begin{tikzcd} & L^*(E, \mathcal{E}_\infty^H; \alpha) \arrow[d, "\Phi"] \\ X \arrow[ur, "\bar{g}"] \arrow[r, "\bar{\phi}"'] & L(B, \mathcal{B}_\infty^H; \alpha) \end{tikzcd}$$

with $\bar{\phi} = \Phi\bar{g}$ gives rise to the commutative diagram of G-spaces

$$\begin{tikzcd} G/H \times X \times E \arrow[r, "g"] \arrow[d, "\mathrm{id}_{G/H} \times \mathrm{id}_X \times q"'] & \mathcal{E}_\infty \arrow[d, "\wp_\infty"] \\ G/H \times X \times B \arrow[r, "\phi"'] & \mathcal{B}_\infty. \end{tikzcd}$$

In the rest, one makes use of the adjointness of the functors $(-)^H$ and $- \times G/H$ and then the proof of [9, Lemma 2] goes over to this case verbatim.

□

Let $\mathfrak{F}(\mathcal{E}_\infty^H)$ denote the subgroup of homotopy classes of homotopy equivalences $f : F \to F$ which extend to fibre homotopy equivalences $\bar{f} : \mathcal{E}_\infty^H \to \mathcal{E}_\infty^H$. Then, one can make use of the previous lemma and mimic the nonequivariant procedure presented in [9, Section 5] to prove

Theorem 4.3 *If $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ is a G-fibration universal for G-fibrations with fibre F then there is an isomorphism $G_1(\mathcal{B}_\infty^H, b) \cong \mathfrak{F}(\mathcal{E}_\infty^H)$ for any subgroup $H \subseteq G$.*

Let now $\mathcal{O}(G)$ be the category of canonical orbits associated with the group G and π an $\mathcal{O}(G)$-diagram of abelian groups. Then, by [4], for any positive integer $n \geq 1$, there is a G-space $K(\pi, n)$ such that the subspace $K(\pi, n)^H$ of fixed-points is an Eilenberg-Maclane space of type $(\pi(G/H), n)$.

Let $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ be a universal G-fibration for G-fibrations with fibre $F = K(\pi, n)$. Then, in view of Theorem 4.2 and [9], the Gottlieb groups $G_1(\mathcal{B}^H_\infty, b)$ are trivial for all subgroups $H \subseteq G$ and b in $\mathcal{B}^H_\infty$. On the other hand, there is an inclusion $\boldsymbol{G}_1(\mathcal{B}^H_\infty, b) \subseteq G_1(\mathcal{B}^H_\infty, b)$. Thus, one has shown an equivariant version of [9, Corollary 1 in Section 5].

Corollary 4.4 *If $\wp_\infty : \mathcal{E}_\infty \to \mathcal{B}_\infty$ is a universal G-fibration for G-fibrations with fibre $F = K(\pi, n)$ then the $\mathcal{O}(G, \mathcal{B}_\infty)$-system $\boldsymbol{G}_1(\mathcal{B}_\infty)$ of Gottlieb groups is trivial.*

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